The New Estimation of Bi-response Nonparametric Regression Curve with Combined Estimators

Helida Nurcahayani, I Nyoman Budiantara, and Ismaini Zain

Abstract— **To date, nonparametric regression studies using a combined estimator model have recently begun to develop significantly. Yet previous research employing this approach is still restricted to models with a single response variable. Hence, this paper offers a novel method to estimate bi-response nonparametric regression using a model that combines Fourier series and truncated spline estimators. This research aims to estimate the regression curve of the proposed model using two-stage estimation. The first stage is completed by the penalized weighted least square optimization followed by utilized the weighted least square optimization. We conduct numerical simulations with various sample sizes and correlations to assess the performance of the proposed model. Using generalized cross-validation as a criterion, the best model was obtained from the scenario model with big sample size and strong correlation. Furthermore, compared to uncombined estimators, the proposed model outperformed when applied to a real dataset of the human development index (HDI) education indicator in the East Java Province, Indonesia.**

Index Terms— **bi-response, combined estimators, Fourier series, nonparametric regression, truncated spline**

I. Introduction

The general relationship between the predictor and response variable is described by a regression curve. Given paired data (y_i, z_i), $i = 1, 2, \ldots, n$ and the relationship between y_i and z_i can be modeled as

$$y_i = \mu(z_i) + \varepsilon_i \,, \tag{1}$$

where $\mu(z_i)$ is the regression curve, ε_i is observation errors, and n is the number of observations. There are two fundamental methods—parametric regression and nonparametric regression—for estimating regression curves [1]. Even though parametric regression has been used extensively, its main problem is that the method relies on the strong assumption regarding the shape of the regression curve should follow a particular form such as linear, quadratic, cubic, etc. However, the actual shape of the relationship between y_i and z_i is unknown. If the parametric regression model is nevertheless applied in this situation, inaccurate results will be drawn. Hence, nonparametric regression has been getting attention as an alternative solution to parametric regression since there isn't any claim made on the regression curve's shape.

In nonparametric regression, a variety of estimators can be used to estimate the regression curve, such as splines [2]–[4], Fourier series [5], [6], kernel [7]–[9], and local polynomials [10], [11]. Moreover, in the models with multiple predictor variables, the relationship between each predictor and the response variable does not always follow the same pattern. Thus, current development in the field of nonparametric regression is not limited to a single estimator but also to combining two or more estimators. Several studies developed combination estimators such as the mixed estimator of spline smoothing and kernel [12], [13], the combination truncated spline and a Fourier series estimators [14]–[16], the mixed estimator of kernel and Fourier series [17], and the combination spline, kernel, and Fourier series estimators [18]. The development of regression using a spatial approach, namely a nonparametric truncated spline in a spatially weighted regression model, is also of interest to other researchers [19], [20].

The research on the mixed estimator is still limited to models with a single response variable, even though real datasets sometimes reflect the presence of two or more correlated response variables. Consequently, by utilizing bi-response variables, this study significantly contributes to the field of nonparametric regression with the combined estimator. The truncated spline is one of several types of estimators that frequently used for its several strengths, including being very flexible and having a good capability to handle data whose behavior changes at specific sub-intervals [13], [21]. Instead, we employed the Fourier series function as an estimator if the data display a periodic pattern at particular intervals [5], [17]. By considering some of the benefits of these two functions, this research proposes a new estimation of the bi-response nonparametric regression curve using a combined Fourier series and truncated spline (bi-response nonparametric regression of the CFS-TS model).

Given the perspective previously described, the primary objective of this research is to estimate the curve of the bi-response nonparametric regression CFS-TS model. The proposed model was estimated using the least square method in two stages. Stage one comprised estimating the Fourier series component through penalized weighted least square (PWLS) optimization. Once the first stage was

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complete, the model proceeded to stage two, which was estimating the truncated spline component by employing weighted least square (WLS) optimization. After obtaining the curve estimation as a theoretical result, we ran computational modeling using simulated data. Moreover, we implement the proposed model into a case study of East Java's HDI education indicator.

The overall structure of the study takes the form of six sections. The topic overview, knowledge gap, and study aims are all presented in Section 1. Meanwhile, the second section describes the nonparametric regression model and the PWLS method. The research's findings are reported in the third section, focusing on the proposed model's curve estimation as well as estimator properties and smoothing parameter selection. The implementation of the obtained estimator under simulation studies and data application is presented in Sections 4 and 5. The last section gives a compact conclusion with recommendations for future work.

II. Material and Methods

In this paper, we only consider two response or bi-response data. Nevertheless, these methods can be extended to a multi-response nonparametric regression model. Let y_{1i} and y_{2i} are responses variable, x_{ji} and t_{ki} are the predictor variables with $i=1,...,n$ is the number of observations. The following bi-response nonparametric regression model is assumed to represent the relationship between response and predictor variables:

$$y_{hi} = \mu_{hi}\left(x_{1i},...,x_{pi},t_{1i},...,t_{qi}\right)+\varepsilon_{hi},\ h=1,2, \tag{2}$$

where μ_{hi} is the regression curve. Moreover, (2) can be written in the following matrix form

$$\mathbf{y}=\boldsymbol{\mu}\left(\mathbf{x},\mathbf{t}\right)+\boldsymbol{\varepsilon}\ . \tag{3}$$

Equation (2) assumes that the regression curve μ_{hi} is an additive model as given in (4).

$$\mu_{hi}\left(x_{1i},...,x_{pi},t_{1i},...,t_{qi}\right)=\sum_{j=1}^{p} f_{hj}(x_{ji})+\sum_{k=1}^{q} g_{hk}(t_{ki})\ , \tag{4}$$

while $f_{hj}(x_{ji})$, j = 1, 2, …, p and $g_{hk}(t_{ki})$ k = 1, 2, …, q were approached by Fourier series and truncated spline estimators respectively. Substituting (4) into (2), the bi-response nonparametric regression of the CFS-TS model can be written as

$$y_{hi}=\sum_{j=1}^{p} f_{hj}(x_{ji})+\sum_{k=1}^{q} g_{hk}(t_{ki})+\varepsilon_{hi}. \tag{5}$$

Two-stage optimization was used to obtain the estimator of μ_{hi}, i.e., the penalized weighted least square (PWLS) and the weighted least square (WLS) method. Firstly, PWLS was conducted to estimate $f_{hj}(x_{ji})$, which resulted in Theorem 1. Once the first stage was completed, the second stage was carried out using WLS to estimate $g_{hk}(t_{ki})$, which led to Theorem 2. Taken together the result from both theorems gave the regression curve estimation of μ_{hi} as in Corollary 1.

