Parameter Estimation for Dothan Model Driven by Small Symmetrical Noise Based on Discrete Observation

Huiping Jiao, Piaopiao Zhou, Heling Li, Chao Wei

Abstract—**In this paper, we consider the parameter estimation problem for discretely observed Dothan model driven by small symmetrical noise. The least square method is applied to obtain the drift parameter estimators. The consistency and asymptotic distribution of the estimators are derived when a small dispersion coefficient $\varepsilon \to 0$. Some simulations are made to demonstrate the applicability of the results.**

Index Terms—**Dothan model; small symmetrical noise; least square estimation; consistency; asymptotic distribution.**

I. Introduction

Due to the widespread application of stochastic differential equations in the field of financial economics, it has attracted a large number of scholars to devote themselves to research in this field ([1], [3], [4], [23], [24], [26]). However, parameters are always unknown. This seems to be a common problem in stochastic model. In the past few decades, parameter estimation problem for economical models has been studied by many authors. For example, Rossi ([16]) applied particle filters and maximum likelihood estimation to solve the parameter estimation for Cox-Ingersoll-Ross model. Tunaru and Zheng([21]) used Bayesian method to discuss parameter estimation risk in financial modelling. Wei et al.([22]) utilized Gaussian estimation method to investigate the parameter estimation for discretely observed Cox-Ingersoll-Ross model. Yang et al.([25]) used α-path method to estimate the unknown parameter of uncertain differential equation from discretely sampled data. However, it is well-known that many financial processes exhibit discontinuous sample paths and heavy tailed properties (e.g. certain moments are infinite). These features cannot be captured by Brownian motion ([6], [7], [11], [14], [17]). Therefore, it is natural to replace the driving Brownian motion by Lévy noises. In recent years, with the development of Lévy process theory and its application in the fields of engineering systems, economic systems and management systems, it has attracted great attention from scholars. Therefore, some authors considered parameter estimation for stochastic differential equations driven by Lévy noises. For example, Li and Ma([12]) discussed the asymptotic properties of estimators in a stable Cox-Ingersoll-Ross model. Long ([13]) analyzed the least squares estimator for discretely observed Ornstein-Uhlenbeck processes with small Lévy noises. Then, Long et al. ([15]) tackled the least squares estimators for discretely observed stochastic processes driven by small Lévy noises. Singh et al. ([18]) utilized a randomized response model under Poisson distribution to estimate a rare sensitive attribute in two-stage sampling. There have been many applications of small noise asymptotics to mathematical finance ([2], [10], [19]). From a practical point of view in parametric inference, it is more realistic and interesting to consider asymptotic estimation for diffusion processes with small noise based on discrete observations.

Manuscript received April 10, 2022; revised August 19, 2022.

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The Dothan model is an interest rates model introduced by Dothan in 1978 ([5]). During the past few decades, some authors have studied the parameter estimation problem for Dothan model. For example, Treepongkaruna ([20]) estimated the parameter of Dothan model by using Quasi-maximum likelihood estimation. Khramov ([9]) used generalized method of moments to estimate the parameter of Dothan model and provided some simulations by real data. However, there is few literature about parameter estimation for Dothan model driven by non-Gauss noise. In this paper, we consider the parameter estimation problem for Dothan model driven by small symmetrical noise. The least square method is utilized to obtain the drift parameter estimators. The consistency and asymptotic distribution of the estimators are derived when a small dispersion coefficient $\varepsilon \to 0$.

This paper is organized as follows. In Section 2, Dothan model driven by small symmetrical noise is introduced and the estimators are obtained. In Section 3, some lemmas are given, the consistency and asymptotic distribution of the estimators are provided. In Section 4, some simulations are given. The conclusion is given in Section 5.

II. Problem Formulation and Preliminaries

A random variable η is said to have a stable distribution with index of stability $\alpha \in (0, 2]$, scale parameter $\sigma \in (0, \infty)$, skewness parameter $\beta \in [-1, 1]$ and location parameter $\mu \in (-\infty, \infty)$ if it has the following characteristic function:

$$\phi_\eta(u) = \begin{cases} -\sigma^\alpha |u|^\alpha (1 - i\beta sgn(u) \tan \dfrac{\alpha\pi}{2}) + i\mu u, if \alpha \neq 1, \\ -\sigma |u| (1 + i\beta \dfrac{2}{\pi} sgn(u) \log |u|) + i\mu u, if \alpha = 1. \end{cases}$$

We denote $\eta \sim S_\alpha(\sigma,\beta,\mu)$. When $\mu=0$, we say η is strictly α-stable, if in addition $\beta=0$, we call η symmetrical α-stable. Throughout this paper, it is assumed that α-stable motion is strictly symmetrical and $\alpha\in(1,2)$.

We investigate the parameter estimation problem for Dothan model described by the following stochastic differential equation:

$$\begin{cases} dX_t = (\theta+\gamma X_t)dt + \varepsilon X_t dZ_t, \quad t\in[0,1] \\ X_0 = x_0, \end{cases} \tag{1}$$

where θ, γ are unknown, Z is a strictly symmetric α-stable motion. It is assumed that $\varepsilon\in(0,1]$.

