Study of Time Fractional Burgers' Equation using Caputo, Caputo-Fabrizio and Atangana-Baleanu Fractional Derivatives

Swapnali Doley, A. Vanav Kumar, Karam Ratan Singh, L. Jino

***Abstract*—In this paper, a numerical study of time-fractional based Burgers' equation (TFBE) by using Caputo fractional derivatives (CFD), Caputo-Fabrizio fractional derivatives (CFFD), and Atangana-Baleanu fractional derivatives (ABFD) is carried out. The TFBE is solved using an implicit upwind scheme based on the finite difference method (IFDM). After verifying the IFDM scheme against the stability and convergence, the numerical solutions that are derived and validated with the exact solution and also compared with the CFD, CFFD, and ABFD models.**

***Index Terms*—Caputo, Caputo-Fabrizio, Atangana-Baleanu, time fractional, Burgers' equation.**

I. Introduction

THE time-fractional based Burgers' equations (TFBE) play an important role in fluid mechanics research as a model of subdiffusive convection equations. Such models are useful in describing and better understanding the flow systems such as shock propagation, electromagnetic waves, turbulence, porous media flows, contaminant flow, temperature and pressure waves, medical sciences, etc [1], [2], [3], [4]. Numerical solutions for ordinary Burgers' equation are achieved by various techniques such as explicit, implicit, and Crank-Nicolson (C-N) schemes respectively [5], [6], [7]. Moreover, numerical solutions still provide an advantage in solving the complicated problems based on fractional derivatives such as time-fractional diffusion equation [8], [9], space fractional equations using C-N method [10], convection-diffusion equation with space and time-fractional equations [11], [12]. For instance, Xu and Agrawal [13] discussed the finite difference method (FDM) based solutions for TFBE equations and found that the FDM solution is simple and stable. Likewise, Esen and Tasbozan [14] were involved in studying the Galerkin-based finite element schemes for TFBE. Li et al. [15] proposed a linear IFDM based solution for Burgers' equation with time-fractional derivative and concluded that the proposed scheme is convergent and globally stable. Yokus and Kaya [16] analyzed solutions of Caputo-based TFBE by using the FDM, Cole-Hopf transformation, and expansion method. In addition, they analyzed the Fourier-von Neumann stability. From the study, it is concluded that the FDM method was stable. Also, Sungnul et al. [17] analyzed the forward time central space scheme based on FDM for a modified Burgers' equation. The analysis signifies that the implicit-based scheme is convergent to the exact solutions.

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Swapnali Doley is a Research Scholar in the Department of Basic and Applied Science, National Institute of Technology Arunachal Pradesh in Arunachal Pradesh, India. (Corresponding author e-mail: swapnalidoley05@gmail.com)

A. Vanav Kumar is an Assistant Professor in the Department of Basic and Applied Science, National Institute of Technology Arunachal Pradesh in Arunachal Pradesh, India. e-mail: vanavkumar.a@gmail.com

Karam Ratan Singh is an Assistant Professor in the Department of Basic and Applied Science, National Institute of Technology Arunachal Pradesh in Arunachal Pradesh, India. e-mail: karamratan@nitap.ac.in

L. Jino is a Research Scholar in the Department of Basic and Applied Science, National Institute of Technology Arunachal Pradesh in Arunachal Pradesh, India. e-mail: jinogojulee@gmail.com

In a recent study, Alsaedi et al. [18] discussed the TFBE with Caputo and Riemann-Liouville fractional derivatives. They approached the blow-up solutions for TFBE based on the suggestion by Mitidieri et al. [19]. Zhang et al. [20] investigated the residual power series (RPS) method of solutions for the TFBE. They considered the Caputo-based time derivatives and found that the RPS solutions are closer to the exact solutions when a fractional derivative is closer to the value 1. Hassani and Naraghirad [21] proposed a new technique by Lagrange multipliers for achieving the solutions for variable-TFBE. Lili and Li [22] investigated and compared the solutions of TFBE by using the separation of variables (Exact solution) and by Legendre-Galerkin spectral with combining L1-scheme on graded meshes (Numerical method). It is clearly noticed that the numerical scheme is more accurate on the higher number of spatial meshes. Yadav and Pandey [23] studied the Atangana-Baleanu based fractional derivative for the TFBE and solved using FDM and included its stability and convergence. Similarly, Onal and Esen [24] considered Caputo sensed TFBE equation, which is solved using the C-N based FDM and also validated with the exact solution. Guesmia and Dail [25] used finite volume technique (method of lines) to obtain the solutions for space fractional Burgers' equation (SFBE) with Caputo derivative. The study suggests the finite volume techniques for solving fractional Burgers' equations to the non-regular domains. Verma et al. [26] obtained a numerical solution for the TFBE with delay using a non-standard FDM-Haar wavelet scheme. Li and Wu [27] explored the solutions of TFBE via FDM with artificial boundary conditions. In addition, the study illustrates FDM-based schemes' boundedness, convergence, accuracy, and feasibility.