III. Results

A. The Curve Estimation of Bi-response Nonparametric Regression of the CFS-TS Model

The section discusses the two-stage optimization to estimate the nonparametric regression curve of μ_{hi} in (4). In the first stage, we must complete PWLS optimization as follows

$$\underset{f_p\in C[0,\pi]}{Min}\left\{(2n)^{-1}\sum_{h=1}^{2}\sum_{i=1}^{n} w_{hi}\left(y_{hi}-\sum_{j=1}^{p} f_{hj}(x_{ji})-\sum_{k=1}^{q} g_{hk}(t_{ki})\right)^2 +\sum_{j=1}^{p}\lambda_j\int_0^{\pi}\frac{2}{\pi}\left(f_j^{''}(x_j)\right)^2 dx_j\right\}, \tag{6}$$

where w_{hi} is a weighted component and λ_j is defined as a smoothing parameter. To make it simpler to carry out, we divided the equation into two components namely Goodness of Fit (GoF) and penalty that is presented in Lemma 1 and Lemma 2 respectively.

Lemma 1. *Suppose $f_{hj}(x_{ji})$ is approached by the Fourier series function, then the Goodness of Fit can be written as*

$$G\left(\mathbf{f}_1,\mathbf{f}_2\right)=(2n)^{-1}\left(\mathbf{y}^*-\mathbf{Xa}\right)^T\mathbf{W}\left(\mathbf{y}^*-\mathbf{Xa}\right).$$

Proof of Lemma 1. To begin with, we assume $f_{hj}(x_{ji})$ in (5) as a smooth function and it is contained in the space of continuous function on the interval $C(0,\pi)$. Additionally, $f_{hj}(x_{ji})$ is approached by the modified Fourier series cosine function used by Bilodeau [5] as follows

$$f_{hj}(x_{ji})=b_{hj}x_{ji}+\frac{a_{0hj}}{2}+\sum_{v=1}^{V} a_{vhj}\cos vx_{ji}\ . \tag{7}$$

For efficiency, the function $f_{hj}(x_{ji})$ can be drawn in the following matrix form

$$\mathbf{f}=\begin{bmatrix}\mathbf{f}_1^T\\ \mathbf{f}_2^T\end{bmatrix}=\begin{bmatrix}\mathbf{X}_1\mathbf{a}_1\\ \mathbf{X}_2\mathbf{a}_2\end{bmatrix}=\begin{bmatrix}\mathbf{X}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_2\end{bmatrix}\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{bmatrix} =\mathbf{Xa} \tag{8}$$

where

$$\mathbf{X}_h=\begin{bmatrix} x_{j1} & 1/2 & \cos 1x_{j1} & \cos 2x_{j1} & \cdots & \cos Vx_{j1}\\ x_{j2} & 1/2 & \cos 1x_{j2} & \cos 2x_{j2} & \cdots & \cos Vx_{j2}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ x_{jn} & 1/2 & \cos 1x_{jn} & \cos 2x_{jn} & \cdots & \cos Vx_{jn}\end{bmatrix},$$

$$\mathbf{a}_h=\begin{bmatrix} b_{hj} & a_{0hj} & a_{1hj} & a_{2hj} & \cdots & a_{Vhj}\end{bmatrix},\ h = 1, 2.$$

Furthermore, we define the Goodness of Fit component in (6) as follows

$$G\left(f_{hj}(x_{j1}),\dots,f_{hj}(x_{jn})\right) =(2n)^{-1}\sum_{h=1}^{2}\sum_{i=1}^{n} w_{hi}\left(y_{hi}-\sum_{j=1}^{p} f_{hj}(x_{ji})-\sum_{k=1}^{q} g_{hk}(t_{ki})\right)^2. \tag{9}$$

Based on the bi-response nonparametric regression model in (5), if we move the $\sum_{k=1}^{q} g_{hk}(t_{ki})$ to the left side then (5) can easily be expressed as below

$$y_{hi} - \sum_{k=1}^{q} g_{hk}(t_{ki}) = \sum_{j=1}^{p} f_{hj}(x_{ji}) + \varepsilon_{hi}. \quad (10)$$

Let $y_{hi}^{*} = y_{hi} - \sum_{k=1}^{q} g_{hk}(t_{ki})$, the Goodness of Fit in (9) is written as follows

$$G\left(f_{hj}(x_{j1}),\ldots,f_{hj}(x_{jn})\right) = (2n)^{-1}\sum_{h=1}^{2}\sum_{i=1}^{n} w_{hi}\left(y_{hi}^{*} - \sum_{k=1}^{q} f_{hj}(x_{ji})\right)^2. \quad (11)$$

The outcome in (11) can be presented in the following matrix form

$$G(\mathbf{f}_1,\mathbf{f}_2) = (2n)^{-1}\mathbf{W}\left(\mathbf{y}^{*} - \mathbf{Xa}\right)^2 = (2n)^{-1}\left(\mathbf{y}^{*} - \mathbf{Xa}\right)^T \mathbf{W}\left(\mathbf{y}^{*} - \mathbf{Xa}\right), \quad \blacksquare$$

where $\mathbf{y}^{*} = \mathbf{y} - \mathbf{g}$, $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2 \end{bmatrix}$, and $\mathbf{a} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix}$.

Lemma 2. *The penalty component is presented by the following equation*

$$P(\lambda_1,\ldots,\lambda_p) = \mathbf{a}^T\mathbf{D}(\lambda)\mathbf{a}.$$

Proof of Lemma 2. With regards to (6), we define the penalty component as below

$$P(\lambda_1,\ldots,\lambda_p) = \sum_{j=1}^{p}\lambda_j \int_0^{\pi} \frac{2}{\pi}\left(f_j^{''}(x_j)\right)^2 dx_j \quad (12)$$

where $f_j(x_j)$ is approached by the Fourier series in (7). To begin with, we solve the second derivative of $f_j(x_j)$ such that we obtain (13).

$$P(\lambda_1,\ldots,\lambda_p) = \sum_{j=1}^{p}\lambda_j \int_0^{\pi} \frac{2}{\pi}\left(f_j^{''}(x_j)\right)^2 dx_j$$
$$= \int_0^{\pi} \frac{2}{\pi}\frac{d}{dx_j}\left[\frac{d}{dx_j}\left(b_j x_j + \frac{a_{0j}}{2} + \sum_{v=1}^{V} a_{vj}\cos v x_j\right)\right]^2 dx_j$$
$$= \sum_{j=1}^{p}\left(\lambda_j \sum_{v=1}^{V} v^4 \, a_{vj}^2\right). \quad (13)$$

As a result, (13) can be written in the following matrix form.

$$P(\lambda_1,\ldots,\lambda_p) = \mathbf{a}^T\mathbf{D}(\lambda)\mathbf{a}, \quad \blacksquare$$

where

$$\mathbf{D}(\lambda) = \begin{bmatrix} \mathbf{D}_1(\lambda) & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2(\lambda) \end{bmatrix},\ \mathbf{D}_h(\lambda) = \begin{bmatrix} \mathbf{d}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{d}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{d}_p \end{bmatrix},$$

$$d_j = diag\begin{bmatrix} 0 & 0 & \mathbf{I}_j \end{bmatrix},\ \mathbf{I}_j = \begin{bmatrix} \lambda_j 1^4 & \lambda_j 2^4 & \cdots & \lambda_j V^4 \end{bmatrix}.$$

In summary, the construction of the Goodness of Fit and penalty has been established in Lemma 1 and Lemma 2. The result was Theorem 1 which presents the first stage of estimation using PWLS optimization.