We assume that the process $\{X_t, t\geq 0\}$ can be observed at discrete point $\{t_i = i\Delta, i=0,1,2,...,n\}$ with $\Delta>0$. Consider the following contrast function:

$$\rho_{n,\varepsilon}(\theta,\gamma) = \sum_{i=1}^{n} \frac{|X_{t_i}-X_{t_{i-1}}-(\theta+\gamma X_{t_{i-1}})\Delta t_{i-1}|^2}{\varepsilon^2 X_{t_{i-1}}^2 \Delta t_{i-1}}, \tag{2}$$

where $\Delta t_{i-1} = t_i - t_{i-1} = \frac{1}{n}$.

Then, we obtain

$$\begin{cases} \widehat{\theta}_{n,\varepsilon} = \dfrac{\sum_{i=1}^n \frac{X_{t_i}-X_{t_{i-1}}}{X_{t_{i-1}}^2}}{\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}^2} - (\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}})^2} - \dfrac{\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}}\sum_{i=1}^n \frac{X_{t_i}-X_{t_{i-1}}}{X_{t_{i-1}}}}{\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}^2} - (\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}})^2}, \\ \widehat{\gamma}_{n,\varepsilon} = \dfrac{\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}^2}\sum_{i=1}^n \frac{X_{t_i}-X_{t_{i-1}}}{X_{t_{i-1}}}}{\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}^2} - (\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}})^2} - \dfrac{\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}}\sum_{i=1}^n \frac{X_{t_i}-X_{t_{i-1}}}{X_{t_{i-1}}^2}}{\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}^2} - (\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}})^2} \end{cases} \tag{3}$$

III. Main Results and Proofs

Since

$$X_{t_i} - X_{t_{i-1}} = \frac{1}{n}\theta + \gamma\int_{t_{i-1}}^{t_i} X_s ds + \varepsilon\int_{t_{i-1}}^{t_i} X_s dZ_s. \tag{4}$$

Substituting (4) into (3) , we have

$$\begin{aligned} \widehat{\theta}_{n,\varepsilon} = \theta &+ \frac{\gamma\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}^2} ds}{\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}^2} - (\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}})^2} \\ &- \frac{\gamma\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}} ds}{\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}^2} - (\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}})^2} \\ &+ \frac{\varepsilon\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}^2} dZ_s}{\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}^2} - (\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}})^2} \\ &- \frac{\varepsilon\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}} dZ_s}{\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}^2} - (\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}})^2}, \end{aligned}$$

$$\begin{aligned} \widehat{\gamma}_{n,\varepsilon} = &\frac{\gamma\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}^2}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}} ds}{\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}^2} - (\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}})^2} \\ &- \frac{\gamma\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}^2} ds}{\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}^2} - (\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}})^2} \\ &+ \frac{\varepsilon\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}^2}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}} dZ_s}{\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}^2} - (\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}})^2} \\ &- \frac{\varepsilon\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}^2} dZ_s}{\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}^2} - (\frac{1}{n}\sum_{i=1}^n \frac{1}{X_{t_{i-1}}})^2}. \end{aligned}$$

Consider the underlying ordinary differential equation:

$$dX_t^0 = (\theta+\gamma X_t^0)dt, \quad X_0^0 = x_0.$$

Firstly, we introduce some lemmas below.

Lemma 1: ([8]) Z is a strictly α-stable Lévy process and $\phi\in L^\alpha_{a.s.}$. Then,

$$\int_0^t \phi(s)dZ_s = Z^{'}\circ\int_0^t \phi_+^\alpha(s)ds - Z^{''}\circ\int_0^t \phi_-^\alpha(s)ds, \quad a.s.$$

If Z is symmetric, that is, $\beta=0$, then, there exists some α-stable Lévy process $Z^{'} \overset{d}{=} Z$, such that

$$\int_0^t \phi(s)dZ_s = Z^{'}\circ\int_0^t |\phi(s)|^\alpha ds, \quad a.s.$$

Lemma 2: When $\varepsilon\to 0$ and $n\to\infty$, we have

$$\sup_{0\leq t\leq 1}|X_t - X_t^0| \xrightarrow{P} 0.$$

Proof: Since

$$X_t - X_t^0 = \gamma\int_0^t (X_s - X_s^0)ds + \varepsilon\int_0^t X_s dZ_s, \tag{5}$$

we have

$$\begin{aligned} &|X_t - X_t^0|^2 \\ &\leq 2\gamma^2|\int_0^t (X_s-X_s^0)ds|^2 + 2\varepsilon^2|\int_0^t X_s dZ_s|^2 \\ &\leq 2\gamma^2 t\int_0^t |X_s - X_s^0|^2 ds + 2\varepsilon^2\sup_{0\leq t\leq 1}|\int_0^t X_s dZ_s|^2. \end{aligned}$$

Applying Gronwall's inequality, we have

$$|X_t - X_t^0|^2 \leq 2\varepsilon^2 e^{2\gamma^2 t^2}\sup_{0\leq t\leq 1}|\int_0^t X_s dZ_s|^2.$$

