Fuzzy Integro Dynamic Equations on Time Scales using Fuzzy Laplace Transform Method

M.N.L. Anuradha, C.H. Vasavi, T. Srinivasa Rao, G. Suresh Kumar

Abstract—**In this paper, we developed the calculus of fuzzy Laplace transforms under Hukuhara delta derivative for the fuzzy valued functions on time scales $\mathbb{T}$. We developed the fundamental properties and related theorems which help to establish the relation between the fuzzy Laplace transforms of a fuzzy valued function on $\mathbb{T}$ and Hukuhara delta derivative to solve first order fuzzy dynamic equations on time scales. These results generalize the results of fuzzy Laplace transforms on fuzzy differential and difference calculus. There are many other time scales than set of Real numbers and integers, hence one can get much more general result. Also, we extended our results to study the fuzzy integro dynamic equation on time scales with kernel of convolution type.**

Index Terms—**Fuzzy Laplace transforms, strongly generalized differentiability, Δ_g-derivative, time scales, fuzzy integro-dynamic equations.**

I. Introduction

Fuzzy differential equations (FDEs) attract the attention of many researchers since they are widely used for the purpose of modeling problems under uncertainty in several important mathematical and physical applications [11]. For applications, we refer [2]-[5], [7]-[11], [12]-[18], [24].

Allahviranloo et al. [1] introduced fuzzy Laplace transform method under strongly generalized differentiability to solve fuzzy linear differential equations, an important application of fuzzy Laplace transform. It solves fuzzy initial value problems, without finding the general solution and non-homogenous problems without first solving the corresponding homogenous equation.

On the other side, the concept of dynamic equations on time scales can bridge the gap between differential and difference equations. The time scale calculus was introduced for the first time in 1988 by Hilger to unify the theory of difference and differential equations [6]. The time scale $\mathbb{T}$ is the arbitrary closed subset of real numbers, combination of real and discrete intervals.

Manuscript received September 06, 2022; revised April 20, 2023.

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Recently, in [19]-[23], the authors developed the calculus and established the results on existence as well as uniqueness of solutions to FDEs on $\mathbb{T}$. In Section 2, we present calculus on fuzzy and $\mathbb{T}$ which are useful for further discussions. In Section 3, properties of derivative and integral were given. In Section 4, fuzzy Laplace transforms on time scales is defined and corresponding properties are investigated.

II. Preliminaries

We present some notations useful for the discussion. For $0 \leq \beta \leq 1$, denote $[t]^\beta = \{\tau \in \mathbb{R}/t(\tau) \geq \beta\}$, the β-level set $[t]^\beta$ $[\underline{t}(\beta), \bar{t}(\beta)]$ is a closed and bounded interval, where $\underline{t}(\beta), \bar{t}(\beta)$ are respectively the left and right endpoints of $[u]^\beta$.

Lemma 2.1: Let $f(\ell)$ defined on $[a, \infty)$ and $[f(\ell)]^\beta = (\underline{f}(\ell,\beta), \overline{f}(\ell,\beta))$. For each $\beta \in [0,1]$, assume $\underline{f}(\ell,\beta), \overline{f}(\ell,\beta)$ are Riemann integrable on [a, b] for every $b \geq a$, there exists $\underline{k}(\beta), \overline{k}(\beta)$ such that $\int_a^b |\underline{f}(\ell,\beta)|dt \leq \underline{k}(\beta)$ and $\int_a^b |\overline{f}(ell,\beta)|dt \leq \overline{k}(\beta)$. Then $f(\ell)$ is improper fuzzy Riemann integrable on $[a,\infty)$. Further,

$$\int_a^\infty f(\ell)dt = \left(\int_a^\infty \underline{f}(\ell,\beta)dt, \int_a^\infty \overline{f}(\ell,\beta)dt\right).$$

Definition 2.1: Let $f(\tau)$ be a function of τ defined for all positive values of τ. Then the Laplace transform of $f(\tau)$ is denoted by $\ell\{f(\tau)\}$ and defined as

$$\ell\{f(\tau)\} = \int_0^\infty e^{-s\tau} f(\tau)d\tau,$$

provided the integral exists and s is a parameter which may be real or complex. To differentiate from fuzzy case, we call this Laplace transform as *classical Laplace transform.*

From this definition, we have

(i) $\ell\{1\}(z) = \dfrac{1}{z}$,

(ii) $\ell\{e^{a\tau}\}(z) = \dfrac{1}{z-a}$,

(iii) $\ell\{f'(\tau)\}(z) = zL\{f(\tau)\}(z) - f(0)$.

Definition 2.2: Let $f(\tau)$ be defined for discrete values $(\tau = 0, 1, 2, \ldots)$ and $f(\tau) = 0$, for $\tau < 0$, then the Z-transform of f is $Z\{f(\tau)\}$ given by

$$Z\{f(\tau)\}(z) = \sum_{\tau=0}^{\infty} \frac{f(\tau)}{z^\tau},$$

for which the infinite sum converges. To differentiate from fuzzy case, we call this Z-transform as *classical Z-transform.*

From this definition, we have

(i) $Z\{1\}(z) = \dfrac{z}{z-1}$

(ii) $Z\{a^\tau\}(z) = \dfrac{z}{z-a}$,

(iii) $Z\{k(\tau+1)\}(z) = z\,[Z\{k(\tau)\}(z) - k_0]$.

Definition 2.3: [1] Suppose that fuzzy valued function $f(\tau)\odot e^{-p\tau}$ is improper fuzzy Riemann integrable on $[0,\infty)$, then $\int_0^\infty f(\tau)\odot e^{-p\tau}$ is called fuzzy Laplace transform and is denoted by

$$\mathcal{L}[f(\tau)] = \int_0^\infty f(\tau)\odot e^{-p\tau}d\tau, (p>0 \text{ and integer}).$$

From Lemma 2.1, we have

$$\int_0^\infty f(\tau)\odot e^{-p\tau}d\tau = \left(\int_0^\infty \underline{f}(\tau,\beta)e^{-p\tau}dt, \int_0^\infty \overline{f}(\tau,\beta)e^{-p\tau}d\tau\right),$$

$$\ell(\underline{f}(\tau,\beta)) = \int_0^\infty \underline{f}(\tau,\beta)e^{-p\tau}d\tau,$$
$$\ell(\overline{f}(\tau,\beta)) = \int_0^\infty \overline{f}(\tau,\beta)e^{-p\tau}d\tau.$$

Then, $\mathcal{L}[f(\tau)] = (\ell(\underline{f}(\tau,\beta), \ell(\overline{f}(\tau,\beta))$.

Definition 2.4: Let $f(t) : T \to \mathbb{E}^1$ be defined for discrete values $(\tau = 0,1,2,\ldots)$, $f(\tau)=0$, for $\tau<0$, then the fuzzy $\mathbf{Z}$-transform of f is $\mathbf{Z}\{f(\tau)\}$ and defined as

$$\mathbf{Z}\{f(\tau)\}(z) = \sum_{\tau=0}^{\infty} f(\tau)\odot\frac{1}{z^\tau},$$

for which the infinite sum converges. From, classical Z-transform:

$$Z(\underline{f}(\tau,\beta)) = \sum_{\tau=0}^{\infty}\frac{\underline{f}(\tau,\beta)}{z^\tau},$$
$$Z(\overline{f}(\tau,\beta) = \sum_{\tau=0}^{\infty}\frac{\overline{f}(\tau,\beta)}{z^\tau}.$$

Then, $\mathbf{Z}[f(\tau)] = (Z(\underline{f}(\tau,\beta), Z(\overline{f}(\tau,\beta))$.

Let $\mathbb{T}_0$ be such that $0\in\mathbb{T}_0$ and $\sup\mathbb{T}_0 = \infty$.