Theorem 1. *The PWLS optimization yielded the following Fourier series estimator as follows*

$$\hat{\mathbf{f}}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t}) = \left(\mathbf{X}\left[\mathbf{X}^T\mathbf{WX} + (2n)\mathbf{D}(\lambda)\right]^{-1}\mathbf{X}^T\mathbf{W}\right)\mathbf{y}^{*}.$$

Proof of Theorem 1. The results from Lemma 1 and Lemma 2 were substituted into the PWLS optimization in (6) to produce the following equation

$$\underset{f_p \in C[0,\pi]}{Min}\left\{(2n)^{-1}\sum_{h=1}^{2}\sum_{i=1}^{n} w_{hi}\left(y_{hi} - \sum_{j=1}^{p} f_{hj}(x_{ji}) - \sum_{k=1}^{q} g_{hk}(t_{ki})\right)^2 + \sum_{j=1}^{p}\lambda_j\int_0^{\pi}\frac{2}{\pi}\left(f_j^{''}(x_j)\right)^2 dx_j\right\},$$
$$= \underset{\mathbf{a}\in R^{(2+V)pr\times 1}}{Min}\left\{(2n)^{-1}\left(\mathbf{y}^{*} - \mathbf{Xa}\right)^T\mathbf{W}\left(\mathbf{y}^{*} - \mathbf{Xa}\right) + \mathbf{a}^T\mathbf{D}(\lambda)\mathbf{a}\right\}$$
$$= \underset{\mathbf{a}\in R^{(2+V)pr\times 1}}{Min}\left\{(2n)^{-1}\mathbf{y}^{*T}\mathbf{W}\mathbf{y}^{*} - 2(2n)^{-1}\mathbf{a}^T\mathbf{X}^T\mathbf{W}\mathbf{y}^{*} + (2n)^{-1}\mathbf{a}^T\mathbf{X}^T\mathbf{WXa} + \mathbf{a}^T\mathbf{D}(\lambda)\mathbf{a}\right\}. \quad (14)$$

By partially deriving (14) against $\mathbf{a}$ and equating the outcome with $\mathbf{0}$ then $\hat{\mathbf{a}}$ is obtained as below

$$\frac{\partial\left[(2n)^{-1}\mathbf{y}^{*T}\mathbf{W}\mathbf{y}^{*} - 2(2n)^{-1}\mathbf{a}^T\mathbf{X}^T\mathbf{W}\mathbf{y}^{*} + (2n)^{-1}\mathbf{a}^T\mathbf{X}^T\mathbf{WXa} + \mathbf{a}^T\mathbf{D}(\lambda)\mathbf{a}\right]}{\partial\mathbf{a}} = \mathbf{0}$$

$$\hat{\mathbf{a}} = \left[\mathbf{X}^T\mathbf{WX} + (2n)\mathbf{D}(\lambda)\right]^{-1}\mathbf{X}^T\mathbf{W}\mathbf{y}^{*}. \quad (15)$$

Thus, by substituting $\hat{\mathbf{a}}$ into (8), we get the following estimator of the Fourier series component

$$\hat{\mathbf{f}}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t}) = \mathbf{X}\hat{\mathbf{a}} \quad (16)$$
$$= \left(\mathbf{X}\left[\mathbf{X}^T\mathbf{WX} + (2n)\mathbf{D}(\lambda)\right]^{-1}\mathbf{X}^T\mathbf{W}\right)\mathbf{y}^{*} \quad \blacksquare$$
$$= \mathbf{H}\mathbf{y}^{*} \quad (17)$$

where $\mathbf{H} = \mathbf{X}\left[\mathbf{X}^T\mathbf{WX} + (2n)\mathbf{D}(\lambda)\right]^{-1}\mathbf{X}^T\mathbf{W}$.

Moving on to obtain the estimation of the truncated spline component, the next stage estimated the regression curve using WLS optimization, as stated in Lemma 3, and the outcome is shown in Theorem 2.

Lemma 3. *The WLS optimization can be presented as the following equation*

$$\left[(\mathbf{I} - \mathbf{H})(\mathbf{y} - \mathbf{J}\boldsymbol{\theta})\right]^T\mathbf{W}\left[(\mathbf{I} - \mathbf{H})(\mathbf{y} - \mathbf{J}\boldsymbol{\theta})\right].$$

Proof of Lemma 3. In this paper, we assume $g_{hk}(t_{ki})$ in (5) is approached by linear truncated spline function with knot $K_{hk1}, K_{hk2}, ..., K_{hku}$ as follows

$$g_{hk}(t_{ki}) = \alpha_{hk}t_{ki} + \sum_{s=1}^{u}\beta_{hks}(t_{ki} - K_{hks})_{+} \quad (18)$$

with the truncated function

$$(t_{ki} - K_{hks})_{+} = \begin{cases} (t_{ki} - K_{hks})_{+} & ,t_{ki} \geq K_{hks} \\ 0 & ,t_{ki} < K_{hks}. \end{cases}$$

The truncated spline function in (18) can be defined as matrix form in (19) .