Then,

$$\sup_{0\leq t\leq 1}|X_t - X_t^0| \leq \sqrt{2}\varepsilon e^{\gamma^2}\sup_{0\leq t\leq 1}|\int_0^t X_s dZ_s|.$$

For any given $\delta>0$, when $\varepsilon\to 0$ and $n\to\infty$, we have

$$\begin{aligned} &P(|\sqrt{2}\varepsilon e^{\gamma^2}\sup_{0\leq t\leq 1}|\int_0^t X_s dZ_s||>\delta) \\ \leq\ & \delta^{-1}\sqrt{2}\varepsilon e^{\gamma^2}\mathbb{E}[\sup_{0\leq t\leq 1}|\int_0^t X_s dZ_s|] \\ \leq\ & C\delta^{-1}\sqrt{2}\varepsilon e^{\gamma^2}\mathbb{E}[(\int_0^1 X_s^\alpha ds)^{\frac{1}{\alpha}}] \\ \leq\ & C\delta^{-1}\sqrt{2}\varepsilon e^{\gamma^2}\mathbb{E}[X_M] \\ \to\ & 0, \end{aligned}$$

where C is a constant and $X_M = \sup_{0\le t\le 1}\{X_t\}$.

Therefore,

$$\sup_{0\le t\le 1}|X_t - X_t^0| \xrightarrow{P} 0. \qquad (6)$$

The proof is complete. ■

Lemma 3: When $\varepsilon \to 0$ and $n \to \infty$,

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}} \xrightarrow{P} \int_0^1 \frac{1}{X_t^0}dt$$

Proof:

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^0} + \frac{1}{n}\sum_{i=1}^{n}(\frac{1}{X_{t_{i-1}}} - \frac{1}{X_{t_{i-1}}^0}). \quad (7)$$

It is easy to check that

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^0} \xrightarrow{P} \int_0^1 \frac{1}{X_t^0}dt. \qquad (8)$$

According to Lemma 2, when $\varepsilon \to 0$ and $n \to \infty$, we have

$$\begin{aligned}
&|\frac{1}{n}\sum_{i=1}^{n}(\frac{1}{X_{t_{i-1}}} - \frac{1}{X_{t_{i-1}}^0})| \\
&= \frac{1}{n}|\sum_{i=1}^{n}\frac{X_{t_{i-1}}^0 - X_{t_{i-1}}}{X_{t_{i-1}}X_{t_{i-1}}^0}| \\
&\le \frac{1}{n}\sum_{i=1}^{n}\frac{|X_{t_{i-1}}^0 - X_{t_{i-1}}|}{|X_{t_{i-1}}X_{t_{i-1}}^0|} \\
&\le \sup_{0\le t\le 1}\frac{|X_t^0 - X_t|}{|X_t X_t^0|} \\
&\le \frac{1}{X_N^2}\sup_{0\le t\le 1}|X_t^0 - X_t| \\
&\xrightarrow{P} 0,
\end{aligned}$$

where $X_N = \inf_{0\le t\le 1}\{X_t\}$.

Then, we obtain

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}} \xrightarrow{P} \int_0^1 \frac{1}{X_t^0}dt \qquad (9)$$

The proof is complete. ■

Remark 1: By using the method in Lemma 3, it is easy to check that $\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2} \xrightarrow{P} \int_0^1 \frac{1}{(X_t^0)^2}dt$.

Now we introduce the main results.

Theorem 1: When $\varepsilon \to 0$, $n \to \infty$, and $\varepsilon n^{1-\frac{1}{\alpha}} \to 0$,

$$\widehat{\theta}_{n,\varepsilon} \xrightarrow{P} \theta, \widehat{\gamma}_{n,\varepsilon} \xrightarrow{P} \gamma.$$

Proof: Observe that

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2} - (\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}})^2 \qquad (10)\\
&\xrightarrow{P} \int_0^1 \frac{1}{(X_t^0)^2}dt - (\int_0^1 \frac{1}{X_t^0}dt)^2.
\end{aligned}$$

When $\varepsilon \to 0$ and $n \to \infty$, we have

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}^2}ds \xrightarrow{P} \int_0^1 \frac{X_t}{(X_t^0)^2}dt, \qquad (11)$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}ds \xrightarrow{P} \int_0^1 \frac{1}{X_t^0}dt\int_0^1 \frac{X_t}{X_t^0}dt. \qquad (12)$$

According to Lemma 2, we obtain

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}^2}ds - \frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}ds \xrightarrow{P} 0. \qquad (13)$$

Since

$$\begin{aligned}
&|\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}^2}dZ_s| \\
&\le \varepsilon\sum_{i=1}^{n}|\frac{1}{X_{t_{i-1}}^2}||\int_{t_{i-1}}^{t_i}X_s dZ_s| \\
&\le \varepsilon\sum_{i=1}^{n}(|\frac{1}{(X_{t_{i-1}}^0)^2}| + |\frac{1}{(X_{t_{i-1}})^2} - \frac{1}{(X_{t_{i-1}}^0)^2}|) \\
&|\int_{t_{i-1}}^{t_i}X_s dZ_s| \\
&\le \varepsilon\sum_{i=1}^{n}|\frac{1}{(X_{t_{i-1}}^0)^2}||\int_{t_{i-1}}^{t_i}X_s dZ_s| \\
&+\varepsilon\sup_{0\le t\le 1}|\frac{1}{(X_t)^2} - \frac{1}{(X_t^0)^2}||\int_{t_{i-1}}^{t_i}X_s dZ_s|
\end{aligned}$$