Definition 2.5: Let $x:\mathbb{T}_0\to\mathbb{R}$ be regulated. Then the Laplace transform of x is denoted by $\mathbb{L}\{x(\tau)\}$ and defined as

$$L[x(\tau)](z) = \int_0^\infty x(\tau)e^{\sigma}_{\ominus z}(\tau,0)\Delta\tau,$$

for any $z\in\mathcal{R}$ for which the improper integral exists.

Remark 2.1:
(i) If $\mathbb{T}_0 = [0,\infty)$, then the Laplace transform is the familiar classical Laplace transform given in Definition 2.1.
(ii) If $\mathbb{T}_0 = \mathbb{N}_0$, then the Laplace transform is the classical Z-transform given in Definition 2.2.

Remark 2.2: Let $x:\mathbb{T}_0\to\mathbb{R}$ be such that x^Δ is regulated. Then

$$L[x^\Delta](z) = L[x](z) - x(0),$$

$z\in\mathcal{R}$, such that $\lim_{\tau\to\infty}\{x(\tau)e_{\ominus z}(\tau,0)\} = 0$.

III. Differentiability of fuzzy valued functions on time scales

For Δ_g-derivative, we refer [21]

Definition 3.1: [21] Let $F:\mathbb{T}\to\mathbb{E}^1$ be a fuzzy valued function and let $\tau\in\mathbb{T}^k$. Then F is said to be Δ_g-differentiable at $\tau\in\mathbb{T}^k$, if there exist an element $F^{\Delta_g(\tau)}\in\mathbb{E}^1$ with the property that for every $\epsilon>0$, there is a neighbourhood $U_\mathbb{T}$ of τ for some $\delta>0$ such that

$$\begin{cases} D\left(\dfrac{1}{h-\mu(\tau)}\odot(F(\tau+h)\ominus F(\sigma(\tau)), F^{\Delta_g}(\tau)\right)\le\epsilon,\\ D\left(\dfrac{1}{h-\mu(\tau)}\odot(F(\sigma(\tau))\ominus F(\tau-h), F^{\Delta_g}(\tau)\right)\le\epsilon,\end{cases}$$

or

$$\begin{cases} D\left(\dfrac{-1}{h-\mu(\tau)}\odot(F(\sigma(\tau))\ominus F(\tau+h)), F^{\Delta_g}(\tau)\right)\le\epsilon,\\ D\left(\dfrac{-1}{h+\mu(\tau)}\odot(F(\tau-h)\ominus F(\sigma(\tau)), F^{\Delta_g}(\tau)\right)\le\epsilon,\end{cases}$$

Remark 3.1: [21] Let $\tau\in\mathbb{T}^k$. Then F is $\Delta_{1,g}$-differentiable if F is Δ_g-differentiable as in (**??**) and $\Delta_{2,g}$-differentiable if F is Δ_g-differentiable as in (**??**).
Denote $[F(\tau)]^\beta = F_\beta(\tau)$. If F is Δ_g-differentiable, then F_β is also Δ_g-differentiable and

$$[F^{\Delta_g}(\tau)]^\beta = [F^{\Delta_g}_\beta(\tau)].$$

Theorem 3.1: Let $F:\mathbb{T}\to\mathbb{E}^1$ be a fuzzy valued function and denote $[F(\tau)]^\beta = [\underline{f}^\beta(\tau), \overline{f}^\beta(\tau)]$, for each $\beta\in[0,1]$, where $\underline{f}_\beta, \overline{f}_\beta$ are delta differentiable at τ.

(i) If F is $\Delta_{1,g}$-differentiable at $\tau\in\mathbb{T}^k$, then F_β is $\Delta_{1,g}$-differentiable at t and
$$F^{\Delta_g}_\beta(\tau) = [\underline{f}^\Delta_\beta(\tau), \overline{f}^\Delta_\beta(\tau)].$$
(ii) If F is $\Delta_{2,g}$-differentiable at $\tau\in\mathbb{T}^k$, then F_β is $\Delta_{2,g}$-differentiable at τ and
$$F^{\Delta_g}_\beta(\tau) = [\overline{f}^\Delta_\beta(\tau), \underline{f}^\Delta_\beta(\tau)].$$

IV. Fuzzy Laplace Transform on Time Scales

Now, we study the properties of fuzzy Laplace transforms on time scales.

Definition 4.1: Let $F:\mathbb{T}_0\to\mathbb{E}^1$ be rd-continuous fuzzy valued function on $\mathbb{T}_0$. Suppose that $F(\tau)\odot e^\sigma_{\ominus z}(\tau,0)$ is fuzzy Riemann integrable on $[0,\infty)$, then the fuzzy Laplace transform of F is

$$\mathcal{L}[F(\tau)](z) = \int_0^\infty F(\tau)\odot e^\sigma_{\ominus z}(\tau,0)\Delta\tau,$$

for any $z\in\mathcal{R}$.

$$\int_0^\infty f(\tau)\odot e^{-p\tau}dt = \left(\int_0^\infty \underline{f}(\tau,\beta)e^{-p\tau}dt, \int_0^\infty \overline{f}(\tau,\beta)e^{-p\tau}d\tau\right),$$

$\mathcal{L}[F(\tau,\beta)] = \big(\ell(\underline{f}(\tau,\beta), \ell(\overline{f}(\tau,\beta)\big)$,
where $\ell(\underline{f}(\tau,\beta), \ell(\overline{f}(\tau,\beta)$ denotes the classical Laplace transform of crisp functions $\underline{f}(\tau), \overline{f}(\tau)$.

Remark 4.1:
(i) If $\mathbb{T}_0 = [0,\infty)$, then this definition coincides with Definition 2.3.
(ii) If $\mathbb{T}_0 = \mathbb{N}_0$, then this definition coincides with Definition 2.4.

Lemma 4.1: Let $F:\mathbb{T}_0\to\mathbb{E}^1$ be rd-continuous fuzzy valued function on $\mathbb{T}_0$ given by $F(t) = \tilde{1}\odot u$, where u is a triangular fuzzy number. Then $\mathcal{L}[\tilde{1}](z) = \dfrac{1}{z}\odot u$, for $z\in\mathcal{R}$, such that $\lim_{\tau\to\infty} e_{\ominus z}(\tau,0) = 0$.

Consider

$$\begin{aligned}\mathcal{L}[\tilde{1}\odot u](z) &= \int_0^\infty (\tilde{1}\odot u\odot e^\sigma_{\ominus z}(\tau,0)\Delta\tau \\ &= \left(\frac{-1}{z}\int_0^\infty ((\ominus z)(\tau)\odot e_{\ominus z}(\tau,0)\Delta\tau\right)\odot u \\ &= \left(\frac{-1}{z}\left[e_{\ominus z}(\tau,0)\right]_{\tau=0}^{t\to\infty}\right)\odot u \\ &= \frac{1}{z}\odot u.\end{aligned}$$

for $z\in\mathcal{R}$, $\lim_{t\to\infty} e_{\ominus z}(t,0)=0$.

Theorem 4.1: Let $F:\mathbb{T}_0\to\mathbb{E}^1$ and α,β are constants, then

(i) $\mathcal{L}[\alpha F(\tau)\oplus\beta G(\tau)](z)=(\alpha\mathcal{L}[F(\tau)]\oplus\beta\mathcal{L}[G(\tau)])$.

(ii) For $\lambda\geq 0$, $\mathcal{L}[\lambda F(\tau)]=\lambda\mathcal{L}[F(t)]$.