$$\mathbf{g} = \begin{bmatrix} \mathbf{g}_1^T \\ \mathbf{g}_2^T \end{bmatrix} = \begin{bmatrix} \mathbf{T}_1\boldsymbol{\alpha}_1 \\ \mathbf{T}_2\boldsymbol{\alpha}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{S}_1\boldsymbol{\beta}_1 \\ \mathbf{S}_2\boldsymbol{\beta}_2 \end{bmatrix} = \left[\mathbf{T} \mid \mathbf{S}\right]\left[\frac{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\right] = \mathbf{J}\boldsymbol{\theta}, \quad (19)$$

where

$$\mathbf{J} = \left[\mathbf{T} \mid \mathbf{S}\right],\ \boldsymbol{\theta} = \left[\frac{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\right],\ \mathbf{T}_i = \begin{bmatrix} t_{11} & t_{21} & \cdots & t_{q1} \\ t_{12} & t_{22} & \cdots & t_{q2} \\ \vdots & \vdots & \ddots & \vdots \\ t_{1n} & t_{2n} & \cdots & t_{qn} \end{bmatrix},$$

$$\mathbf{S}_i = \begin{bmatrix} (t_{11}-K_{h11})_+ & \cdots & (t_{11}-K_{h1u})_+ & \cdots & (t_{p1}-K_{hq1})_+ & \cdots & (t_{11}-K_{hqu})_+ \\ (t_{12}-K_{h11})_+ & \cdots & (t_{12}-K_{h1u})_+ & \cdots & (t_{p2}-K_{hq1})_+ & \cdots & (t_{12}-K_{hqu})_+ \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ (t_{1n}-K_{h11})_+ & \cdots & (t_{1n}-K_{h1u})_+ & \cdots & (t_{pn}-K_{hq1})_+ & \cdots & (t_{1n}-K_{hqu})_+ \end{bmatrix},$$

$\boldsymbol{\beta}_h = \begin{bmatrix} \beta_{h11} & \beta_{h12} & \cdots & \beta_{h1u} & \cdots & \beta_{hq1} & \beta_{hq2} & \cdots & \beta_{hqu} \end{bmatrix}^T$,

$\boldsymbol{\alpha}_h = \begin{bmatrix} \alpha_{1h} & \cdots & \alpha_{qh} \end{bmatrix}^T$. Thereafter, the equation in (17) and (19) are substituted into (3), hence we retrieve the result as in (20).

$$\begin{aligned} \mathbf{y} &= \mathbf{f} + \mathbf{g} + \boldsymbol{\varepsilon} \text{; where } \mathbf{f} = \mathbf{H}\mathbf{y}^* \\ \boldsymbol{\varepsilon} &= \mathbf{y} - \mathbf{H}\mathbf{y}^* - \mathbf{g} \\ &= (\mathbf{I}-\mathbf{H})(\mathbf{y}-\mathbf{J}\boldsymbol{\theta}). \end{aligned} \quad (20)$$

The estimator $\hat{\boldsymbol{\theta}}$ is derived by solving the subsequent WLS optimization given by (21)

$$\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} = \left[(\mathbf{I}-\mathbf{H})(\mathbf{y}-\mathbf{J}\boldsymbol{\theta})\right]^T \mathbf{W}\left[(\mathbf{I}-\mathbf{H})(\mathbf{y}-\mathbf{J}\boldsymbol{\theta})\right]. \quad \blacksquare \ (21)$$

Therefore, the following step was to solve the optimization using WLS given by Theorem 2 as follows.

Theorem 2. *As a result of WLS optimization, the model's truncated spline estimator is given by the equation below.*

$$\hat{\mathbf{g}}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t}) = \mathbf{J}\mathbf{K}^{-1}\mathbf{L}\mathbf{y}.$$

Proof of Theorem 2. Using (21) as a reference, suppose $\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} = \mathbf{M}(\boldsymbol{\theta})$ such that the following equation obtained.

$$\mathbf{M}(\boldsymbol{\theta}) = [\mathbf{y}-\mathbf{H}\mathbf{y}-\mathbf{J}\boldsymbol{\theta}+\mathbf{H}\mathbf{J}\boldsymbol{\theta}]^T \mathbf{W}[\mathbf{y}-\mathbf{H}\mathbf{y}-\mathbf{J}\boldsymbol{\theta}+\mathbf{H}\mathbf{J}\boldsymbol{\theta}]. \quad (22)$$

By partially deriving $\mathbf{M}(\boldsymbol{\theta})$ on $\boldsymbol{\theta}$ and equating the outcome with $\mathbf{0}$, we obtain the equation below.

$$\frac{\partial \mathbf{M}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \frac{\partial}{\partial \boldsymbol{\theta}}\left([\mathbf{y}-\mathbf{H}\mathbf{y}-\mathbf{J}\boldsymbol{\theta}+\mathbf{H}\mathbf{J}\boldsymbol{\theta}]^T \mathbf{W}[\mathbf{y}-\mathbf{H}\mathbf{y}-\mathbf{J}\boldsymbol{\theta}+\mathbf{H}\mathbf{J}\boldsymbol{\theta}]\right)$$

$$\hat{\boldsymbol{\theta}} = \left[\mathbf{J}^T\left(\mathbf{I}-2\mathbf{H}^T\right)\mathbf{W}\mathbf{J}+\mathbf{J}^T\mathbf{H}^T\mathbf{W}\mathbf{H}\mathbf{J}\right]^{-1}\left[\mathbf{J}^T\left(\mathbf{H}^T-\mathbf{I}\right)\mathbf{W}\left(\mathbf{H}-\mathbf{I}\right)\right]\mathbf{y}.$$

$$\hat{\boldsymbol{\theta}} = \mathbf{K}^{-1}\mathbf{L}\mathbf{y}. \quad (23)$$

where $\mathbf{K} = \left[\mathbf{J}^T\left(\mathbf{I}-2\mathbf{H}^T\right)\mathbf{W}\mathbf{J}+\mathbf{J}^T\mathbf{H}^T\mathbf{W}\mathbf{H}\mathbf{J}\right]$,

$\mathbf{L} = \left[\mathbf{J}^T\left(\mathbf{H}^T-\mathbf{I}\right)\mathbf{W}\left(\mathbf{H}-\mathbf{I}\right)\right]$. Substituting (23) into (19) yields

$$\begin{aligned} \hat{\mathbf{g}}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t}) &= \mathbf{J}\hat{\boldsymbol{\theta}} \\ &= \mathbf{J}\mathbf{K}^{-1}\mathbf{L}\mathbf{y} = \mathbf{A}_{(\mathbf{K},\lambda,\mathbf{V})}\mathbf{y}. \end{aligned} \quad \blacksquare \ (24)$$

Thus, we substitute (24) into (15) such that we attain the equation below.

$$\begin{aligned} \hat{\mathbf{a}} &= \left[\mathbf{X}^T\mathbf{W}\mathbf{X}+(2n)\mathbf{D}(\lambda)\right]^{-1}\mathbf{X}^T\mathbf{W}\mathbf{y}^* \\ &= \left[\mathbf{X}^T\mathbf{W}\mathbf{X}+(2n)\mathbf{D}(\lambda)\right]^{-1}\mathbf{X}^T\mathbf{W}\left(\mathbf{I}-\mathbf{J}\mathbf{K}^{-1}\mathbf{L}\right)\mathbf{y}. \end{aligned} \quad (25)$$