Saad et al. [28] described the TFBE using various fractional operators such as Mittag-Leffler, Liouville-Caputo, and Caputo-Fabrizio respectively. In addition, the solutions for the TFBE are achieved using the Homotopy Analysis Transform scheme and quoted that the applied scheme is effective and accurate. Sulaima et al. [29] used the Laplace homotopy perturbation scheme to solve the TFBE of different fractional operators such as Atangana-Baleanu, Liouville-Caputo, and Yang-Srivastava-Machado respectively. The study also incorporated the existence and uniqueness of the solution. It is noted that the Atangana-Baleanu based fractional operators are effective over the other fractional operators. Further, Malyk et al. [30] derived the analytical solution for the TFBE with fractional operators such as Atangana-Baleanu, Liouville-Caputo, Caputo-Fabrizio, and Yang-Abdel-Cattani. They also compared the solutions with various fractional operators. Recently, Doley et al. [31] discussed the Lax-Friedrichs-based implicit scheme to solve the SFBE and compared it with the normal implicit scheme. It is found that the implicit-Lax-Friedrichs based solutions are more accurate than the normal implicit scheme. Kamran et al. [32] carried out the numerically simulated results for fractional-based BBM-Burgers' equation. The study considers function based on B-Spline to solve the Caputo fractional equation. Chen and Lu [33] illustrated the FDM-based time approximation and Fourier spectral-spatial approximation to solve the TFBE equation. The results indicate that the hybrid scheme is efficient by comparing numerical solution against the exact solution.

II. Time fractional Burgers' equation in Caputo sense

The Burgers' equation is an important nonlinear partial differential equation for fluid mechanics and various other phenomena such as a mathematical model of turbulence. Moreover, Burges' equation has been used in many applications such as science and engineering including sound waves in multiple media, magnetohydrodynamic waves, shock waves, gas dynamics and other waves in fluid dynamics.

In this paper, we consider the following non-linear fractional partial differential equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2} + f(x,t), \quad x \in [0,1] \;\; and \;\; t \in [0,T], \tag{1}$$

with initial condition

$$u(x,0) = U_0(x), \tag{2}$$

and boundary condition

$$u(0,t) = B_1, u(L,t) = B_2, \tag{3}$$

where $D > 0$ is the kinetic viscosity, $U_0(x)$ is the smooth function, B_1 and B_2 are the known functions and $u(x,t)$ is the unknown function. This equation appears in many areas of applied mathematics such as modeling of fluid dynamics, boundary layer behavior, and turbulence formulations. Moreover, many researchers have worked on various problems through Burgers' equation and have obtained key insight through their investigations.

In this study, the solutions of TFBE are achieved by using an implicit upwind scheme (IFDM) with the fractional derivatives of CFD, CFFD, and ABFD models respectively. The study also discusses the stability of an implicit upwind scheme using the induction method. The derived solutions using FDM-based implicit upwind solutions with various fractional derivatives are compared with the exact solution by estimating the numerical errors for different fractional models.

III. Preliminaries

In this section, we provide three types of basic definitions and mathematical preliminaries of fractional derivatives that are required to establish our results [34].

A. *Definition*

The Caputo fractional derivative of order $\alpha \in \mathbb{R}$ is of a function f is given by,

$${}_a^C D_t^\alpha f(t_n) = \frac{1}{\Gamma(n-\alpha)} \int_0^{t_n} \frac{d}{ds} f'(s)(t_n - s)^{-\alpha} ds$$

where $\Gamma(.)$ is a Gamma function.

B. *Definition*

The function $f(t)$ is stated as Caputo-Fabrizio's fractional derivative in Caputo sense for time derivative term in the order $\alpha \in \mathbb{R}$.

$${}_a^{CF} D_t^\alpha f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t f'(r) exp\Big(\frac{-\alpha(t-r)}{1-\alpha}\Big) dr$$

in which $t > \alpha$, $0 < \alpha < 1$ and $M(\alpha)$ is called the normalization function and it satisfies $M(0) = M(1) = 1$.

C. *Definition*

Also, the new Atanagana Beleanu fractional derivatives is defined based on Caputo sense [35] as follows:

$${}_a^{AB} D_t^\alpha f(t) = \frac{AB(\alpha)}{(1-\alpha)} \int_a^t f'(x) E_\alpha\Big[-\alpha \frac{(t-x)^\alpha}{1-\alpha}\Big] dx$$

IV. Numerical approximations of fractional upwind scheme

With the above three fractional derivatives, the numerical approximations of TFBE are carried out using an implicit upwind scheme via finite difference approximations. The numerical computation uses the upwind scheme for non-linear terms with first-order spatial derivative and central difference for the second-order spatial derivatives respectively. The implemented implicit scheme for TFBE is investigated for stability as well as convergence. Thus, the non-linear term $u\frac{\partial u}{\partial x}$ is discretized as follows

$$u\frac{\partial u}{\partial x} \approx \frac{u_j^{n+1}}{2}\Big(\frac{u_j^n - u_{j-1}^n}{\delta x}\Big) + \frac{u_j^n}{2}\Big(\frac{u_j^{n+1} - u_{j-1}^{n+1}}{\delta x}\Big), \tag{4}$$

and diffusive term as

$$\frac{\partial^2 u}{\partial x^2} = \frac{(u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1})}{\delta x^2}. \tag{5}$$

A. Discretization of Upwind Caputo time fractional derivatives

The Caputo based time derivatives $\dfrac{\partial^\alpha u}{\partial t^\alpha}$ in the order less than one are approximated numerically as [14],

$$\frac{\partial^\alpha u}{\partial t^\alpha}\Big|_{t_n} \approx \frac{(\delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{m-1} b_k^\alpha \big[u_j^{n-k} - u_j^{n-k-1}\big], \tag{6}$$

where $b_k^\alpha = (k+1)^{1-\alpha} - k^{1-\alpha}$.