Proof:

(i) For $z\in\mathcal{R}$, consider

$$\begin{aligned}&\mathcal{L}[\alpha F(\tau)\oplus\beta G(\tau)](z)\\ &=\int_0^\infty(\alpha F(\tau)\oplus\beta G(\tau)](z))\odot e^\sigma_{\ominus z}(\tau,0)\Delta\tau\\ &=\int_0^\infty \alpha F(\tau)(z)\odot e^\sigma_{\ominus z}(\tau,0)\Delta\tau\\ &\qquad\oplus\int_0^\infty\beta G(\tau)(z)\odot e^\sigma_{\ominus z}(\tau,0)\Delta\tau\\ &=\left(\alpha\int_0^\infty F(\tau)(z)\odot e^\sigma_{\ominus z}(\tau,0)\Delta\tau\right)\\ &\qquad\oplus\left(\beta\int_0^\infty G(\tau)(z)\odot e^\sigma_{\ominus z}(\tau,0)\Delta\tau\right)\\ &=\alpha\mathcal{L}[F(\tau)](z)\oplus\beta\mathcal{L}[G(\tau)](z).\end{aligned}$$

Hence $\mathcal{L}[\alpha F(\tau)\oplus\beta G(\tau)](z)=(\alpha\mathcal{L}[F(\tau)]\oplus\beta\mathcal{L}[G(\tau)])$.

(ii) For $\lambda\geq 0$, consider

$$\begin{aligned}\mathcal{L}[\lambda F(\tau)] &= \int_0^\infty(\lambda F(\tau)\odot e^\sigma_{\ominus z}(\tau,0))\Delta\tau\\ &=\lambda\int_0^\infty F(\tau)\odot e^\sigma_{\ominus z}(\tau,0)\Delta\tau\\ &=\lambda\mathcal{L}[F(\tau)].\end{aligned}$$

Hence, $\mathcal{L}[\lambda F(\tau)]=\lambda\mathcal{L}[F(\tau)]$.

■

Theorem 4.2: Let $F:\mathbb{T}_0\to\mathbb{E}^1$ and $z\in\mathcal{R}$.

(i) If F is $\Delta_{1,g}$-differentiable, then

$$\mathcal{L}[F^{\Delta_{1,g}}(\tau)](z)=z\odot\mathcal{L}[F(\tau)](z)\ominus F(0). \tag{1}$$

or

(ii) If F is $\Delta_{2,g}$-differentiable, then

$$\mathcal{L}[F^{\Delta_{2,g}}(\tau)](z)=(-F(0))z\ominus(-z\odot\mathcal{L}[F(\tau)](z)). \tag{2}$$

Proof:

(i) Let F be $\Delta_{1,g}$-differentiable. For $\beta\in[0,1]$, since $\underline{f},\overline{f}$ are delta differentiable, we have

$$\ell[\overline{f}^\Delta(\tau,\beta)]=z\ell[\overline{f}(\tau,\beta)]-\overline{f}(0,\beta)$$

and

$$\ell[\underline{f}^\Delta(\tau,\beta)]=z\ell[\underline{f}(\tau,\beta)]-\underline{f}(0,\beta).$$

consider

$$\begin{aligned}&z\odot\mathcal{L}[F(\tau)](z)\ominus F(0)\\ &=(z\ell[\underline{f}(\tau,\beta)]-\underline{f}(0,\beta),z\ell[\overline{f}(\tau,\beta)]-\overline{f}(0,\beta)\\ &=\left(\ell[\underline{f}^\Delta(\tau,\beta)],\ell[\overline{f}^\Delta(\tau,\beta)]\right),\ \text{by the linearlity of }\mathbb{L}\\ &=\mathcal{L}[F^{\Delta_{1,g}}(\tau)](z).\end{aligned}$$

(ii) Assume F is $\Delta_{2,g}$-differentiable. For $\beta\in[0,1]$, we have

$$\begin{aligned}&(-F(0))(z)\ominus(-z\odot\mathcal{L}[F(\tau)](z))\\ &=(-\overline{f}(0,\beta)(z)+z\ell[\overline{f}(\tau,\beta)],-\underline{f}(0,\beta)(z)+z\ell[\underline{f}(\tau,\beta)])\\ &=\left(\ell[(\overline{f}^\Delta(\tau,\beta)],\ell[\underline{f}^\Delta(\tau,\beta))]\right)\\ &=\mathcal{L}[F^{\Delta_{2,g}}(\tau)](z).\end{aligned}$$

■

Theorem 4.3: Let $F:\mathbb{T}_0\to\mathbb{E}^1$ be rd-continuous fuzzy valued function on $\mathbb{T}_0$, where $F(t)=e_p(\tau,0)$, $p\in\mathcal{R}$ and $\mathcal{L}[F(\tau)]=F(z)$, then $\mathcal{L}[e_p(\tau,0)\odot F(\tau)]=F(z-p)$.

Proof: For $p\in\mathcal{R}$, consider

$$\begin{aligned}&\ell[e_p(\tau,0)\odot F(\tau)](z)\\ &=\int_0^\infty e_p(\tau,0)\odot e^\sigma_{\ominus z}(\tau,0)\odot F(\tau)\Delta\tau\\ &=\int_0^\infty\frac{1}{1+\mu(\tau)z}e_p(\tau,0)\odot e_{\ominus z}(\tau,0)\odot F(\tau)\Delta\tau\\ &=\int_0^\infty\frac{1}{1+\mu(\tau)z}e_{p\ominus z}(\tau,0)\odot F(\tau)\Delta\tau\\ &=\frac{1}{p-z}\int_0^\infty\frac{p-z}{1+\mu(\tau)z}e_{p\ominus z}(\tau,0)\odot F(\tau)\Delta\tau\\ &=\frac{-1}{z-p}\int_0^\infty(p\ominus z)(t)e_{p\ominus z}(\tau,0)\odot F(\tau)\Delta\tau\\ &=\int_0^\infty e^\sigma_{p\ominus z}(\tau,0)\odot F(\tau)\Delta\tau\\ &=\left(\int_0^\infty e^\sigma_{p\ominus z}(\tau,0)\underline{f}(\tau,\beta)\Delta\tau,\int_0^\infty e^\sigma_{p\ominus z}(\tau,0)\overline{f}(\tau,\beta)\Delta\tau\right)\\ &=\mathcal{L}[e_p(\tau,0)\odot F(\tau)]=F(z-p).\end{aligned}$$

■

Now, we will show the importance of the present work in the following examples.

Example 4.1: Consider the following fuzzy dynamic equation on time scale.

$$\begin{cases}y^\Delta(\tau)=y(\tau)\\ y(0,\beta)=(\underline{y}(0,\beta),\overline{y}(0,\beta))=(-1,0,1)\in\mathbb{E}^1,\end{cases} \tag{3}$$

Now, we will study (3), by considering different time scales $[0,\infty),\mathbb{N}_0,2^{\mathbb{N}_0}$.

(i) If $\mathbb{T}_0=[0,\infty)$, (3) becomes

$$\begin{cases}y'(\tau)=y(\tau)\\ y(0,\beta)=(\underline{y}(0,\beta),\overline{y}(0,\beta))=(\beta-1,1-\beta)\in\mathbb{E}^1,\end{cases} \tag{4}$$

From [1], if y is (i)-differentiable, then $\mathcal{L}[y'(\tau)](z)=z\odot\mathcal{L}[y(\tau)](z)\ominus y(0)$, hence, (4) can be equivalently written as

$$\begin{cases}\underline{y}'(\tau,\beta)=\underline{y}(\tau,\beta), & \underline{y}(0,\beta)=\beta-1,\\ \overline{y}'(\tau,\beta)=\overline{y}(\tau,\beta), & \overline{y}(0,\beta)=1-\beta,\end{cases}$$

$$\mathcal{L}[y'(\tau)]=\mathcal{L}[y(\tau)]$$

then

$$\begin{cases} \ell[\underline{y}(\tau,\beta)] = z\ell[\underline{y}(\tau,\beta)] - \ell[\underline{y}(0,\beta)], \\ \ell[\overline{y}(\tau,\beta)] = z\ell[\overline{y}(\tau,\beta)] - \ell[\overline{y}(0,\beta)], \end{cases}$$

Hence the solution of (4) is

$$\begin{cases} \ell[\underline{y}(\tau,\beta)] = \dfrac{1}{z-1}\ell[\underline{y}(0,\beta)] \\ \ell[\overline{y}(\tau,\beta)] = \dfrac{1}{z-1}\ell[\overline{y}(0,\beta)]. \end{cases}$$

Therefore,

$$\underline{y}(\tau,\beta) = \underline{y}(0,\beta)\ell^{-1}\left[\frac{1}{z-1}\right] = (\beta-1)e^{\tau}.$$

Similarly, $\overline{y}(\tau,\beta) = \overline{y}(0,\beta)L^{-1}\left[\dfrac{1}{z-1}\right] = (1-\beta)e^{\tau}$. Hence $y(\tau) = [(\beta-1)e^{\tau}, (1-\beta)e^{\tau}]$, for all $\tau \in [0,\infty)$.