Following that, the outcome on (25) is substituted into (16). Consequently, we get the estimator of $g_{hk}(t_{ki})$ which no longer contains $f_{hj}(x_{ji})$ as follows

$$\begin{aligned} \hat{\mathbf{f}}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t}) &= \mathbf{X}\hat{\mathbf{a}} \\ &= \mathbf{X}\left(\left[\mathbf{X}^T\mathbf{W}\mathbf{X}+(2n)\mathbf{D}(\lambda)\right]^{-1}\mathbf{X}^T\mathbf{W}\left(\mathbf{I}-\mathbf{J}\mathbf{K}^{-1}\mathbf{L}\right)\mathbf{y}\right) \\ &= \mathbf{H}\left(\mathbf{I}-\mathbf{J}\mathbf{K}^{-1}\mathbf{L}\right)\mathbf{y} = \mathbf{B}_{(\mathbf{K},\lambda,\mathbf{V})}\mathbf{y} \end{aligned} \quad (26)$$

Another main finding in the theoretical result is the regression curve estimation of μ_{hi}, as shown in Corollary 1.

Corollary 1. *Utilizing the findings in (24) and (26) such that we obtain the regression curve estimation of bi-response nonparametric regression with the CFS-TS model as follows*

$$\hat{\boldsymbol{\mu}}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t}) = \left[\mathbf{H}\left(\mathbf{I}-\mathbf{J}\mathbf{K}^{-1}\mathbf{L}\right)+\mathbf{J}\mathbf{K}^{-1}\mathbf{L}\right]\mathbf{y}.$$

Proof of Corollary 1. The model's regression curve estimation can be expressed in the following matrix form by employing an additive model in (4) of the equation.

$$\hat{\boldsymbol{\mu}}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t}) = \hat{\mathbf{f}}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t}) + \hat{\mathbf{g}}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t}). \quad (27)$$

According to the results of truncated spline and Fourier series estimators on (24) and (26) respectively, (27) can be written as

$$\begin{aligned} \hat{\boldsymbol{\mu}}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t}) &= \left[\mathbf{H}\left(\mathbf{I}-\mathbf{J}\mathbf{K}^{-1}\mathbf{L}\right)+\mathbf{J}\mathbf{K}^{-1}\mathbf{L}\right]\mathbf{y} \quad \blacksquare \\ &= \mathbf{C}_{(\mathbf{K},\lambda,\mathbf{V})}\mathbf{y}. \end{aligned}$$

B. The Properties of Bi-response Nonparametric Regression of the CFS-TS Model

Regarding the outcome in the previous section, the following procedure was to verify the properties of the proposed model. If we assign the mathematical expectation to (27) then,

$$\begin{aligned} E\left(\hat{\boldsymbol{\mu}}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t})\right) &= E\left(\mathbf{C}_{(\mathbf{K},\lambda,\mathbf{V})}\mathbf{y}\right) \\ &= \mathbf{C}_{(\mathbf{K},\lambda,\mathbf{V})}E(\mathbf{y}) \\ &= \mathbf{C}_{(\mathbf{K},\lambda,\mathbf{V})}E\left(\hat{\mathbf{f}}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t})+\hat{\mathbf{g}}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t})\right) \\ &\neq \boldsymbol{\mu}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t}). \end{aligned}$$

Since $E\left(\hat{\boldsymbol{\mu}}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t})\right) \neq \boldsymbol{\mu}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t})$, this leads to the conclusion that the obtained estimator is biased. Nevertheless, the estimator is linear in the observation as proved in the following equation

$$\begin{aligned} \hat{\boldsymbol{\mu}}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t}) &= \hat{\mathbf{f}}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t})+\hat{\mathbf{g}}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t}) \\ &= \mathbf{A}_{(\mathbf{K},\lambda,\mathbf{V})}\mathbf{y}+\mathbf{B}_{(\mathbf{K},\lambda,\mathbf{V})}\mathbf{y} \\ &= \mathbf{C}_{(\mathbf{K},\lambda,\mathbf{V})}\mathbf{y}. \end{aligned}$$

C. Smoothing Parameter Selection

Smoothing parameter selection is fundamental in nonparametric regression analysis since it will influence the estimated results of the regression curve. In this paper, we selected the optimal knot, oscillation parameter, and smoothing parameter to find the best model. From a variety of knots, oscillation, and smoothing parameter, the smallest GCV value can be used as a criterion for determining the best model.

For the proposed model described in section B, the modified GCV is presented in (29).

$$\mathrm{GCV}_{(\mathbf{K},\lambda,\mathbf{T})} = \frac{\mathrm{MSE}_{(\mathbf{K},\lambda,\mathbf{V})}}{\left[(2n)^{-1}trace\left(\mathbf{I}-\mathbf{C}_{(\mathbf{K},\lambda,\mathbf{V})}\right)\right]^2}, \quad (28)$$

where

$$\mathrm{MSE}_{(\mathbf{K},\lambda,\mathbf{V})} = (2n)^{-1}(\mathbf{y}-\hat{\boldsymbol{\mu}})^T(\mathbf{y}-\hat{\boldsymbol{\mu}}) = (2n)^{-1}\left\|\mathbf{I}-\mathbf{C}_{(\mathbf{K},\lambda,\mathbf{V})}\mathbf{y}\right\|^2.$$

Hence, (28) can be written as follows

$$GCV_{(\mathbf{K},\lambda,\mathbf{T})}=\frac{(2n)^{-1}\left\|\mathbf{I}-\mathbf{C}_{(\mathbf{K},\lambda,\mathbf{V})}\mathbf{y}\right\|^{2}}{\left[(2n)^{-1}trace\left(\mathbf{I}-\mathbf{C}_{(\mathbf{K},\lambda,\mathbf{V})}\right)\right]^{2}}. \qquad (29)$$

IV. Simulation Study

So far, this paper has focused on curve estimation as a theoretical result of the proposed model. The following section provides a simulation study of the bi-response nonparametric regression of the CFS-TS model to assess the performance of the obtained estimator. The simulation was carried out with three data sample sizes (*n* = 20, 50, 100) which were repeated 100 times for each sample size. For each response resulting from the formula, we constructed models that describe two distinct functions that stand in for the truncated spline and Fourier series functions.