Also, the IFDM based TFBE is given by using Eq.(4),(5) and (6) in (1), we get the following system of an algebraic equation,

$$\begin{aligned}\frac{(\delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{m-1} b_k^\alpha \Big(u_j^{n+1-k} - u_j^{n-k}\Big) \\ +\frac{u_j^{n+1}}{2}\Big(\frac{u_j^n - u_{j-1}^n}{\delta x}\Big)+ \\ \frac{u_j^n}{2}\Big(\frac{u_j^{n+1} - u_{j-1}^{n+1}}{\delta x}\Big) \\ = \frac{D}{\delta x^2}(u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}) + f_i^{n+1}\end{aligned} \tag{7}$$

where $S = (\delta t)^\alpha \Gamma(2-\alpha), b_0^\alpha = 1$, when $k \geq 1$,

$$\begin{aligned}\Big(1 + \frac{2DS}{(\delta x)^2} + \frac{S}{2(\delta x)}(u_j^n - u_{j-1}^n) + \frac{S}{2\delta x}u_j^u\Big)u_j^{n+1} \\ = \Big(\frac{SD}{(\delta x)^2} + \frac{S}{2}\frac{u_j^n}{(\delta x)}\Big)u_{j-1}^{n+1} + \Big(\frac{DS}{(\delta x)^2}\Big)u_{j+1}^{n+1} \\ +u_j^n - \sum_{k=1}^{m-1} b_k^\alpha \Big(u_j^{n+1-k} - u_i^{n-k}\Big) + Sf_i^{n+1}.\end{aligned} \tag{8}$$

B. Discretization of Upwind Caputo Fabrizio's in time fractional derivative

The Caputo Fabrizio's derivatives on TFBE is used because of the availability of the exponential decay [36]. Also, the CFFD based implicit method is given as,

$$\begin{aligned}{}_0^{CF}D_t^\alpha u(t)\Big|_{t=t_n} &= \frac{M(\alpha)}{1-\alpha}\int_0^{t_n} u'(r)exp\Big(\frac{-\alpha(t_n - r)}{1-\alpha}\Big)dr \\ &\approx \frac{M(\alpha)}{1-\alpha}\sum_{k=0}^{n-1}\Big[\frac{u_{k+1}-u_k}{\delta t} + O(\delta t)\Big]\int_{t_k}^{t_{k+1}} exp\Big(\frac{-\alpha(t_n - r)}{1-\alpha}\Big)dr \\ &= \frac{M(\alpha)}{\delta t(1-\alpha)}\frac{1-\alpha}{(\alpha)}\sum_{k=0}^{n-1}\Big(u_{k+1} - u_k + O(\delta t)\Big) \\ &\qquad exp\Big[\frac{-\alpha(t_n - r)}{1-\alpha}\Big]\Big|_{t_k}^{t_{k+1}} \\ &= \frac{M(\alpha)}{(\delta t(1-\alpha)}\frac{1-\alpha}{(\alpha)}\sum_{k=0}^{n-1}\Big(u_{k+1} - u_k\Big) \\ &\quad *\Big[exp\Big(\frac{-\alpha(t_n - t_{k+1})}{1-\alpha}\Big) - exp\Big(\frac{-\alpha(t_n - t_k)}{1-\alpha}\Big)\Big]\end{aligned} \tag{9}$$

where $\delta t = \dfrac{T}{n}, t_n = n\delta t$.

By using above equation (4),(5) and (9) in Eq.(1) We get as follows:

$$\begin{aligned}\frac{M(\alpha)}{\alpha\delta t}\Big(e^{\frac{\alpha\delta t}{1-\alpha}} - 1\Big)\sum_{k=0}^{n}\Big(u_j^{n-k+1} - u_j^{n-k}\Big) * e^{-\frac{\alpha k\delta t}{1-\alpha}} \\ +\frac{u_j^{n+1}}{2}\Big(\frac{u_j^n - u_{j-1}^n}{\delta x}\Big) + \frac{u_j^n}{2}\Big(\frac{u_j^{n+1} - u_{j-1}^{n+1}}{\delta x}\Big) \\ = \frac{D}{\delta x^2}(u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}) + f_i^{n+1}\end{aligned} \tag{10}$$

Suppose, $R' = \dfrac{\alpha\delta t}{M(\alpha)(e^{\frac{\alpha*\delta t}{1-\alpha}} - 1)}$, then, when $k \geq 1$

$$\begin{aligned}\Big(1 + \frac{2DR'}{\delta x^2} + \frac{R'}{2\delta x}(u_j^n - u_{j-1}^n) + \frac{R'}{2\delta x}u_j^n\Big)u_j^{n+1} \\ = \Big(\frac{R'D}{\delta x^2} + \frac{R'u_j^n}{2\delta x}\Big)u_{j-1}^{n+1} + \Big(\frac{R'D}{\delta x^2}\Big)u_{j+1}^{n+1} + u_j^n \\ -\sum_{k=1}^{n}\Big(u_j^{n+1-k} - u_i^{n-k}\Big) * e^{\frac{\alpha k\delta t}{1-\alpha}} + R'f_i^{n+1}\end{aligned} \tag{11}$$

C. Discretization of upwind Atangana-Baleanu in time-fractional derivative

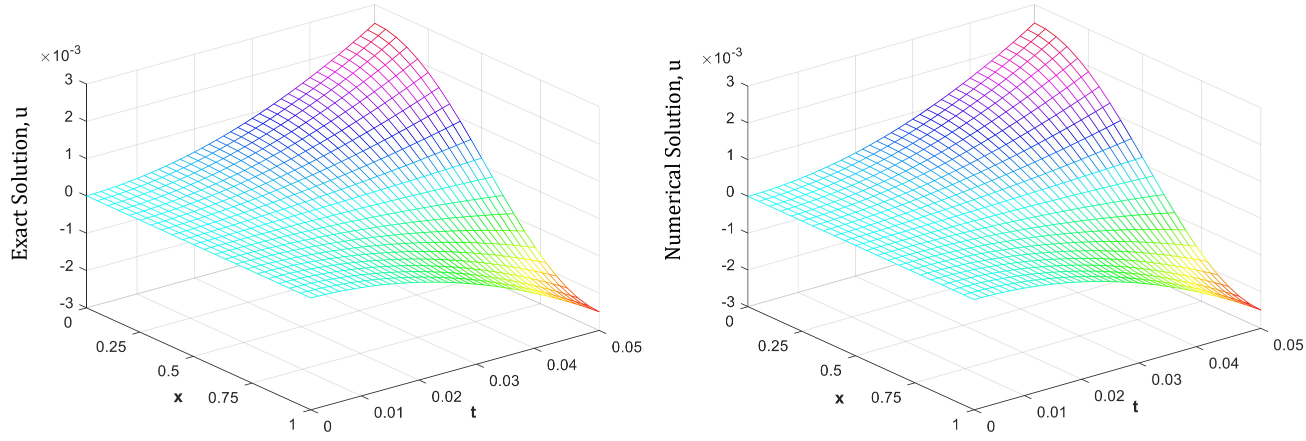
In this study, the ABFD is incorporated for the TFBE because of a nonlocal fractional derivative/non-singular kernel [23], [37]. The ABFD based on Caputo sense is discretized as