(ii) If $\mathbb{T}_0 = \mathbb{N}_0$, (3) becomes

$$\begin{cases} \Delta y(\tau) = y(\tau) \\ y(0,\beta) = (\underline{y}(0,\beta), \overline{y}(0,\beta)) = (\beta-1, 1-\beta) \in \mathbb{E}^1, \end{cases} \tag{5}$$

(5) can be equivalently written as

$$\begin{cases} \underline{y}(\tau+1) = 2\underline{y}(\tau), & \underline{y}(0,\beta) = \beta-1, \\ \overline{y}(\tau+1) = 2\overline{y}(\tau), & \overline{y}(0,\beta) = 1-\beta. \end{cases}$$

From (5),

$$Z[y(\tau+1,\beta)] = 2Z[y(\tau,\beta)]$$

then from Remark 2.1,

$$\begin{cases} Z[\underline{y}(\tau+1,\beta)] = zZ[\underline{y}(\tau,\beta)(z) - \underline{y}(0,\beta)], \\ Z[\overline{y}(\tau+1,\beta)] = zZ[\overline{y}(\tau,\beta) - \overline{y}(0,\beta)], \end{cases}$$

Hence

$$\begin{cases} zZ[\underline{y}(\tau,\beta)(z) - \underline{y}(0,\beta)] = 2Z[\underline{y}(\tau,\beta)] \\ zZ[\overline{y}(\tau,\beta) - \overline{y}(0,\beta)] = 2Z[\overline{y}(\tau,\beta)]. \end{cases}$$

Therefore,

$$\begin{aligned} \underline{y}(\tau,\beta)](z) &= Z^{-1}\left[\frac{z}{z-2}\right]\underline{y}(0,\beta) \\ &= (\beta-1)2^{\tau}. \end{aligned}$$

Similarly, $\overline{y}(\tau,\beta)](z) = (1-\beta)2^{\tau}$.

(iii) If $\mathbb{T}_0 = 2^{\mathbb{N}_0}$, then (3) becomes

$$\begin{cases} y(2\tau) - y(\tau) - \tau y(\tau) = 0, \\ y(0,\beta) = (\beta-1, 1-\beta) \in \mathbb{E}^1, \\ \text{where } \ \Delta y(\tau) = \dfrac{y(2\tau) - y(\tau)}{\tau}. \end{cases}$$

Hence,

$$\underline{y}(\tau,\beta)](z) = (\beta-1)\Pi_{s\in\mathbb{T}\cap(0,\tau)}(1-2s), \quad \text{for all } \ \tau \in 2^{\mathbb{N}_0}.$$

Similarly,

$$\overline{y}(\tau,\beta)](z) = (1-\beta)\Pi_{s\in\mathbb{T}\cap(0,\tau)}(1-2s), \quad \text{for all } \ \tau \in 2^{\mathbb{N}_0}.$$

Now, we generalize the above problem (3) by considering the dynamic model

$$\begin{cases} y^{\Delta}(\tau) = y(\tau) \\ y(0,\beta) = (\beta-1, 1-\beta), \end{cases}$$

where Δ represents Δ_g-derivative. If y is $\Delta_{1,g}$-differentiable, then (3) can be equivalently written as

$$\begin{cases} \underline{y}^{\Delta}(\tau,\beta) = \underline{y}(\tau,\beta), & \underline{y}(0,\beta) = \beta-1, \\ \overline{y}^{\Delta}(\tau,\beta) = \overline{y}(\tau,\beta), & \overline{y}(0,\beta) = 1-\beta, \end{cases}$$

$$\mathcal{L}[y^{\Delta}(t)] = \mathcal{L}[y(t)]$$

and from (1), $\mathcal{L}[y^{\Delta_{1,g}}(\tau)](z) = z \odot \mathcal{L}[y(\tau)](z) \ominus y(0)$, if y is $\Delta_{1,g}$-differentiable,

$$\begin{cases} \ell[\underline{y}^{\Delta}(\tau,\beta)] = z\ell[\underline{y}(\tau,\beta)] - \ell[\underline{y}(0,\beta)], \\ \ell[\overline{y}^{\Delta}(\tau,\beta)] = z\ell[\overline{y}(\tau,\beta)] - \ell[\overline{y}(0,\beta)], \end{cases}$$

Hence the solution is

$$\begin{cases} \ell[\underline{y}(\tau,\beta)] = \dfrac{1}{z-1}\ell[\underline{y}(0,\beta)] \\ \ell[\overline{y}(\tau,\beta)] = \dfrac{1}{z-1}\ell[\overline{y}(0,\beta)]. \end{cases}$$

Therefore,

$$\underline{y}(\tau,\beta) = \underline{y}(0,\beta)\ell^{-1}\left[\frac{1}{z-1}\right] = (\beta-1)e_1(\tau,0).$$

Similarly, $\overline{y}(\tau,\beta) = \overline{y}(0,\beta)\ell^{-1}\left[\dfrac{1}{z-1}\right] = (1-\beta)e_1(\tau,0)$. Hence $y(\tau) = [(\beta-1)e_1(\tau,0), (1-\beta)e_1(\tau,0)]$, for all $\tau \in \mathbb{T}_0$, where

$$e_1(\tau,0) = \begin{cases} e^{\tau}, & \text{if } \ \mathbb{T}_0 = \mathbb{R} \\ 2^{\tau}, & \text{if } \ \mathbb{T}_0 = \mathbb{Z} \\ \Pi_{s\in\mathbb{T}\cap(0,\tau)}(1+s), & \text{if } \ \mathbb{T}_0 = 2^{\mathbb{N}_0}. \end{cases}$$

Example 4.2: Consider fuzzy dynamic equation on time scale

$$\begin{cases} y^{\Delta}(\tau) = -y(\tau) \\ y(0,\beta) = (\underline{y}(0,\beta), \overline{y}(0,\beta)) = (-1,0,1) \in \mathbb{E}^1, \end{cases} \tag{6}$$

where Δ represents $\Delta_{2,g}$-derivative. If y is $\Delta_{2,g}$-differentiable, then (3) can be equivalently written as

$$\begin{cases} \overline{y}^{\Delta}(\tau,\beta) = -\underline{y}(\tau,\beta), & \underline{y}(0,\beta) = \beta-1, \\ \underline{y}^{\Delta}(\tau,\beta) = \overline{y}(\tau,\beta), & \overline{y}(0,\beta) = 1-\beta, \end{cases}$$

from (2), $L[F^{\Delta_{2,g}}(\tau)](z) = (-F(0))z \ominus (-z \odot L[F(\tau)](z))$. By using fuzzy Laplace transform, (4) can be written as

$$\mathcal{L}[y^{\Delta_{2,g}}(\tau)] = \mathcal{L}[-y(\tau)].$$

Since, $-\mathcal{L}[y(\tau)] = (-y(0))(z) \ominus (-z \odot \mathcal{L}[y(\tau)](z))$,

$$\begin{cases} -\ell[\underline{y}(\tau,\beta)] = (-\underline{y}(0,\beta))(z) \ominus (-z \odot \ell[\underline{y}(\tau,\beta)](z)), \\ -\ell[\overline{y}(\tau,\beta)] = (-\overline{y}(0,\beta))(z) \ominus (-z \odot \ell[\overline{y}(\tau,\beta)](z)). \end{cases}$$

$$\begin{cases} \ell[\underline{y}(\tau,\beta)] = \underline{y}(0,\beta))(z)\dfrac{1}{1+z}, \\ \ell[\overline{y}(\tau,\beta)] = \overline{y}(0,\beta))(z)\dfrac{1}{1+z}. \end{cases}$$