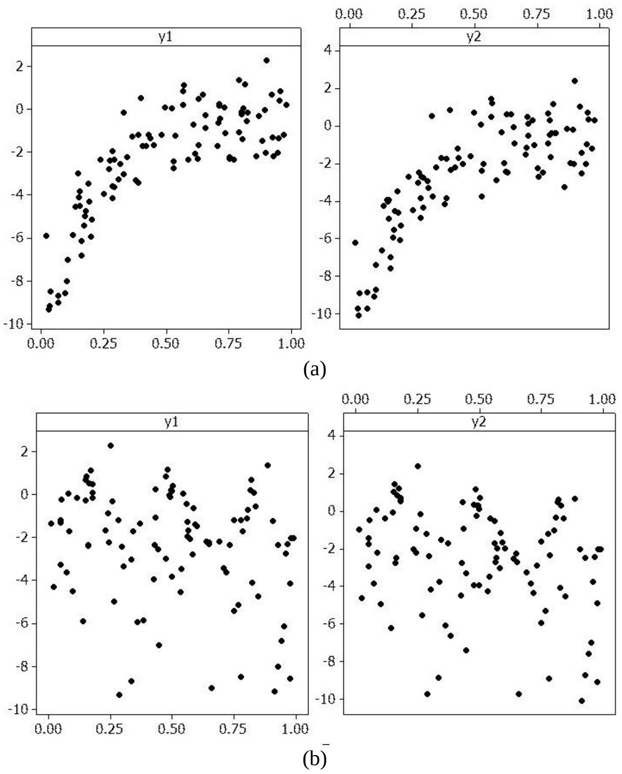

Fig. 1. Partially scattered plot for each predictor variable to the response variable, which represents a truncated spline function (a) and a Fourier series function (b)

Fig. I (a) illustrate how a trigonometric function reflects a trend for the Fourier series function. Meanwhile Fig. I (b) display partially scattered plot of polynomial function which represent the truncated spline. Thus, the model used in this numerical study is as follows

$$y_{1i}=-1.5x_i-\cos\left(6\pi x_i\right)-17\left(0.8-t_i\right)^3+\varepsilon_{1i},$$
$$y_{2i}=-1.75x_i-1.2\cos\left(6\pi x_i\right)-18\left(0.8-t_i\right)^3+\varepsilon_{2i}. \qquad (30)$$

In (30), we use a Uniform (0,1) distribution to generate these two predictor variables while the multivariate normal distribution is used to generate the random error ε_{hi}. The correlation between ε_{1i} and ε_{2i} id defined as $corr\left(\varepsilon_{1i},\varepsilon_{2i}\right)=\rho$. Moreover, since the study discusses the bi-response model, the simulation was carried out with three different correlations ρ = 0, 0.5, and 0.9. According to the partial scatterplot identification in Fig. 1 and with the intention to simplify the computational process, we executed the simulation with combination of three knots (K = 1, 2, and 3) and three oscillation parameters (V = 1, 2, and 3).

TABLE I
Comparison GCV of the proposed model using simulation data with the number of knots is 1

n	ρ	**Number of oscillations**	**GCV**
20	0	1	1.31016
	0	2	1.30794
	0	3	1.29848
	0.5	1	1.29660
	0.5	2	1.29636
	0.5	3	1.29479
	0.9	1	1.29209
	0.9	2	1.29122
	0.9	3	1.28670*)
50	0	1	1.23039
	0	2	1.23000
	0	3	1.22970
	0.5	1	1.19910
	0.5	2	1.19881
	0.5	3	1.19676
	0.9	1	1.19033
	0.9	2	1.19026
	0.9	3	1.18662*)
100	0	1	1.20418
	0	2	1.20403
	0	3	1.20355
	0.5	1	1.18326
	0.5	2	1.18322
	0.5	3	1.18165
	0.9	1	1.18118
	0.9	2	1.18108
	0.9	3	1.18067*)

Furthermore, Table I compares the GCV of the proposed models using simulated data with one knot on various data sample sizes, correlation, and the number of oscillations. A complete summary of the statistical result for the proposed models using two and three knots is provided in Appendix (Table A and Table B). Table I demonstrates that the smallest GCV of the sample size *n* = 20, 50, and 100 occurs for ρ = 0.9 with the combination of one knot and three oscillation model. Additionally, *n* = 100 yields the GCV with the least value (1.18067) among the three sample sizes. Surprisingly, we obtain the same outcome by using models with two and three knots, as shown in Table A and Table B (Appendix). Consequently, a high correlation with a large sample size will result in a smaller GCV, which indicates that the regression curve will be better estimated as well.

V. Data Application

This section continues with the application of an actual dataset utilizing bi-response nonparametric regression of the CFS-TS model. In this article, we employed 2019 data from 38 regencies/cities in East Java Province, Indonesia, for the education indicator of the human development index (HDI). Both indicators, expected years of schooling (EYS) and mean years of schooling (MYS), were used in this case study as the response variables and thus were denoted as y_1 and y_2, respectively. According to earlier studies, several predictor variables, including the percentage of poverty [16], [22], the unemployment rate [22], [23], and the per capita gross regional domestic product (GRDP) [22], [24] potentially influence Indonesia's HDI. Thus, the per capita gross regional domestic product (per capita GRDP) and the percentage of people who lived in poverty were utilized as predictor variables in this study. Despite being one of Indonesia's largest provinces, East Java's HDI problem nonetheless attracted a lot of criticism, notably in the education dimension. Additionally, when compared to the other provinces on the island of Java, East Java has the lowest HDI; this ranking is based on the province's HDI accomplishments from 2015 to 2019. It is therefore possible to discuss further studies on the HDI in East Java.

In this study, we used the Pearson correlation to measure the strength and direction of the association between y_1 and y_2. The outcome revealed a statistical value of 0.769 for the Pearson correlation (r), indicating a positive and generally strong association between the two response variables. Additionally, the *p-value* is 0.00, which implies that the null hypothesis is rejected since the *p-value* $< \alpha$ (0.05). This finding implies that there is a significant correlation between y_1 and y_2, hence bi-response nonparametric regression can be applied. A partial scatterplot, as seen in Fig. 2, was used to determine the relationship between each predictor variable and the response variable.