$$\begin{aligned}{}_a^{AB}D_t^\alpha u(x,t) \approx \frac{AB(\alpha)}{1-\alpha}\sum_{k=0}^{n-1}\int_{t_{k-1}}^{t_k}\Big[\frac{u_{k+1}-u_k}{\delta t}\Big]* \\ E_\alpha(\frac{-\alpha}{1-\alpha}(t_n - s)^\alpha]ds\end{aligned} \tag{12}$$

where $E_\alpha(z) = \sum_{n=0}^{\infty}\dfrac{z^n}{\Gamma(n\alpha+1)}$

$$\begin{aligned}\frac{\partial^\alpha u(x,t_n)}{\partial t^\alpha} &= \frac{AB(\alpha)}{1-\alpha}\sum_{k=0}^{n-1}\Big[\frac{u_{k+1}-u_k}{\delta t}\Big]* \\ &\quad \int_{t_{j-1}}^{t_j}\sum_{n=0}^{\infty}\Big(\frac{-\alpha}{1-\alpha}\Big)^n\frac{(t_n - s)^{\alpha n}}{\Gamma(n\alpha+1)}ds \\ &= \frac{AB(\alpha)}{(1-\alpha)}\sum_{k=0}^{n-1}\Big[\frac{u_{k+1}-u_k}{\delta t}\Big]\sum_{n=0}^{\infty}\frac{\Big(\frac{-\alpha}{1-\alpha}\Big)^n}{\Gamma(n\alpha+1)}* \\ &\quad \int_{t_{j-1}}^{t_j}(t_n - s)^{\alpha n}ds \\ &= \frac{AB(\alpha)}{1-\alpha}\sum_{k=0}^{n-1}\Big(\frac{u_{k+1}-u_k}{\delta t}\Big)\Big[(t_n - t_{j-1})* \\ &\quad E_{\alpha,2}\Big(\Big(\frac{-\alpha}{1-\alpha}\Big)^n(t_n - t_{j-1})^\alpha\Big) \\ &\quad -(t_n - t_j)E_{\alpha,2}\Big(\Big(\frac{-\alpha}{1-\alpha}\Big)^n(t_n - t_j)^\alpha\Big)\Big] \\ &= \frac{AB(\alpha)}{\delta t(1-\alpha)}\sum_{k=0}^{n}\Big(u_{k+1} - u_k\Big)\delta_{n,k}^\alpha,\end{aligned} \tag{13}$$

where $\delta_{n,k}^\alpha = \Big[(n-k)E_{\alpha,2}\{-\frac{\alpha\delta t}{(1-\alpha)}(n-k)\}$ $-(n-k-1)E_{\alpha,2}\{-\frac{\alpha\delta t}{(1-\alpha)}(n-k-1)\}\Big]$



(a) (b)

Fig. 1: comparison of numerical solution and exact solution (example 1)

Thus, the discretized equations for TFBE Eq.(1) using above approximation for ABFD and by using Eq. (4) and (5), we get

$$
\begin{aligned}
&\frac{AB(\alpha)}{\delta t(1-\alpha)}\sum_{k=0}^{n}\left(u_j^{k+1}-u_j^{k}\right)\delta_{n,k}^{\alpha}\\
&=\frac{D}{\delta x^2}\left(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}\right)\\
&-\Big[\frac{u_j^{n+1}}{2}\Big(\frac{u_j^n-u_{j-1}^n}{\delta x}\Big)+\frac{u_j^n}{2}\Big(\frac{u_j^{n+1}-u_{j-1}^{n+1}}{\delta x}\Big)\Big]\\
&+f(x_j,t_n).
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&\frac{AB(\alpha)}{\delta t(1-\alpha)}\left(u_j^{n+1}-u_j^{n}\right)\delta_{n,n}^{\alpha}\\
&+\frac{AB(\alpha)}{\delta t(1-\alpha)}\sum_{k=0}^{n-1}\left(u_j^{k+1}-u_j^{k}\right)\delta_{n,k}^{\alpha}\\
&=\frac{D}{\delta x^2}\left(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}\right)\\
&-\Big[\frac{u_j^{n+1}}{2}\Big(\frac{u_j^n-u_{j-1}^n}{\delta x}\Big)\\
&+\frac{u_j^n}{2}\Big(\frac{u_j^{n+1}-u_{j-1}^{n+1}}{\delta x}\Big)\Big]+f(x_j,t_n).
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&\Big(\frac{AB(\alpha)}{\delta t(1-\alpha)}\delta_{n,n}^{\alpha}+\frac{2D}{\delta x^2}+\frac{u_j^n}{2\delta x}\\
&+\frac{1}{2}\Big(\frac{u_j^n-u_{j-1}^n}{\delta x}\Big)\Big)u_j^{n+1}\\
&+\Big(\frac{-D}{\delta x^2}-\frac{u_j^n}{2\delta x}\Big)u_{j-1}^{n+1}-\frac{D}{\delta x^2}u_{j+1}^{n+1}\\
&=\frac{AB(\alpha)}{\delta t(1-\alpha)}\delta_{n,n}^{\alpha}u_j^n-\frac{AB(\alpha)}{\delta t(1-\alpha)}\\
&\sum_{k=0}^{n-1}\left(u_j^{k+1}-u_j^{k}\right)\delta_{n,k}^{\alpha}+f(x_j,t_n).
\end{aligned}
\tag{16}
$$