Hence,

$$\begin{cases} \underline{y}(\tau,\beta) = \underline{y}(0,\beta))(z)\ell^{-1}\left[\dfrac{1}{1+z}\right], \\ \overline{y}(\tau,\beta) = \overline{y}(0,\beta))(z)\ell^{-1}\left[\dfrac{1}{1+z}\right]. \end{cases}$$

Therefore,

$$\underline{y}(\tau,\beta) = \underline{y}(0,\beta)\ell^{-1}\left[\frac{1}{z-1}\right] = (\beta-1)e_{-1}(\tau,0).$$

Similarly, $\overline{y}(\tau,\beta)=\overline{y}(0,\beta)\ell^{-1}\left[\dfrac{1}{z-1}\right]=(1-\beta)e_{-1}(\tau,0)$.

Remark 4.2: If $\mathbb{T}=\mathbb{R}$, then $e_1 t, 0=e^{-t}$, if y is $\Delta_{2,g}$-differentiable, then the diam(supp $y(t)=2e^{-t}\to 0$) as $t\to\infty$. The solutions of the FDEs on time scales (3), (4) are dependent on selection of the derivative. Although the unicity of the solution is lost, we can prefer the solution which yields better results in many real world problems.

Example 4.3: Consider fuzzy dynamic equation on time scale

$$\begin{cases} y^{\Delta}(\tau)=-y(\tau)+\tau+1 \\ y(0,\beta)=(\underline{y}(0,\beta),\overline{y}(0,\beta))=(-1,0,1)\in\mathbb{E}^1, \end{cases} \tag{7}$$

where Δ represents $\Delta_{2,g}$-derivative. By using fuzzy Laplace transform, (5) can be written as

$$\mathcal{L}[y^{\Delta_{2,g}}(\tau)]=\mathcal{L}[-y(\tau)]+\mathcal{L}[\tau]+\mathcal{L}[1].$$

Since, $\mathcal{L}[y^{\Delta_{2,g}}(\tau)]=(-y(0))(z)\ominus(-z\odot\mathcal{L}[y(\tau)](z))$,
$\mathcal{L}[-y(\tau)]+\mathcal{L}[\tau]+\mathcal{L}[1]=(-y(0))(z)\ominus(-z\odot\mathcal{L}[y(\tau)](z))$.
Hence,

$$\begin{aligned} &-\ell[\underline{y}(\tau,\beta)]+\ell[\tau]+\ell[1] \\ &=(-\underline{y}(0,\beta))(z)\ominus(-z\odot\underline{y}(\tau,\beta)(z)), \end{aligned}$$

$$\begin{aligned} &-\ell[\overline{y}(\tau,\beta)]+\ell[\tau]+\ell[1] \\ &=(-\overline{y}(0,\beta))(z)\ominus(-z\odot\overline{y}(\tau,\beta)(z)) \end{aligned}$$

Thus,

$$\begin{cases} \ell[\underline{y}(\tau,\beta)]=\underline{y}(0,\beta)\dfrac{1}{1+z}+\dfrac{1}{1+z}+\dfrac{1}{z^2(1+z)} \\ \ell[\overline{y}(\tau,\beta)]=\overline{y}(0,\beta)\dfrac{1}{1+z}+\dfrac{1}{1+z}+\dfrac{1}{z^2(1+z)}. \end{cases}$$

Since,

$$\begin{cases} \ell[\dfrac{1}{1+z}]=e_{-1}(\tau,0), \\ \ell[\dfrac{1}{z(1+z)}]=1-e_{-1}(\tau,0)], \\ \ell[\dfrac{1}{z^2(1+z)}]=e_{-1}(\tau,0)-1+\tau. \end{cases}$$

Therefore

$$\begin{cases} \underline{y}(\tau,\beta)=(\beta-1)e_{-1}(\tau,0)+\tau, \\ \overline{y}(\tau,\beta)=(1-\beta)e_{-1}(\tau,0)+\tau. \end{cases}$$

If $\mathbb{T}=\mathbb{R}$, $e_{-1}(\tau,0)=e^(-\tau)$, if y is $\Delta_{2,g}$-differentiable, then $diam(\sup(y(t)))=2e^(-\tau)\to 0$ as $\tau\to\infty$.

V. Fuzzy Laplace Transform for Solving Fuzzy Integro Dynamic Equation on Time Scales

Now, we will solve fuzzy integro dynamic equation on time scales (FIDET) under Δ_g-derivative using fuzzy Laplace transform method.

$$y^{\Delta}(\tau)=F(\tau,y(\tau))+\int_0^{\tau}G(\tau,s,y(s))\Delta s,\ \tau_0,s\in[\tau_0,\tau_0+a]_{\mathbb{T}}, \tag{1}$$

$$y(0)=y_0=(\underline{y}_0,\overline{y}_0)\in\mathbb{E}^1,$$

where $F:[\tau_0,\tau_0+a]_{\mathbb{T}}\times\mathbb{E}^1\to\mathbb{E}^1$ and $G:[\tau_0,\tau_0+a]^2_{\mathbb{T}}\times\mathbb{E}^1\to\mathbb{E}^1$ are rd-continuous fuzzy valued functions, y^{Δ} is the Δ_{gH}-derivative of y, $\tau\in\mathbb{T}$, $y_0\in\mathbb{E}^1$.

In order to solve FIDET (1), we replace it by equivalent crisp system of IDETs

(i) If $y(\tau)$ is $\Delta_{1,gH}$-differentiable, then $[y^{\Delta_{gH}}(\tau)]_\beta=[\underline{y}^{\Delta}_{\beta}(\tau),\overline{y}^{\Delta}_{\beta}(\tau)]$ and FIDET (1) is translated into

$$\begin{aligned} &\underline{y}^{\Delta_{gH}}(\tau) \\ &=\underline{F}_\beta(\tau,\underline{y}(\tau),\overline{y}(\tau))+\int_0^{\tau}\underline{G}_\beta(\tau,s,\underline{y}(s),\overline{y}(s))\Delta s, \\ &\overline{y}^{\Delta_{gH}}(\tau) \\ &=\overline{F}_\beta(\tau,\underline{y}(\tau),\overline{y}(\tau))+\int_0^{\tau}\overline{G}_\beta(\tau,s,\underline{y}(s),\overline{y}(s))\Delta s, \end{aligned} \tag{2}$$

subject to $\underline{y}_\beta(0)=\underline{y_0}_\beta,\overline{y}_\beta(\tau_0)=\overline{y_0}_\beta$.