By the Markov inequality, for any given $\delta > 0$, when $\varepsilon \to 0$, $n \to \infty$, and $\varepsilon n^{1-\frac{1}{\alpha}} \to 0$, we obtain

$$\begin{aligned}
&P(|\varepsilon\sum_{i=1}^{n}|\frac{1}{(X_{t_{i-1}}^0)^2}||\int_{t_{i-1}}^{t_i}X_s dZ_s|| > \delta) \\
&\le \delta^{-1}\varepsilon\sum_{i=1}^{n}|\frac{1}{(X_{t_{i-1}}^0)^2}|\mathbb{E}[|\int_{t_{i-1}}^{t_i}X_s dZ_s|] \\
&\le 2C\delta^{-1}\varepsilon\sum_{i=1}^{n}|\frac{1}{(X_{t_{i-1}}^0)^2}|\mathbb{E}[\int_{t_{i-1}}^{t_i}(X_s)^\alpha]^{\frac{1}{\alpha}} \\
&\le 2C\delta^{-1}\varepsilon n^{1-\frac{1}{\alpha}}\frac{1}{n}\sum_{i=1}^{n}|\frac{1}{(X_{t_{i-1}}^0)^2}|\mathbb{E}[X_M] \\
&\to 0,
\end{aligned}$$

where C is constant.

From Lemma 2, when $\varepsilon \to 0$ and $n \to \infty$,

$$\varepsilon\sup_{0\le t\le 1}|\frac{1}{(X_t)^2} - \frac{1}{(X_t^0)^2}||\int_{t_{i-1}}^{t_i}X_s dZ_s| \xrightarrow{P} 0. \qquad (14)$$

Thus, we have

$$\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}^2}dZ_s \xrightarrow{P} 0. \qquad (15)$$

Then, it can be checked that

$$\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}dZ_s \xrightarrow{P} 0. \qquad (16)$$

Then, we have

$$\widehat{\theta}_{n,\varepsilon} \xrightarrow{P} \theta. \qquad (17)$$

According to Lemma 1, we obtain

$$\begin{aligned}
&\gamma\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}ds \qquad (18)\\
&\xrightarrow{P} \gamma\int_0^1 \frac{1}{(X_t^0)^2}dt\int_0^1 \frac{X_t}{X_t^0}dt,
\end{aligned}$$

and

$$\gamma\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}^2}ds \tag{19}$$
$$\xrightarrow{P}\gamma\int_0^1\frac{1}{X_t^0}dt\int_0^1\frac{X_t}{(X_t^0)^2}dt.$$

Together with Lemma 2, we have

$$\frac{\gamma\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}ds}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}-(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}})^2} \tag{20}$$
$$-\frac{\gamma\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}^2}ds}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}-(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}})^2}\xrightarrow{P}\gamma. \tag{21}$$

Since

$$\frac{\varepsilon\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}dZ_s}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}-(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}})^2}\xrightarrow{P}0, \tag{22}$$

and

$$\frac{\varepsilon\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}^2}dZ_s}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}-(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}})^2}\xrightarrow{P}0. \tag{23}$$

Therefore, we obtain

$$\widehat{\gamma}_{n,\varepsilon}\xrightarrow{P}\gamma. \tag{24}$$

The proof is complete. ■

Theorem 2: When $\varepsilon\to 0$, $n\to\infty$, and $n\varepsilon\to\infty$,

$$\varepsilon^{-1}(\widehat{\theta}_{n,\varepsilon}-\theta)$$
$$\xrightarrow{d}\frac{(\int_0^1(\frac{1}{X_t^0})^\alpha dt)^{\frac{1}{\alpha}}S_\alpha(1,0,0)-\int_0^1\frac{1}{X_t^0}dtS_\alpha(1,0,0)}{\int_0^1\frac{1}{(X_t^0)^2}dt-(\int_0^1\frac{1}{X_t^0}dt)^2},$$

$$\varepsilon^{-1}(\widehat{\gamma}_{n,\varepsilon}-\gamma)$$
$$\xrightarrow{d}\frac{\int_0^1\frac{1}{(X_t^0)^2}dtS_\alpha(1,0,0)}{\int_0^1\frac{1}{(X_t^0)^2}dt-(\int_0^1\frac{1}{X_t^0}dt)^2}$$
$$-\frac{\int_0^1\frac{1}{X_t^0}dt(\int_0^1(\frac{1}{X_t^0})^\alpha dt)^{\frac{1}{\alpha}}S_\alpha(1,0,0)}{\int_0^1\frac{1}{(X_t^0)^2}dt-(\int_0^1\frac{1}{X_t^0}dt)^2}$$