V. NUMERICAL RESULTS

The implicit upwind numerical schemes were developed for the time-fractional Burgers' equation in Eq. (1) by various time-fractional derivatives: Caputo fractional (CFD), Caputo-Fabrizo's in Caputo sense (CFFD), and Atangana-Baleno in Caputo sense (ABFD) fractional derivatives.

For the above TFBE with the time fractional terms such as CFD, CFFD and ABFD, an accuracy of the presented implicit upwind methods are examined by using the error norm L_2 and L_∞

$$
\begin{aligned}
L_2 &= \|U^{exact}-(u_N)_j\|\\
&=\sqrt{\delta x\sum_{j=0}^{m}\left(U_j^{exact}-(u_N)\right)^2}
\end{aligned}
\tag{17}
$$

And also the maximum error norm as L_∞

$$
\begin{aligned}
L_\infty &= \|U^{Exact}-(u_N)_j\|_\infty\\
&=Max_j|U^{Exact}-(u_N)_j|
\end{aligned}
\tag{18}
$$

A. Example.1

Consider the fractional Burgers' equation (1) with initial conditions is as follows:

$$u(x,0)=0, 0\le x\le 1 \tag{19}$$

and the boundary conditions as

$$u(0,t)=t^2, u(1,t)=-t^2, t\ge 0 \tag{20}$$

The source term $f(x,t)$ can be found from [14] in the form given by below:

$$
\begin{aligned}
f(x,t)=\frac{2t^{2-\alpha}cos(\pi x)}{\Gamma(3-\alpha)}-\pi t^4 sin(\pi x)cos(\pi x)\\
+D\pi^2t^2cos(\pi x)
\end{aligned}
\tag{21}
$$

and analytical solution of the fractional Burgers' equation is given by

$$u(x,t)=t^2cos(\pi x) \tag{22}$$

The comparison is done for exact results available and numerical results for TFBE by implicit upwind based FDA. It is noted that the computed results are in good agreement with the analytical results which can be observed in Fig. 1.

TABLE I: The error norms of Example 1 at $\alpha = 0.9, D = 1, \delta t = 0.00125, \delta x = 0.05$ and $t_{max} = 0.05$.

Distance (X)	CFD	CFFD	ABFD
0.1	0.00000797	0.00025464	0.00018407
0.2	0.00001110	0.00034507	0.00024919
0.3	0.00001029	0.00031233	0.00022578
0.4	0.00000665	0.00019841	0.00014364
0.5	0.00000140	0.00004144	0.00003002
0.6	0.00000417	0.00012391	0.00008977
0.7	0.00000876	0.00026377	0.00019085
0.8	0.00001110	0.00034200	0.00024709
0.9	0.00001001	0.00031838	0.00022998
L_2	0.00000080	0.00024850	0.00001796
L_∞	0.00000111	0.00034559	0.00024958

TABLE II: The error norms of Example 1 at $\alpha = 0.5, D = 1, \delta t = 0.00125, \delta x = 0.05$ and $t_{max} = 0.05$.

Distance (X)	CFD	CFFD	ABFD
0.1	0.00000088	0.00118032	0.00042754
0.2	0.00000123	0.00144828	0.00058873
0.3	0.00000115	0.00122525	0.00054146
0.4	0.00000077	0.00074790	0.00034805
0.5	0.00000021	0.00015382	0.00007297
0.6	0.00000039	0.00046171	0.00021842
0.7	0.00000087	0.00101067	0.00046061
0.8	0.00000109	0.00138206	0.00058860
0.9	0.00000092	0.00139774	0.00053882
L_2	0.00000083	0.00103383	0.00004263
L_∞	0.00000123	0.00144986	0.00058980

The numerical error for the space length $0.1 \leq x \leq 0.9$ are listed and the errors are compared with various order of time derivatives ($\alpha = 0.9, 0.5$ and 0.1) as shown in Tables. I-III. Along with the spatial errors, the accuracy of derivatives/scheme is shown using L_2 and maximum error as L_∞.

B. Example.2

Consider the fractional Burgers' equation (1) with initial conditions as follows,

$$u(x, 0) = 0, 0 \leq x \leq 1 \tag{23}$$

and the boundary conditions as

$$u(0, t) = t^2, u(1, t) = et^2, t \geq 0 \tag{24}$$

TABLE III: The error norms of Example 1 at $\alpha = 0.1, D = 1, \delta t = 0.00125, \delta x = 0.05$ and $t_{max} = 0.05$.

Distance (X)	CFD	CFFD	ABFD
0.1	0.00000068	0.00204968	0.00009175
0.2	0.00000097	0.00200022	0.00012976
0.3	0.00000093	0.00152608	0.00012174
0.4	0.00000065	0.00089434	0.00007920
0.5	0.00000023	0.00018166	0.00001653
0.6	0.00000022	0.00054723	0.00005023
0.7	0.00000058	0.00122714	0.00010470
0.8	0.00000075	0.00179216	0.0001315
0.9	0.00000062	0.00212104	0.00011759
L_2	0.00000062	0.00148794	0.00094648
L_∞	0.00000098	0.00212104	0.00013150

TABLE IV: The error norms of Example 2 at $\alpha = 0.9, D = 1, \delta t = 0.00125, \delta x = 0.05$ and $t_{max} = 0.05$.