(ii) If $y(\tau)$ is $\Delta_{2,gH}$-differentiable, then $[y^{\Delta_{gH}}(\tau)]_\beta=[\overline{y}^{\Delta}_{\beta}(\tau),\underline{y}^{\Delta}_{\beta}(\tau)]$ and FIDET (1) is translated into

$$\begin{aligned} &\underline{y}^{\Delta_{gH}}(\tau) \\ &=\overline{F}_\beta(\tau,\underline{y}(\tau),\overline{y}(\tau))+\int_0^{\tau}\overline{G}_\beta(\tau,s,\underline{y}(s),\overline{y}(s))\Delta s, \\ &\overline{y}^{\Delta_{gH}}(\tau) \\ &=\underline{F}_\beta(\tau,\underline{y}(\tau),\overline{y}(\tau))+\int_0^{\tau}\underline{G}_\beta(\tau,s,\underline{y}(s),\overline{y}(s))\Delta s, \end{aligned} \tag{3}$$

subject to $\underline{y}_\beta(0)=\underline{y_0}_\beta,\overline{y}_\beta(\tau_0)=\overline{y_0}_\beta$.

Obviously, $[\underline{y}_\beta(\tau),\overline{y}_\beta(\tau)]$ and its Δ_{gH}-derivative $[\underline{y}^{\Delta_{gH}}(\tau),\overline{y}^{\Delta_{gH}}(\tau)]$ are valid level sets for each $\beta\in[0,1]$.

Assume G is of the form $G(\tau,s,y(s))=k(\tau,s)G(y(s))$, where the kernel function $k(\tau,s)$ is non-negative for $0\le s\le\tau$. Therefore, IDETs (7), (10) can be expressed in the equivalent form

$$\begin{aligned} &\underline{y}^{\Delta_{gH}}(\tau) \\ &=\underline{F}_\beta(\tau,\underline{y}(\tau),\overline{y}(\tau))+\int_0^{\tau}k(\tau,s)\underline{G}_\beta(s,\underline{y}_\beta(s),\overline{y}_\beta(s))\Delta s, \\ &\overline{y}^{\Delta_{gH}}(\tau) \\ &=\overline{F}_\beta(\tau,\underline{y}(\tau),\overline{y}(\tau))+\int_0^{\tau}k(\tau,s)\overline{G}_\beta(s,\underline{y}(s),\overline{y}(s))\Delta s. \end{aligned} \tag{4}$$

$$\begin{aligned} &\underline{y}^{\Delta_{gH}}(\tau) \\ &=\overline{F}_\beta(\tau,\underline{y}(\tau),\overline{y}(\tau))+\int_0^{\tau}k(\tau,s)\overline{G}_\beta(s,\underline{y}(s),\overline{y}(s))\Delta s, \\ &\overline{y}^{\Delta_{gH}}(\tau) \\ &=\underline{F}_\beta(\tau,\underline{y}(\tau),\overline{y}(\tau))+\int_0^{\tau}k(\tau,s)\underline{G}_\beta(s,\underline{y}(s),\overline{y}(s))\Delta s, \end{aligned} \tag{5}$$

Lemma 5.1: Let $y:[\tau_0,\tau_0+a]_{\mathbb{T}}\to\mathbb{E}^1$ be a triangular fuzzy valued function and let $y(\tau)=[\underline{y}(\tau),y_c(\tau),\overline{y}(\tau)]$.

(i) If $y(\tau)$ is $\Delta_{1,gH}$-differentiable, then $[y^{\Delta_{gH}}(\tau)]_\beta=[\underline{y}^{\Delta}_{\beta}(\tau),y_{c\beta}^{\Delta},\overline{y}^{\Delta}_{\beta}(\tau)]$.

(ii) If $y(\tau)$ is $\Delta_{2,gH}$-differentiable, then $[y^{\Delta_{gH}}(\tau)]_\beta=[\overline{y}^{\Delta}_{\beta}(\tau),y_{c\beta}^{\Delta},\underline{y}^{\Delta}_{\beta}(\tau)]$

Lemma 5.2: Let $y:[\tau_0,\tau_0+a]_{\mathbb{T}}\to\mathbb{E}^1$ be a triangular fuzzy valued function and let $y(\tau)=[\underline{y}(\tau),y_c(\tau),\overline{y}(\tau)]$. Then

$\int_{\tau_0}^{\tau_0+a} y(\tau)\Delta\tau \in \mathbb{E}^1$ and $\underline{y}_\beta(\tau), y_{c\beta}, \overline{y}_\beta(\tau)$ are Δ-integrable functions on $[\tau_0, \tau_0 + a]_{\mathbb{T}}$ and

$$\int_{\tau_0}^{\tau_0+a} y(\tau)\Delta\tau = \left(\int_{\tau_0}^{\tau_0+a} \underline{y}_\beta(\tau)\Delta\tau, \int_{\tau_0}^{\tau_0+a} y_{c\beta}(\tau)\Delta\tau, \int_{\tau_0}^{\tau_0+a} \overline{y}(\tau)\Delta\tau\right)$$

Consider (1) subject to the fuzzy initial condition $y(\tau_0) = y_0$, where $y(\tau) = (\overline{y}(\tau), y_c(\tau), \underline{y}(\tau))$ and $y(\tau_0) = (\underline{y}_0, y_{c0}, \overline{y}_0)$, $F : [\tau_0, \tau_0 + a]_{\mathbb{T}} \times \mathbb{E}^1 \to \mathbb{E}^1$ and $G : [\tau_0, \tau_0 + a]^2_{\mathbb{T}} \times \mathbb{E}^1 \to \mathbb{E}^1$ are rd-continuous triangular fuzzy valued functions such that

$$\begin{aligned} F(\tau, y(\tau)) = \big(&\underline{F}(\tau, \underline{y}(\tau), y_c(\tau), \overline{y}(\tau)),\\ &F_c(\tau, \underline{y}(\tau), y_c(\tau), \overline{y}(\tau)),\\ &\overline{F}(\tau, \underline{y}(\tau), y_c(\tau), \overline{y}(\tau))\big)\,, \end{aligned}$$

$$\begin{aligned} G(\tau, s, y(s)) = \big(&\underline{G}(\tau, s, \underline{y}(s), y_c(s), \overline{y}(s)),\\ &G_c(\tau, s, \underline{y}(s), y_c(s), \overline{y}(s)),\\ &\overline{G}(\tau, s, \underline{y}(s), y_c(s), \overline{y}(s))\big)\,, \end{aligned}$$

then we can express (1) into a system of crisp FIDETs as:

$$\begin{aligned} &\underline{y}^{\Delta_{gH}}(\tau)\\ &= \underline{F}(\tau, \underline{y}(\tau), y_c(\tau), \overline{y}(\tau)) + \int_0^\tau \underline{G}(\tau, s, \underline{y}(s), y_c(s)\overline{y}(s))\Delta s,\\ &{y_c}^{\Delta_{gH}}(\tau)\\ &= F_c(\tau, \underline{y}(\tau), y_c(\tau), \overline{y}(\tau)) + \int_0^\tau G_c(\tau, s, \underline{y}(s), y_c(s)\overline{y}(s))\Delta s,\\ &\overline{y}^{\Delta_{gH}}(\tau)\\ &= \overline{F}(\tau, \underline{y}(\tau), y_c(\tau), \overline{y}(\tau)) + \int_0^\tau \overline{G}(\tau, s, \underline{y}(s), y_c(\tau), \overline{y}(s))\Delta s, \end{aligned} \tag{6}$$

subject to the crisp initial conditions $\underline{y}(\tau_0) = \underline{y}_0, y_c(\tau_0) = y_{c0}, \overline{y}(\tau_0) = \overline{y}_0$.