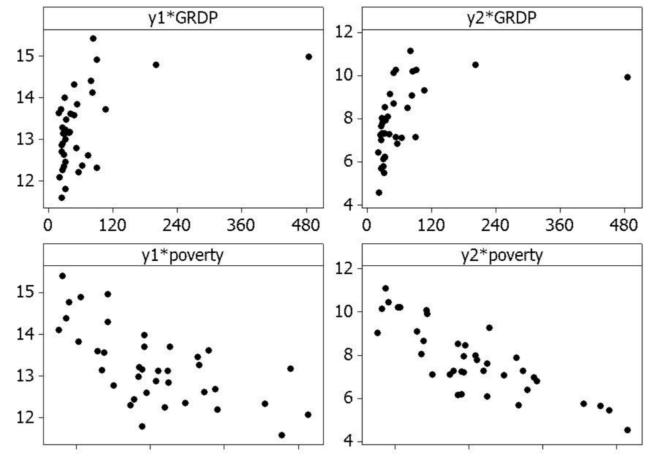

Fig. 2. Partial scatter plot between response variable with two predictor variables: per capita GRDP and percentage of people living in poverty.

The partial scatter plot between the per capita GRDP (t) and both response variables revealed changes at a particular subinterval that fit the truncated spline function. Meanwhile, the partial scatterplot between the response variable and the percentage of persons living in poverty (x) followed a pattern that is repeated with a particular trend and at a specific interval; as a result, the Fourier series function was used to approach this variable.

TABLE II
GCV VALUE AROUND OPTIMUM VALUED

λ	GCV	λ	GCV
0.0001	0.52821	0.01	0.52463
0.0003	0.52812	0.03	0.52103
0.0005	0.52803	0.05	0.51990
0.0007	0.52795	0.07	0.51986*)
0.0009	0.52786	0.09	0.51994
0.001	0.52781	0.3	0.52331
0.003	0.52699	0.5	0.52542
0.005	0.52624	0.7	0.52665
0.007	0.52555	0.9	0.52745
0.009	0.52493	1	0.52775

Additionally, we chose the best model using the smallest GCV as the criterion. The results of the GCV value from several value λ of are summarized in Table II and Fig. 3. What is interesting about the data in this table is that the smallest GCV not always obtained from the smallest λ. As shown in Fig.3, we obtained the minimum GCV from λ=0.07.

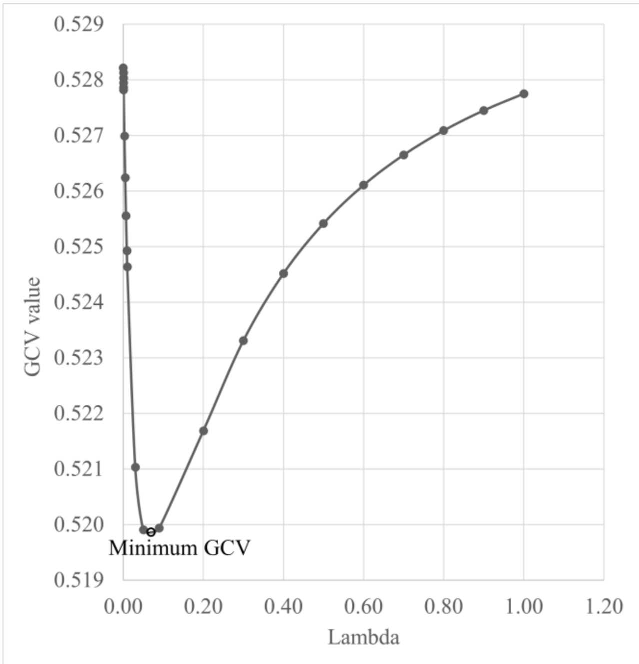

Fig. 3. The GCV value on various λ.

We compared the proposed model which uses a combined estimator (Model 1) to a single estimator or uncombined estimator (Model 2 and Model 3), as illustrated in Table III. When these three models were evaluated, compared to the single or uncombined estimator model, it became clear that the proposed model (Model 1) had the lowest GCV (0.51986). Also, Model 1 generally has a lower GCV value than Models 2 and 3 for all combinations number of oscillation and knots. Consequently, it can be concluded that the proposed model is generally more recommended to be used in modeling the real dataset of 2019 East Java Province EYS and MYS data.

TABLE III
SUMMARY OF GCV AND MSE FOR THE DATA APPLICATION

Model 1	*Bi-response nonparametric regression of CTS-FS*			
	Number of Oscillation	Number of Knots	GCV	MSE
	1	1	0.53752	0.41454
		2	0.52876	0.38401
		3	0.51986*)	0.35706
	2	1	0.54028	0.41685
		2	0.53143	0.38387
		3	0.52135	0.35843
	3	1	0.53913	0.41305
		2	0.53084	0.38311
		3	0.52120	0.35817
Model 2	*Bi-response nonparametric regression of truncated spline*			
	Number of Knots		GCV	MSE
	1		0.59074	0.44551
	2		0.59983	0.39919
	3		0.56248	0.35058
Model 3	*Bi-response nonparametric regression of Fourier series*			
	Number of Oscillation		GCV	MSE
	1		0.66312	0.50009
	2		0.70051	0.46620
	3		0.77373	0.45063

Another result from Table III, the best model obtained from the bi-response nonparametric regression of CTS-FS model with one oscillation and three knot. Based on these models, we can calculate the parameter estimation of East Java's education human development index (HDI) model as the following equation.

$$\begin{aligned}\hat{y}_1 &= 0.004x_i + \frac{1}{2}(25.864) - 0.108\cos x_i + 0.348t_i \\ &\quad -0.779(t_i - 119.939)_+ + 0.7(t_i - 132.511)_+ \\ &\quad -0.329(t_i - 157.654)_+ \\ \hat{y}_2 &= 0.005x_i - \frac{1}{2}(8.413) - 0.206\cos x_i + 3.439t_i \\ &\quad -4.301(t_i - 119.939)_+ + 1.278(t_i - 132.511)_+ \\ &\quad -0.676(t_i - 157.654)_+\end{aligned}$$

Comparison of the actual and prediction from both response variables presented in Fig. 4 and Fig. 5. On the graph, we can notice that, with a few exceptions, all fitted values generally follow the pattern of the true data. Consequently, the suggested model can result in a prediction model.