Proof: From the explicit decomposition for $\widehat{\theta}_{n,\varepsilon}$, we derive that

$$\varepsilon^{-1}(\widehat{\theta}_{n,\varepsilon}-\theta)$$
$$=\frac{\varepsilon^{-1}\gamma\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}^2}ds}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}-(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}})^2}$$
$$-\frac{\varepsilon^{-1}\gamma\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}ds}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}-(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}})^2}$$
$$+\frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}^2}dZ_s}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}-(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}})^2}$$
$$-\frac{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}dZ_s}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}-(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}})^2}.$$

When $\varepsilon\to 0$, $n\to\infty$, and $n\varepsilon\to\infty$, we have

$$|\varepsilon^{-1}\gamma\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}^2}ds|$$
$$\leq\varepsilon^{-1}\gamma\sum_{i=1}^{n}|\frac{1}{X_{t_{i-1}}^2}||\int_{t_{i-1}}^{t_i}X_sds|$$
$$\leq n^{-1}\varepsilon^{-1}\gamma\sum_{i=1}^{n}(|\frac{1}{(X_{t_{i-1}})^2}-\frac{1}{(X_{t_{i-1}}^0)^2}|+|\frac{1}{(X_{t_{i-1}}^0)^2}|)$$
$$\sup_{t_{i-1}\leq t\leq t_i}|X_t|$$
$$\xrightarrow{P}0.$$

Then, it is easy to check that

$$\varepsilon^{-1}\gamma\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}ds\xrightarrow{P}0. \tag{25}$$

Since

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}^2}dZ_s$$
$$=\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\frac{1}{(X_{t_{i-1}}^0)^2}+\frac{1}{(X_{t_{i-1}})^2}-\frac{1}{(X_{t_{i-1}}^0)^2})X_sdZ_s$$
$$=\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{1}{(X_{t_{i-1}}^0)^2}X_sdZ_s$$
$$+\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\frac{1}{(X_{t_{i-1}})^2}-\frac{1}{(X_{t_{i-1}}^0)^2})X_sdZ_s.$$

Together with Markov inequality and Lemma 2, for any given $\delta>0$, when $\varepsilon\to 0$, $n\to\infty$ and $n^{1-\frac{1}{\alpha}}\to 0$, we have

$$P(|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\frac{1}{(X_{t_{i-1}})^2}-\frac{1}{(X_{t_{i-1}}^0)^2})X_sdZ_s|>\delta)$$
$$\leq\quad\delta^{-1}\sum_{i=1}^{n}\mathbb{E}[|\int_{t_{i-1}}^{t_i}(\frac{1}{(X_{t_{i-1}})^2}-\frac{1}{(X_{t_{i-1}}^0)^2})X_sdZ_s|]$$
$$=\quad\delta^{-1}\sum_{i=1}^{n}\mathbb{E}[|\int_{t_{i-1}}^{t_i}(\frac{1}{X_{t_{i-1}}}+\frac{1}{X_{t_{i-1}}^0})$$
$$(\frac{1}{X_{t_{i-1}}}-\frac{1}{X_{t_{i-1}}^0})X_sdZ_s|]$$
$$\leq\quad\delta^{-1}\sum_{i=1}^{n}\mathbb{E}[|\frac{2}{X_N}|\int_{t_{i-1}}^{t_i}|(\frac{1}{X_{t_{i-1}}}-\frac{1}{X_{t_{i-1}}^0})X_s|dZ_s]$$
$$\leq\quad\delta^{-1}\sum_{i=1}^{n}(\mathbb{E}[|\frac{2}{X_N}|]^2)^{\frac{1}{2}}$$
$$(\mathbb{E}[|\int_{t_{i-1}}^{t_i}|\frac{1}{X_{t_{i-1}}}-\frac{1}{X_{t_{i-1}}^0})X_s|dZ_s]^2)^{\frac{1}{2}}$$
$$\leq\quad\delta^{-1}C(\mathbb{E}[\frac{4}{X_N^2}])^{\frac{1}{2}}$$
$$\sum_{i=1}^{n}(\mathbb{E}[\int_{t_{i-1}}^{t_i}|\frac{1}{X_{t_{i-1}}}-\frac{1}{X_{t_{i-1}}^0})X_s|^\alpha ds]^{\frac{2}{\alpha}})^{\frac{1}{2}}$$
$$\leq\quad\delta^{-1}C(\mathbb{E}[\frac{4}{X_N^2}])^{\frac{1}{2}}\sum_{i=1}^{n}(\mathbb{E}[\sup_{t_{i-1}\leq t\leq t_i}|X_t-X_t^0|^2n^{-\frac{2}{\alpha}}])^{\frac{1}{2}}$$
$$\leq\quad\delta^{-1}Cn^{1-\frac{1}{\alpha}}(\mathbb{E}[\frac{4}{X_N^2}])^{\frac{1}{2}}(\mathbb{E}[\sup_{0\leq t\leq 1}|X_t-X_t^0|^2])^{\frac{1}{2}}$$
$$\to\quad 0.$$

Then, we obtain

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\frac{1}{(X_{t_{i-1}})^2}-\frac{1}{(X_{t_{i-1}}^0)^2}\right)X_s dZ_s \xrightarrow{P} 0. \tag{26}$$

Since

$$\begin{aligned}&\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{(X_{t_{i-1}}^0)^2}dZ_s\\ &=\int_0^1\sum_{i=1}^{n}\frac{X_s}{(X_{t_{i-1}}^0)^2}1_{(t_{i-1},t_i]}(s)dZ_s\\ &=Z'\circ\int_0^1\sum_{i=1}^{n}\frac{X_s}{(X_{t_{i-1}}^0)^2}1_{(t_{i-1},t_i]}(s))^{\alpha}ds,\end{aligned}$$

where $Z' \overset{d}{=} Z$.