Similarly, if $y(\tau)$ is $\Delta_{2,gH}$-differentiable, then (1) can be expressed into a system of crisp FIDETs as:

$$\begin{aligned} &\underline{y}^{\Delta_{gH}}(\tau)\\ &= \overline{F}(\tau, \underline{y}(\tau), y_c(\tau), \overline{y}(\tau)) + \int_0^\tau \overline{G}(\tau, s, \underline{y}(s), y_c(s)\overline{y}(s))\Delta s,\\ &{y_c}^{\Delta_{gH}}(\tau)\\ &= F_c(\tau, \underline{y}(\tau), y_c(\tau), \overline{y}(\tau)) + \int_0^\tau G_c(\tau, s, \underline{y}(s), y_c(s)\overline{y}(s))\Delta s,\\ &\underline{y}^{\Delta_{gH}}(\tau)\\ &= \overline{F}(\tau, \underline{y}(\tau), y_c(\tau), \overline{y}(\tau)) + \int_0^\tau \underline{G}(\tau, s, \underline{y}(s), y_c(\tau), \overline{y}(s))\Delta s, \end{aligned}$$

Theorem 5.1: Consider FIDETs (1) where $F : [\tau_0, \tau_0 + a]_{\mathbb{T}} \times \mathbb{E}^1 \to \mathbb{E}^1$ and $G : [\tau_0, \tau_0 + a]^2_{\mathbb{T}} \times \mathbb{E}^1 \to \mathbb{E}^1$ are rd-continuous triangular fuzzy valued functions such that

(i)

$$\begin{aligned} F(\tau, y(\tau)) = \big(&\underline{F}(\tau, \underline{y}(\tau), y_c(\tau), \overline{y}(\tau)),\\ &F_c(\tau, \underline{y}(\tau), y_c(\tau), \overline{y}(\tau)),\\ &\overline{F}(\tau, \underline{y}(\tau), y_c(\tau), \overline{y}(\tau))\big)\,, \end{aligned}$$

and

$$\begin{aligned} G(\tau, s, y(s)) = \big(&\underline{G}(\tau, s, \underline{y}(s), y_c(s), \overline{y}(s)),\\ &G_c(\tau, s, \underline{y}(s), y_c(s), \overline{y}(s)),\\ &\overline{G}(\tau, s, \underline{y}(s), y_c(s), \overline{y}(s))\big)\,, \end{aligned}$$

(ii) $\underline{F}, F_c, \overline{F}$ and $\underline{G}, G_c, \overline{G}$ are rd-continuous real valued functions.

(iii) there exists constants M_1, M_2 such that

$$\begin{aligned} &\|F(\tau, \underline{y}(\tau), y_c(\tau), \overline{y}(\tau)) - F(\tau, \underline{z}(\tau), z_c(\tau), \overline{z}(\tau))\|\\ &\le M_1 \max\{|\underline{y}(\tau) - \underline{z}(\tau)|, |y_c(\tau) - y_c(\tau), |\overline{y}(\tau) - \overline{z}(\tau)|\} \end{aligned}$$

$$\begin{aligned} &\|G(\tau, s, \underline{y}(s), y_c(s), \overline{y}(s)) - G(\tau, s, \underline{z}(s), z_c(s), \overline{z}(s))\|\\ &\le M_2 \max\{|\underline{y}(\tau) - \underline{z}(\tau)|, |y_c(\tau) - y_c(\tau), |\overline{y}(\tau) - \overline{z}(\tau)|\}, \end{aligned}$$

for each $\tau, s \in [\tau_0, \tau_0 + a]$ and $y(\tau), z(\tau) \in \mathbb{E}^1$. Then the solutions of FIDETs (1) are triangular fuzzy functions $y(\tau) = (\underline{y}(\tau), y_c(\tau), \overline{y}(\tau))$. On the other hand, FIDETs (1) is equivalent to the system of crisp IDETs (7) in the sense of $\Delta_{1,gH}, \Delta_{2,gH}$-differentiability. The FIDETs (1) and the system of crisp IDETs (7) are equivalent.

Proof: The proof is similar to the proof of Theorem 4.1 in [4]. ■

Definition 5.1: Let $k : [0, \infty) \to \mathbb{R}$ be a crisp continuous function and $F : [\tau_0, \tau_0 + a]_{\mathbb{T}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy valued rd-continuous mapping, then the convolution of two functions is defined by

$$(k * F)(\tau) = \int_0^\tau k(\tau - t)F(\tau)\Delta\tau, \ \ \text{for} \tau \ge 0,$$

Suppose that $e^{-p\tau}F(\tau), e^{-p\tau}k(\tau)$ are integrable on $[0, \infty)$. As the kernel function $k(\tau, s)$ is non-negative for $0 \le s \le \tau$, then

$$(k * F)(\tau) = (\int_0^\tau k(\tau - t)\underline{f}(\tau)\Delta\tau, \int_0^\tau k(\tau - t)\overline{f}(\tau)\Delta\tau).$$

$$(k * F)(\tau) = \big((k * \underline{f})(\tau), (k * \overline{f})(\tau)\big)\,.$$

$$\mathcal{L}[(k * F)(\tau)] = \ell[k(\tau)]\mathcal{L}[F(\tau)].$$

Now, we will solve FIDETs (1) using Fuzzy Laplace transforms using Δ_g-derivative.Applying Fuzzy Laplace Transform to (1), we have

$$\mathcal{L}[y]^\Delta(\tau)] = \mathcal{L}[F(\tau), y(\tau)] + \ell[k(\tau)]\mathcal{L}[G(\tau, y(\tau))] \tag{7}$$

(i) If y is $\Delta_{1,g}$-differentiable, then $y^{\Delta_g}(\tau) = (\underline{y}^\Delta_\beta, \overline{y}^\Delta_\beta)$,

From (1), $\mathcal{L}[y^{\Delta_{1,g}}(\tau)](z) = z\mathcal{L}[y(\tau)](z) \ominus y(0)$. From (7), it follows that

$$z\mathcal{L}[y(\tau)] = y(0) + \mathcal{L}[F(\tau), y(\tau)] + \ell[k(\tau)]\mathcal{L}[G(\tau, y(\tau))]$$

As the kernel function $\ell[k(\tau, s)]$ is non-negative for $0 \le s \le \tau$

$$\begin{aligned} z\ell[\underline{y}_\beta(\tau)] &= \underline{y}_0(\beta) + \ell[\underline{F}_\beta(\tau, \underline{y}_\beta(\tau), \overline{y}_\beta(\tau))]\\ &\quad + \ell[k(\tau)]\ell[\underline{G}_\beta(\tau, \underline{y}_\beta(\tau), \overline{y}_\beta(\tau))], \end{aligned}$$

$$\begin{aligned} z\ell[\overline{y}_\beta(\tau)] &= \overline{y}_0(\beta) + \ell[\overline{F}_\beta(\tau, \underline{y}_\beta(\tau), \overline{y}_\beta(\tau))]\\ &\quad + \ell[k(\tau)]\ell[\overline{G}_\beta(\tau, \underline{y}_\beta(\tau), \overline{y}_\beta(\tau))], \end{aligned}$$

Hence,

$$\ell[\underline{y}_\beta(\tau)] = \frac{\underline{y}_0(\beta) + \ell[\underline{F}_\beta(\tau, \underline{y}_\beta(\tau), \overline{y}_\beta(\tau))]}{z - \ell[k(\tau)}] = M_1(z, \beta),$$

$$\ell[\overline{y}_\beta(\tau)] = \frac{\overline{y}_0(\beta) + \ell[\overline{F}_\beta(\tau), \underline{y}_\beta(\tau), \overline{y}_\beta(\tau)]}{z - \ell[k(\tau)}] = N_1(z, \beta).$$

From inverse Laplace transform, we get

$$\begin{aligned} \underline{y}_\beta(\tau)] &= \mathcal{L}^{-1}[M_1(z, \beta)], \\ \overline{y}_\beta(\tau)] &= \mathcal{L}^{-1}[N_1(z, \beta)]. \end{aligned} \tag{8}$$