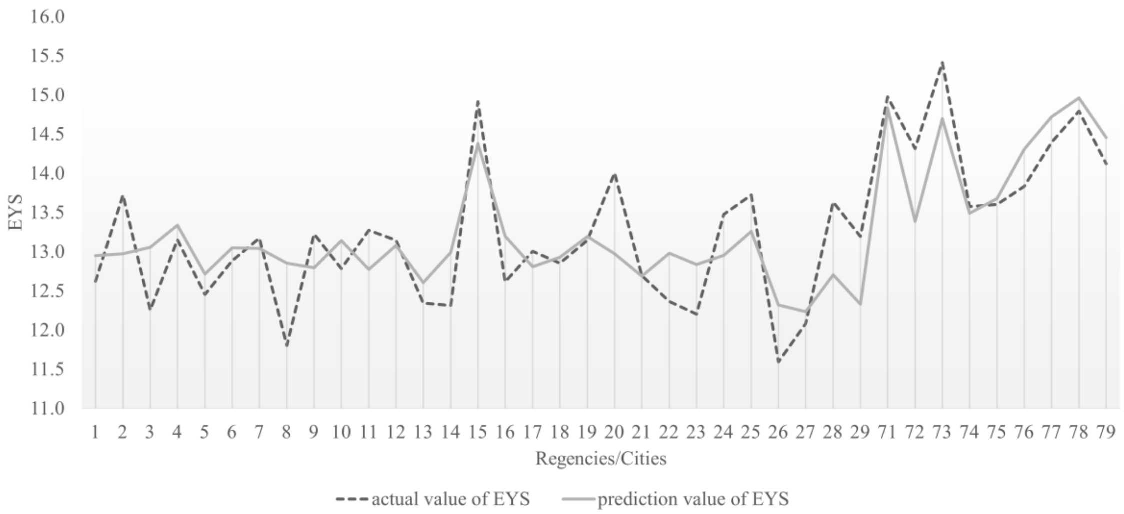

Fig. 4. Comparison between the actual and prediction of expected years of schooling (EYS) variable

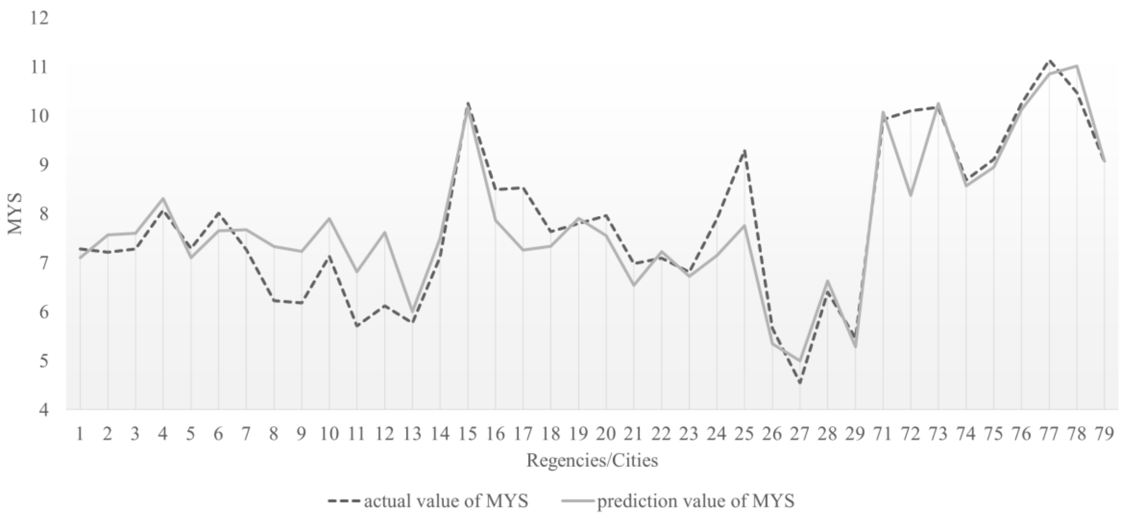

Fig. 5. Comparison between the actual and prediction of mean years of schooling (MYS) variable

VI. Conclusion

Through utilizing PWLS and WLS optimization, this article gives the findings of regression curve estimation using a combined truncated spline and Fourier series estimator in bi-response nonparametric regression as follows

$$\hat{\boldsymbol{\mu}}_{(\mathbf{K},\lambda,\mathbf{V})}(\mathbf{x},\mathbf{t}) = \left[\mathbf{H}\left(\mathbf{I}-\mathbf{J}\mathbf{K}^{-1}\mathbf{L}\right)+\mathbf{J}\mathbf{K}^{-1}\mathbf{L}\right]\mathbf{y}$$

According to the numerical simulations, the findings of this study suggest that the proposed model be employed for data with a large number of samples and a strongly correlated response variable. Additionally, in the application with HDI education indicator data in East Java, the proposed model outperformed compared to the model with a single estimator.

Due to the limited scope of the regression curve estimation, the main weakness of this study was the absence of hypothesis testing and confidence interval of the proposed model. Hence, further work needs to be conducted to obtain the test statistics for hypothesis testing and critical region. Despite its shortcoming, this work contributes to the current literature on nonparametric regression. Firstly, we demonstrate how to estimate the regression curve of combined estimators for bi-response variables through two-stage estimation using PWLS and WLS methods. Additionally, the proposed model is expandable for data with more than two response variables or multi-response model. Secondly, the methods used for this model may be applied to other combined estimators which open future research opportunities. Besides, more numerical simulations with the additional variation of correlation and error variance would reveal more information along with a comprehension conclusion of the proposed model's performance.

Appendix

TABLE A
Comparison GCV of the proposed model using simulation data with the number of knots is 2

n	ρ	Number of oscillations	GCV
20	0	1	1.28684
	0	2	1.28411
	0	3	1.28360
	0.5	1	1.28317
	0.5	2	1.28145
	0.5	3	1.27457
	0.9	1	1.24394
	0.9	2	1.24374
	0.9	3	1.24150*)
50	0	1	1.18191
	0	2	1.18171
	0	3	1.18092
	0.5	1	1.14388
	0.5	2	1.14374
	0.5	3	1.14087
	0.9	1	1.12542
	0.9	2	1.12514
	0.9	3	1.12293*)
100	0	1	1.13866
	0	2	1.13850
	0	3	1.13811
	0.5	1	1.12871
	0.5	2	1.12863
	0.5	3	1.12802
	0.9	1	1.12795
	0.9	2	1.12778
	0.9	3	1.12628*)

TABLE B
Comparison GCV of the proposed model using simulation data with the number of knots is 3

n	ρ	Number of oscillations	GCV
20	0	1	1.36142
	0	2	1.36122
	0	3	1.35880
	0.5	1	1.34759
	0.5	2	1.34526
	0.5	3	1.33274
	0.9	1	1.28127
	0.9	2	1.28080
	0.9	3	1.27912*)
50	0	1	1.17640
	0	2	1.17613
	0	3	1.17543
	0.5	1	1.13806
	0.5	2	1.13801
	0.5	3	1.13656
	0.9	1	1.11365
	0.9	2	1.11337
	0.9	3	1.11096*)
100	0	1	1.13035
	0	2	1.13020
	0	3	1.12984
	0.5	1	1.12013
	0.5	2	1.12009
	0.5	3	1.11964
	0.9	1	1.11941
	0.9	2	1.11939
	0.9	3	1.11762*)

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