According to Lemma 2, we obtain

$$\int_0^1\sum_{i=1}^{n}\frac{X_s}{(X_{t_{i-1}}^0)^2}1_{(t_{i-1},t_i]}(s))^{\alpha}ds \xrightarrow{P} \int_0^1\left(\frac{1}{X_t^0}\right)^{\alpha}dt. \tag{27}$$

From Lemma 1, we have

$$Z'\circ\int_0^1\sum_{i=1}^{n}\frac{X_s}{(X_{t_{i-1}}^0)^2}1_{(t_{i-1},t_i]}(s))^{\alpha}ds \xrightarrow{P} Z'\circ\int_0^1\left(\frac{1}{X_t^0}\right)^{\alpha}dt. \tag{28}$$

Then,

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{(X_{t_{i-1}}^0)^2}dZ_s \xrightarrow{d} \left(\int_0^1\left(\frac{1}{X_t^0}\right)^{\alpha}dt\right)^{\frac{1}{\alpha}}S_\alpha(1,0,0). \tag{29}$$

Thus,

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}dZ_s \xrightarrow{d} \int_0^1\frac{1}{X_t^0}dtS_\alpha(1,0,0). \tag{30}$$

Therefore,

$$\begin{aligned}&\varepsilon^{-1}(\widehat{\theta}_{n,\varepsilon}-\theta) \\ &\xrightarrow{d} \frac{(\int_0^1(\frac{1}{X_t^0})^{\alpha}dt)^{\frac{1}{\alpha}}S_\alpha(1,0,0)-\int_0^1\frac{1}{X_t^0}dtS_\alpha(1,0,0)}{\int_0^1\frac{1}{(X_t^0)^2}dt-(\int_0^1\frac{1}{X_t^0}dt)^2}.\end{aligned} \tag{31}$$

Since

$$\begin{aligned}&\varepsilon^{-1}(\widehat{\gamma}_{n,\varepsilon}-\gamma)\\ &=\frac{\varepsilon^{-1}\gamma\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}ds}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}-(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}})^2}\\ &-\frac{\varepsilon^{-1}\gamma\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}^2}ds}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}-(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}})^2}\\ &-\varepsilon^{-1}\gamma+\frac{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}dZ_s}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}-(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}})^2}\\ &-\frac{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}^2}dZ_s}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}-(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}})^2}.\end{aligned}$$

From above results, it can be checked that

$$\begin{aligned}&\frac{\varepsilon^{-1}\gamma\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}ds}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}-(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}})^2}\\ &-\frac{\varepsilon^{-1}\gamma\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}^2}ds}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}-(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}})^2}-\varepsilon^{-1}\gamma\\ &\xrightarrow{P} 0.\end{aligned} \tag{32}$$

Moreover,

$$\begin{aligned}&\frac{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}dZ_s}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}-(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}})^2}\\ &-\frac{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}^2}dZ_s}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^2}-(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}})^2}\\ &\xrightarrow{d}\frac{\int_0^1\frac{1}{(X_t^0)^2}dtS_\alpha(1,0,0)}{\int_0^1\frac{1}{(X_t^0)^2}dt-(\int_0^1\frac{1}{X_t^0}dt)^2}\\ &-\frac{\int_0^1\frac{1}{X_t^0}dt(\int_0^1(\frac{1}{X_t^0})^{\alpha}dt)^{\frac{1}{\alpha}}S_\alpha(1,0,0)}{\int_0^1\frac{1}{(X_t^0)^2}dt-(\int_0^1\frac{1}{X_t^0}dt)^2}\end{aligned}$$

Therefore,

$$\begin{aligned}&\varepsilon^{-1}(\widehat{\gamma}_{n,\varepsilon}-\gamma)\\ &\xrightarrow{d}\frac{\int_0^1\frac{1}{(X_t^0)^2}dtS_\alpha(1,0,0)-}{\int_0^1\frac{1}{(X_t^0)^2}dt-(\int_0^1\frac{1}{X_t^0}dt)^2}\\ &-\frac{\int_0^1\frac{1}{X_t^0}dt(\int_0^1(\frac{1}{X_t^0})^{\alpha}dt)^{\frac{1}{\alpha}}S_\alpha(1,0,0)}{\int_0^1\frac{1}{(X_t^0)^2}dt-(\int_0^1\frac{1}{X_t^0}dt)^2}.\end{aligned}$$

The proof is complete. ■

IV. Simulation

We use the discrete sample $(X_{t_i})_{i=0,1,\ldots,n}$ to compute the estimator $\widehat{\theta}_{n,\varepsilon}$ and $\widehat{\gamma}_{n,\varepsilon}$. In Tables 1 and 2, we let $x_0 = 0.2$, $\alpha = 1.6$. In Tables 3 and 4, we let $x_0 = 0.1$, $\alpha = 0.8$. In Tables 1 and 3, $\varepsilon = 0.1$, the size is increasing from 1000 to 5000. In Tables 2 and 4, $\varepsilon = 0.05$, the size is increasing from 10000 to 50000. We provide the value of least squares estimator "$\widehat{\theta}_{n,\varepsilon}$", "$\widehat{\gamma}_{n,\varepsilon}$", and the absolute errors (AE)"$|\theta - \widehat{\theta}_{n,\varepsilon}|$", "$|\gamma - \widehat{\gamma}_{n,\varepsilon}|$".