Distance (X)	CFD	CFFD	ABFD
0.1	0.00001746	0.00043555	0.00032392
0.2	0.00003089	0.00073151	0.00054628
0.3	0.00004116	0.00093817	0.00070373
0.4	0.00004881	0.00109240	0.00082157
0.5	0.00005403	0.00121372	0.00091251
0.6	0.00005656	0.00130275	0.00097607
0.7	0.00005564	0.00133797	0.00099660
0.8	0.00004998	0.00126960	0.00093926
0.9	0.00003777	0.00101243	0.00074481
L_2	0.00004224	0.00100426	0.00074920
L_∞	0.00005659	0.00133797	0.00099659

TABLE V: The error norms of Example 2 at $\alpha = 0.5, D = 1, \delta t = 0.00125, \delta x = 0.05$ and $t_{max} = 0.05$.

Distance (X)	CFD	CFFD	ABFD
0.1	0.00000106	0.00164851	0.00089125
0.2	0.00000195	0.00249946	0.00155058
0.3	0.00000267	0.00300811	0.00204323
0.4	0.00000323	0.00339530	0.00241115
0.5	0.00000363	0.00375287	0.00267142
0.6	0.00000384	0.00410127	0.00281407
0.7	0.00000384	0.00410127	0.00281407
0.8	0.00000367	0.00449021	0.00254826
0.9	0.00000322	0.00398502	0.00194481
L_2	0.00000299	0.00337734	0.00211604
L_∞	0.00000388	0.00449020	0.00282992

The source term $f(x, t)$ can be found from [14] in the form given by below,

$$f(x, t) = \frac{2t^{2-\alpha}e^x}{\Gamma(3-\alpha)} + t^4 e^{2x} - Dt^2 e^x \tag{25}$$

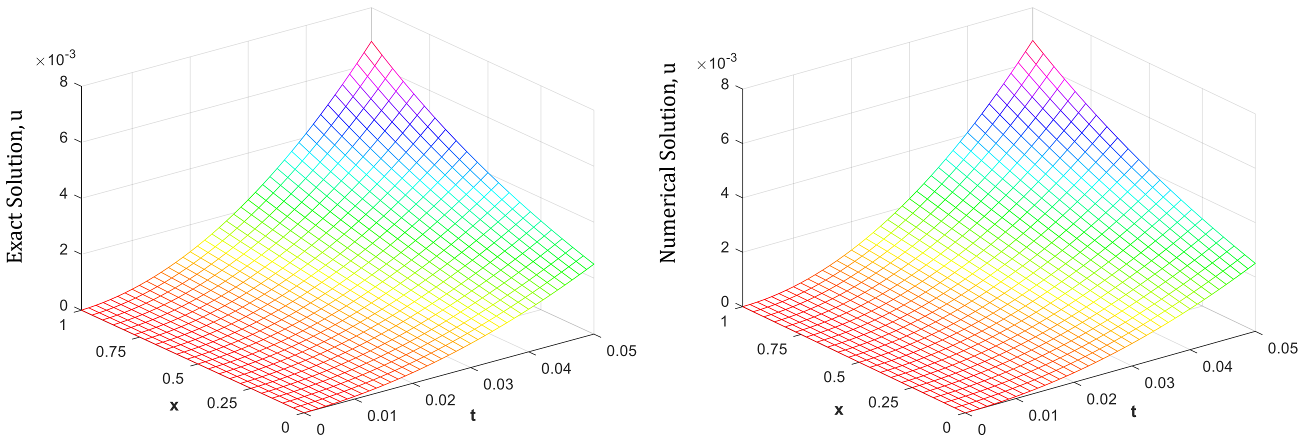
and analytical solution of the fractional Burgers' equation is given by

$$u(x, t) = t^2 e^x \tag{26}$$

In example 2, a good comparison of the numerical results and exact solutions for TFBE are illustrated in Fig. 2. Moreover, it is noted that the plots are in good agreement with the computed results are comparable to the analytical results. The spatial error, L_2 and L_∞ are compared with the CFD, CFFD and ABFD based time fraction derivatives. The errors are determined for time-fractional order $\alpha = 0.9,\ 0.5,\ 0.1$ as listed in the Table. IV-VI.

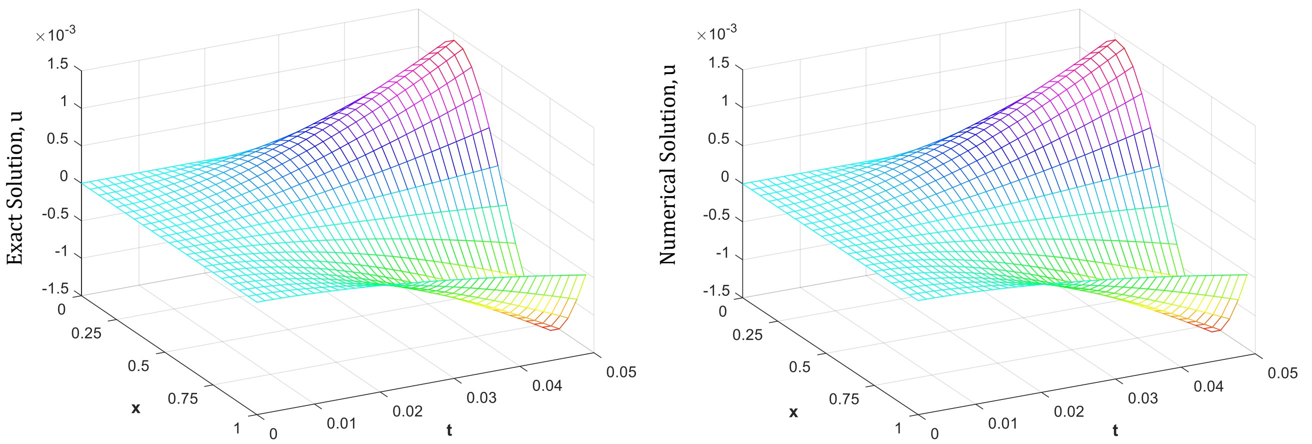
TABLE VI: The error norms of Example 2 at $\alpha = 0.1, D = 1, \delta t = 0.00125, \delta x = 0.05$ and $t_{max} = 0.05$.