(i) If y is $\Delta_{2,g}$-differentiable, then $y^{\Delta_g}(\tau) = (\overline{y}^{\Delta}_\beta, \underline{y}^{\Delta}_\beta)$,
From (1), $\mathcal{L}[y^{\Delta_{2,g}}(\tau)](z) = -y(0) \ominus (-z\mathcal{L}[y(\tau)])(z)$. From (7), it follows that

$$\begin{aligned} &- y(0) \ominus (-z\mathcal{L}[y(\tau)]) \\ &= +\mathcal{L}[F(\tau), y(\tau)] + \ell[k(\tau)]\mathcal{L}[G(\tau, y(\tau))] \end{aligned}$$

As the the kernel function $\ell[k(\tau, s)]$ is non-negative for $0 \leq s \leq \tau$

$$\begin{aligned} &- \overline{y}_0(\beta) + z\ell[\overline{y}_\beta(\tau)] \\ &= \ell[\underline{F}_\beta(\tau), y(\tau)] + \ell[k(\tau)]\ell[\underline{G}_\beta(\tau, y(\tau))], \end{aligned}$$

$$\begin{aligned} &- \underline{y}_0(\beta) + z\ell[\underline{y}_\beta(\tau)] \\ &= \ell[\overline{F}_\beta(\tau), y(\tau)] + \ell[k(\tau)]\ell[\overline{G}_\beta(\tau, y(\tau))]. \end{aligned}$$

$$\ell[k(\tau)]\ell[\underline{G}_\beta(\tau, y(\tau))] - z\ell\overline{y}_\beta(\tau)] = m_1(z, \beta),$$

$$z\ell[\underline{y}_\beta(\tau)] + \ell[k(\tau)]\ell[\overline{G}_\beta(\tau, y(\tau))] = n_1(z, \beta),$$

Therefore,

$$\ell[\underline{y}_\beta(\tau)] = \frac{\ell[k(\tau)]m_1(z, \beta) + zn_1(z, \beta)}{(\ell[k(\tau)]^2) - z^2} = M_2(z, \beta)$$

$$\ell[\overline{y}_\beta(\tau)] = \frac{\ell[k(\tau)]n_1(z, \beta) + zm_1(z, \beta)}{(\ell[k(\tau)]^2) - z^2} = n_2(z, \beta)$$

From inverse Laplace transform, we get

$$\begin{aligned} \underline{y}_\beta(\tau)] &= \ell^{-1}[M_2(z, \beta)], \\ \overline{y}_\beta(\tau)] &= \ell^{-1}[N_2(z, \beta)]. \end{aligned} \tag{9}$$

Example 5.1: Consider the following FIDETs with initial condition is given by

$$y^{\Delta}(\tau) = (1 + \tau) \odot \tilde{1} + \int_0^\tau y(s)\Delta s, \tau_0, s \in [\tau_0, \tau_0 + a]_{\mathbb{T}},$$

$$y(0) = y_0 = (0, 0) \in \mathbb{E}^1,$$

Here $k(\tau, s) = 1$ and $\tilde{1} = (2 + \beta, 3 - \beta)$.

If F is $\Delta_{1,g}$-differentiable, and from Theorem 4.2,

$$z \odot \mathcal{L}(1 + \tau) \ominus y(0) = \mathcal{L}\left[(1 + \tau) \odot \tilde{1} + \int_0^\tau y(s)\Delta s\right] \tag{10}$$

In terms of β-cuts, (10) can be written as

$$\mathcal{L}\left\{\underline{y}^{\Delta}(\tau) = \mathcal{L}(1 + \tau)(2 + \beta)\right\} + \mathcal{L}\left\{\int_0^\tau \underline{y}(s, \beta)\Delta s\right\},$$

$$\mathcal{L}\left\{\overline{y}^{\Delta}(\tau) = \mathcal{L}(1 + \tau)(3 - \beta)\right\} + \mathcal{L}\left\{\int_0^\tau \overline{y}(s, \beta)\Delta s\right\},$$

If $\mathbb{T} = \mathbb{R}$, and from Theorem 4.2,

$$\begin{aligned} &z\mathcal{L}\left\{\underline{y}(\tau, \beta)\right\} - \underline{y}(0, \beta) \\ &= \left(\frac{1}{z} + \frac{1}{z^2}\right)(2 + \beta) + \mathcal{L}\left\{\int_0^\tau \underline{y}(s, \beta)\Delta s\right\}, \end{aligned}$$

$$\begin{aligned} &z\mathcal{L}\left\{\overline{y}(\tau, \beta)\right\} - \overline{y}(0, \beta) \\ &= \left(\frac{1}{z} + \frac{1}{z^2}\right)(3 - \beta) + \mathcal{L}\left\{\int_0^\tau \overline{y}(s, \beta)\Delta s\right\}, \end{aligned}$$

From initial condition and using convolution theorem,

$$\begin{aligned} &z\mathcal{L}\left\{\underline{y}(\tau, \beta)\right\} \\ &= \left(\frac{1}{z} + \frac{1}{z^2}\right)(2 + \beta) + \mathcal{L}\left\{1\right\}\mathcal{L}\left\{\underline{y}(s, \beta)\Delta s\right\}, \end{aligned}$$

$$\begin{aligned} &z\mathcal{L}\left\{\overline{y}(\tau, \beta)\right\} \\ &= \left(\frac{1}{z} + \frac{1}{z^2}\right)(3 - \beta) + \mathcal{L}\left\{1\right\}\mathcal{L}\left\{\overline{y}(s, \beta)\Delta s\right\}, \end{aligned}$$

$$\begin{aligned} &\mathcal{L}\left\{\underline{y}(\tau, \beta)\right\} \\ &= \left(\frac{1}{z^2} + \frac{1}{z^3}\right)(2 + \beta) + \frac{1}{z^2}\mathcal{L}\left\{\underline{y}(s, \beta)\Delta s\right\}, \end{aligned}$$

$$\begin{aligned} &z\mathcal{L}\left\{\overline{y}(\tau, \beta)\right\} \\ &= \left(\frac{1}{z} + \frac{1}{z^2}\right)(3 - \beta) + \frac{1}{z^2}\mathcal{L}\left\{\overline{y}(s, \beta)\Delta s\right\}, \end{aligned}$$

Applying Inverse Laplace Transform,

$$\begin{aligned} &\underline{y}(\tau, \beta) \\ &= \left(\tau + \frac{\tau^2}{2!}\right)(2 + \beta) + \mathcal{L}^{-1}\left\{\frac{1}{z^2}\underline{y}(s, \beta)\Delta s\right\}, \end{aligned}$$

$$\begin{aligned} &\overline{y}(\tau, \beta) \\ &= \left(\tau + \frac{\tau^2}{2!}\right)(2 + \beta) + \mathcal{L}^{-1}\left\{\frac{1}{z^2}\overline{y}(s, \beta)\Delta s\right\}, \end{aligned}$$

Expanding and comparing the terms we get,

$$\underline{y}(\tau, \beta) = (e_1(\tau, 0) - 1)(2 + \beta),$$

$$\overline{y}(\tau, \beta) = (e_1(\tau, 0) - 1)(3 - \beta),$$

where $e_1(\tau, 0) = \begin{cases} (e^\tau - 1), & \mathbb{T} = \mathbb{R}, \\ (2^\tau - 1), & \mathbb{T} = \mathbb{Z}. \end{cases}$

When $\mathbb{T} = \mathbb{R}$,

$$\underline{y}(\tau, \beta) = (e^\tau - 1)(2 + \beta),$$

$$\overline{y}(\tau, \beta) = (e^\tau - 1)(3 - \beta),$$

When $\mathbb{T} = \mathbb{Z}$,

$$\underline{y}(\tau, \beta) = (2^\tau - 1)(2 + \beta),$$

$$\overline{y}(\tau, \beta) = (2^\tau - 1)(3 - \beta),$$

Thus, we have applied Fuzzy Laplace transform method for solving FIDET under Δ_g-derivative with crisp kernel of convolution type. For future research, we will apply Fuzzy Laplace transform method for solving FIDET using nabla derivative.

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