According to the simulation results, when n is large enough and ε is small enough, the estimator is very close to the true parameter value. If we let n converge to the infinity and ε converge to zero, the estimator will converge to the true value.

V. Conclusion

The aim of this paper is to estimate the parameters of Dothan model driven by small symmetrical noise from discrete observation. The least squares estimation has been used to obtain the parameter estimators. The consistency and asymptotic distribution of the estimators have been derived as well. Further research tops will include parameter estimation for partially observed stochastic differential equation driven by small Lévy noise.

TABLE I
LEAST SQUARES ESTIMATOR SIMULATION RESULTS OF θ AND γ

True		Aver		AE	
(θ, γ)	Size n	$\widehat{\theta}_{n,\varepsilon}$	$\widehat{\gamma}_{n,\varepsilon}$	$\lvert\widehat{\theta}_{n,\varepsilon} - \theta\rvert$	$\lvert\widehat{\gamma}_{n,\varepsilon} - \gamma\rvert$
(1,1)	1000	1.1623	1.1536	0.1623	0.1536
	2000	1.0857	1.0815	0.0857	0.0815
	5000	1.0046	1.0039	0.0046	0.0039
(2,3)	1000	2.1735	3.1625	0.1735	0.1625
	2000	2.0937	3.0817	0.0937	0.0817
	5000	2.0051	3.0045	0.0051	0.0045

TABLE II
LEAST SQUARES ESTIMATOR SIMULATION RESULTS OF θ AND γ

True		Aver		AE	
(θ, γ)	Size n	$\widehat{\theta}_{n,\varepsilon}$	$\widehat{\gamma}_{n,\varepsilon}$	$\lvert\widehat{\theta}_{n,\varepsilon} - \theta\rvert$	$\lvert\widehat{\gamma}_{n,\varepsilon} - \gamma\rvert$
(1,1)	10000	1.1206	1.1028	0.1206	0.1028
	20000	1.0227	1.0169	0.0227	0.0169
	50000	1.0008	1.0007	0.0008	0.0007
(2,3)	10000	2.1135	3.0946	0.1135	0.0946
	20000	2.0218	3.0176	0.0218	0.0176
	50000	2.0010	3.0005	0.0010	0.0005

TABLE III
LEAST SQUARES ESTIMATOR SIMULATION RESULTS OF θ AND γ

True		Aver		AE	
(θ, γ)	Size n	$\widehat{\theta}_{n,\varepsilon}$	$\widehat{\gamma}_{n,\varepsilon}$	$\lvert\widehat{\theta}_{n,\varepsilon} - \theta\rvert$	$\lvert\widehat{\gamma}_{n,\varepsilon} - \gamma\rvert$
(1,1)	1000	1.1035	1.1126	0.1035	0.1126
	2000	1.0639	1.0702	0.0639	0.0702
	5000	1.0028	1.0017	0.0028	0.0017
(2,3)	1000	2.1169	3.1014	0.1169	0.1014
	2000	2.0531	3.0681	0.0531	0.0681
	5000	2.0026	3.0019	0.0026	0.0019

TABLE IV
LEAST SQUARES ESTIMATOR SIMULATION RESULTS OF θ AND γ

True		Aver		AE	
(θ, γ)	Size n	$\widehat{\theta}_{n,\varepsilon}$	$\widehat{\gamma}_{n,\varepsilon}$	$\lvert\widehat{\theta}_{n,\varepsilon} - \theta\rvert$	$\lvert\widehat{\gamma}_{n,\varepsilon} - \gamma\rvert$
(1,1)	10000	1.0734	1.0849	0.0734	0.0849
	20000	1.0265	1.0206	0.0265	0.0206
	50000	1.0006	1.0008	0.0006	0.0008
(2,3)	10000	2.0827	3.0713	0.0827	0.0713
	20000	2.0134	3.0201	0.0134	0.0201
	50000	2.0011	3.0007	0.0011	0.0007

REFERENCES

[1] J. P. N. Bishwal, "Parameter estimation in stochastic differential equations", Berlin: Springer-Verlag, 2008.

[2] J. W. Cai, P. Chen, X. Mei, "Range-based threshold spot volatility estimation for jump diffusion models", *IAENG International Journal of Applied Mathematics*, vol. 47, no. 1, pp. 43-48, 2017.