Distance (X)	CFD	CFFD	ABFD
0.1	0.00000022	0.00245059	0.00035110
0.2	0.00000043	0.00299675	0.00064353
0.3	0.00000063	0.00332217	0.00087513
0.4	0.00000083	0.00365577	0.00104231
0.5	0.00000102	0.00402113	0.00113977
0.6	0.00000121	0.00442289	0.00116033
0.7	0.00000139	0.00486391	0.00109472
0.8	0.00000157	0.00533335	0.00093142
0.9	0.00000175	0.00564164	0.00065648
L_2	0.00000115	0.00406894	0.00085054
l_∞	0.00000191	0.00564163	0.00116033



(a) (b)

Fig. 2: comparison of numerical solution and exact solution (example 2)



(a) (b)

Fig. 3: comparison of numerical solution and exact solution (example 3)

C. *Example.3*

Further, consider a fractional Burgers' equation (1) with initial condition,

$$u(x,0) = 0, 0 \leq x \leq 1 \tag{27}$$

and the boundary conditions as

$$u(0,t) = 0, u(1,t) = 0, t \geq 0 \tag{28}$$

The source term $f(x,t)$ can be found from [14] in the form given by bellow:

$$f(x,t) = \frac{2t^{2-\alpha} sin(2\pi x)}{\Gamma(3-\alpha)} + 2\pi t^4 sin(2\pi x) cos(2\pi x) \\ +4D\pi^2 t^2 sin(2\pi x) \tag{29}$$

and analytical solution of the fractional Burgers' equation is given by

$$u(x,t) = t^2 sin(2\pi x) \tag{30}$$

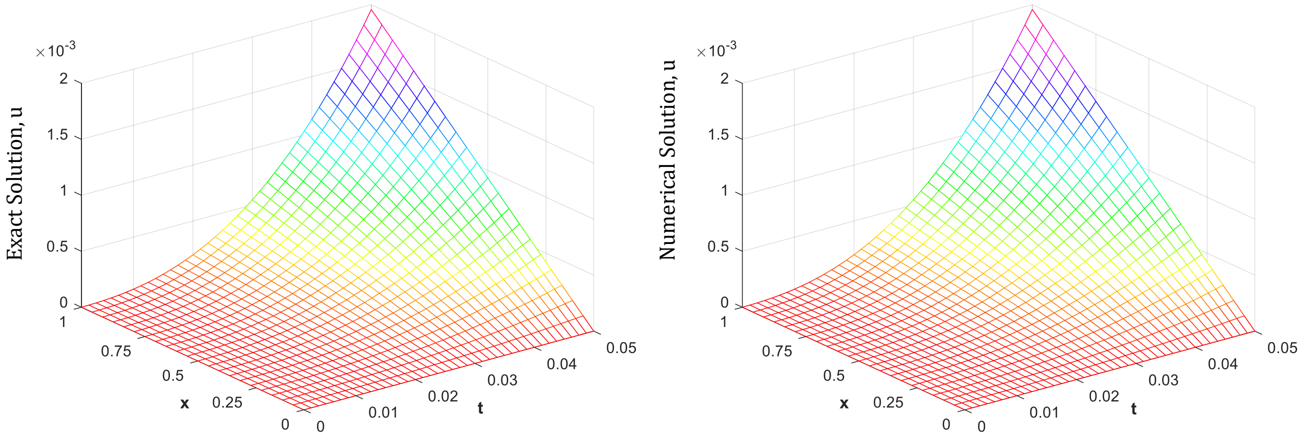
Figure. 3 represents the validation of numerical solutions of TFBE with the analytical one. The spatial error, L_2 and L_∞ are compared with the various time fractional derivatives such as CFD, CFFD and ABFD respectively. The errors are determined for time fractional order $\alpha = 0.9,\ 0.5$ and 0.1 as listed in the Table. VII-IX.

TABLE VII: The error norms of Example 3 at $\alpha = 0.9, D = 1, \delta t = 0.00125, \delta x = 0.05$ and $t_{max} = 0.05$.

Distance (X)	CFD	CFFD	ABFD
0.1	0.00005617	0.00006868	0.00006532
0.2	0.00009281	0.00011349	0.00010793
0.3	0.00009718	0.00011882	0.00011300
0.4	0.00006774	0.00008283	0.00007877
0.5	0.00001475	0.00001802	0.00001714
0.6	0.00004336	0.00005304	0.00005044
0.7	0.00008636	0.00010565	0.00010046
0.8	0.00009929	0.00012147	0.00011551
0.9	0.00007761	0.00009500	0.00009032
L_2	0.00007042	0.00008614	0.00008191
L_∞	0.00099414	0.00012151	0.00011567

D. *Example 4*

Consider the fractional Burgers equation (1) with initial and boundary as follows:



(a) (b)

Fig. 4: comparison of numerical solution and exact solution (example 4)

TABLE VIII: The error norms of Example 3 at $\alpha = 0.5, D = 1, \delta t = 0.00125, \delta x = 0.05$ and $t_{max} = 0.05$.