[3] J. Cox, J. Ingersoll, S. Ross, "An intertemporal general equilibrium model of asset prices", *Econometrica*, vol. 53, no. 1, pp. 363-384, 1985.

[4] J. Cox, J. Ingersoll, S. Ross, "A theory of the term structure of interest rates", *Econometrica*, vol. 53, no. 2, pp. 385-408, 1985.

[5] L. U. Dothan, "On the term structure of interest rates", *Journal of Financial Economics*, vol. 6, no. 1, pp. 59C69, 1978.

[6] Y. Z. Hu, H. W. Long, "Parameter estimator for Ornstein-Uhlenbeck processes driven by α-stable Lévy motions", *Communications on Stochastic Analysis*, vol. 1, no. 2, pp. 175-192, 2007.

[7] Y. Z. Hu, H. W. Long, "Least squares estimator for Ornstein-Uhlenbeck processes driven by α-stable motions", *Stochastic Processes and Their Applications*, vol. 119, no. 1, pp. 2465-2480, 2009.

[8] O. Kallenberg, "Some time change representations of stable integrals, via predictable transformations of local martingales", *Stochastic Processes and Their Applications*, vol. 40, no. 2, pp. 199-223, 1992.

[9] M. V. Khramov, "Estimating parameters of short-term real interest rate models". International Monetary Fund, 2013.

[10] N. Kunitomo, A. Takahashi, "The asymptotic expansion approach to the valuation of interest rate contingent claims", *Mathematical Finance*, vol. 11, no. 1, pp. 117-151, 2001.

[11] C. P. Li, H. B. Hao, "Likelihood and Bayesian estimation in stress strength model from generalized exponential distribution containing outliers," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 2, pp. 155-159, 2016.

[12] Z. H. Li, C. H. Ma, "Asymptotic properties of estimators in a stable Cox-Ingersoll-Ross mode", *Stochastic Processes and Their Applications*, vol. 125, no. 8, pp. 3196-3233, 2015.

[13] H. W. Long. " Least squares estimator for discretely observed

Ornstein-Uhlenbeck processes with small Lévy noises", *Statistics and Probability Letters*, vol. 79, no. 1, pp. 2076-2085, 2009.

[14] H. W. Long. "Parameter estimation for a class of stochastic differential equations driven by small stable noises from discrete observations", *Acta Mathematica Scientia*, vol. 30, no. 4, pp. 645-663, 2010.

[15] H. W. Long, Y. Shimizu, W. Sun. "Least squares estimators for discretely observed stochastic processes driven by small Lévy noises", *Journal of Multivariate Analysis*. vol. 116, no. 1, pp. 422-439, 2013.

[16] G. D. Rossi, "Maximum likelihood estimation of the Cox-Ingersoll-Ross model using particle filters", *Computational Economics*, vol. 36, no. 1, pp. 1-16, 2010.

[17] Y. Shu, Q. Feng, E. Kao, et al., "Lévy-driven non-Gaussian Ornstein-Uhlenbeck processes for degradation-based reliability analysis", *IIE Transactions*, vol. 48, no. 5, pp. 993-1003, 2016.

[18] G. N. Singh, S. Suman, C. Singh, "Estimation of a rare sensitive attribute in two-stage sampling using a randomized response model under Poisson distribution", *Mathematical Population Studies: An International Journal of Mathematical Demography*, vol. 27, no. 2, pp. 81-114, 2020.

[19] A. Takahashi, "An asymptotic expansion approach to pricing contingent claims", *Asia Pacific Financial Markets*, vol. 6, no. 2, pp. 115-151, 1999.

[20] S. Treepongkaruna, "Quasi-maximum likelihood estimates of Kiwi short-term interest rate", *Applied Economics Letters*, vol. 10, no. 15, pp. 937-942, 2003.

[21] R. Tunaru, T. Zheng. "Parameter estimation risk in asset pricing and risk management: a Bayesian approach", *International Review of Financial Analysis*, vol. 53, no. 1, pp. 80-93, 2017.

[22] C. Wei, H. S. Shu, Y. R. Liu. "Gaussian estimation for discretely observed Cox-Ingersoll-Ross model", *International Journal of General Systems*, vol. 45, no. 1, pp. 561-574, 2016.

[23] J. H. Wen, X. J. Wang, S. H. Mao, X. P. Xiao, "Maximum likelihood estimation of McKeanCVlasov stochastic differential equation and its application", *Applied Mathematics and Computation*, vol. 274, no. 4, pp. 237-246, 2015.

[24] W. L. Xiao, W. G. Zhang, W. D. Xu, "Parameter estimation for fractional Ornstein-Uhlenbeck processes at discrete observation", *Applied Mathematical Modelling*, vol. 35, no. 9, pp. 4196-4207, 2011.

[25] X. Yang, Y. Liu, G. Park. "Parameter estimation of uncertain differential equation with application to financial market", *Chaos, Solitons and Fractals*, vol. 139, no. 2, pp. 1-12, 2020.

[26] X. B. Zhang, R. D. Brooks, M. L. King, "A Bayesian approach to bandwidth selection for multivariate kernel regression with an application to state-price density estimation", *Journal of Econometrics*, vol. 153, no. 1, pp. 21-32, 2009.