Distance (X)	CFD	CFFD	ABFD
0.1	0.00008972	0.00012378	0.00010344
0.2	0.00014825	0.00020451	0.00017092
0.3	0.00015524	0.00021412	0.00017896
0.4	0.00010823	0.00014925	0.00012475
0.5	0.00002359	0.00003247	0.00002714
0.6	0.00006923	0.00009560	0.00007989
0.7	0.00013794	0.00019043	0.00015913
0.8	0.00015862	0.00021902	0.00018299
0.9	0.00012409	0.00017141	0.00014318
L_2	0.00011252	0.00015529	0.00012976
L_∞	0.00015889	0.00021906	0.00018308

TABLE IX: The error norms of Example 3 at $\alpha = 0.1, D = 1, \delta t = 0.000125, \delta x = 0.05$ and $t_{max} = 0.05$.

Distance (X)	CFD	CFFD	ABFD
0.1	0.00010157	0.00013823	0.00010344
0.2	0.00016783	0.00022839	0.00017092
0.3	0.00017574	0.00023911	0.00017896
0.4	0.00012253	0.00016667	0.00012475
0.5	0.00002671	0.00003626	0.00002714
0.6	0.00007836	0.00010676	0.00007989
0.7	0.00015615	0.00021266	0.00015913
0.8	0.00017958	0.00024458	0.00018299
0.9	0.00014050	0.00019144	0.00014318
L_2	0.00012738	0.00017342	0.00012976
l_∞	0.00017978	0.00024463	0.00018308

$$u(x,0) = 0, 0 \le x \le 1 \tag{31}$$

and the boundary conditions as

$$u(0,t) = 0, u(1,t) = t^{\frac{5}{2}}, t \ge 0 \tag{32}$$

The source term $f(x,t)$ can be found from [21] in the form given by bellow:

$$f(x,t) = \Big(\frac{\Gamma(\frac{7}{2})t^{\frac{5}{2}-\alpha(x,t)}}{\Gamma(\frac{7}{2}-\alpha(x,t))}\Big)x^{\frac{7}{2}} + \frac{7}{2}t^5x^6 - \frac{35}{4}t^{\frac{5}{2}}x^{\frac{3}{2}} \tag{33}$$

and and exact solution is given by

$$u(x,t) = t^{5/2}x^{7/2} \tag{34}$$

The CFD, CFFD, and ABFD based numerical solutions are compared with the ES and the error values are listed in the TableX-XII. Also, the graphical representation of TFBE behavior of numerical solution and exact solution. Both the solutions are well comparable to each other as shown in Fig. 4.

TABLE X: The error norms of Example 4 at $\alpha = 0.9, D = 1, \delta t = 0.00125, \delta x = 0.05$ and $t_{max} = 0.05$.

Distance (X)	CFD	CFFD	ABFD
0.1	0.00000041	0.00000130	0.00000116
0.2	0.00000060	0.00000279	0.00000240
0.3	0.00000036	0.00004649	0.00003785
0.4	0.00000042	0.00000698	0.00000533
0.5	0.00000181	0.00000982	0.00000702
0.6	0.00000369	0.00001295	0.00000870
0.7	0.00000582	0.00001581	0.00001004
0.8	0.00000783	0.00001721	0.00001041
0.9	0.00000936	0.00001522	0.00000885
L_2	0.00000518	0.00001056	0.00000675
L_∞	0.00001048	0.00001721	0.00000675

TABLE XI: The error norms of Example 4 at $\alpha = 0.5, D = 1, \delta t = 0.00125, \delta x = 0.05$ and $t_{max} = 0.05$.

Distance (X)	CFD	CFFD	ABFD
0.1	0.00000379	0.00000178	0.00000445
0.2	0.00000741	0.00000475	0.00000954
0.3	0.00001065	0.00001009	0.00001575
0.4	0.00001332	0.00001896	0.00002338
0.5	0.00001523	0.00003232	0.00003224
0.6	0.00001613	0.00005068	0.00004153
0.7	0.00001578	0.00007301	0.00004941
0.8	0.00001384	0.00009433	0.00005263
0.9	0.00000994	0.00010040	0.00004600
L_2	0.00001157	0.00005487	0.00003304
L_∞	0.00001613	0.00010074	0.00005263

VI. Conclusion

The study illustrates the comparison of the CFD, CFFD and ABFD based time derivatives for the TFBE. The three types of derivatives are compared in terms of numerical error and accuracy. All the derivatives are cracked using FDM by means of an upwind implicit scheme. The stability is checked

TABLE XII: The error norms of Example 4 at $\alpha = 0.1, D = 1, \delta t = 0.000125, \delta x = 0.05$ and $t_{max} = 0.05$.

Distance (X)	CFD	CFFD	ABFD
0.1	0.00000667	0.00000054	0.00000736
0.2	0.00001299	0.00000238	0.00001455
0.3	0.00001863	0.00000680	0.00002134
0.4	0.00002322	0.00001510	0.00002744
0.5	0.00002640	0.00002856	0.00003240
0.6	0.00002778	0.00004849	0.00003563
0.7	0.00002694	0.00007614	0.00003632
0.8	0.00002346	0.00001225	0.00003337
0.9	0.00001686	0.00015092	0.00002539
L_2	0.00001993	0.00007491	0.00002578
l_∞	0.00002778	0.000015785	0.00003635

against the proposed numerical scheme for TFBE using the various form of derivatives based on Caputo sense and found that the scheme is unconditionally stable. It is noted that the CFD-upwind-implicit scheme provides better accuracy of numerical results over the CFFD/ABFD-upwind-implicit scheme.

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