Formulating Few Key Identities Relating Infinite Series of Eisenstein and Borweins' Cubic Theta Functions

Vidya Harekala Chandrashekara and Smitha Ganesh Bhat*

***Abstract*—Few new Eisenstein series identities involving Borwein's theta functions are formulated using $(p,\ k)$- parametrization technique which is introduced by Alaca. As an application, we have evaluated the convolution sum of divisor functions which are related to Eisenstein series.**

***Index Terms*—Cubic Theta Functions, Eisenstein Series, Convolution Sum, Digital Signal Processing.**

I. Introduction

CONVOLUTION is a data computation activity in science and engineering that includes multiplying two vectors and then aggregating the results. In the realms of numerical linear algebra, probability theory, numerical analysis, deep learning, as well as the design and application of finite impulse response filters in signal processing where it is used for combining two signals to generate a third output signal [17], convolution emerges as an indispensable and ubiquitous technique. Convolution, being the theoretical cornerstone of digital signal processing, holds paramount importance in this field. Additionally, its applications extend to communication systems, further highlighting its versatility and widespread utility across multiple domains. Mathematicians regularly employ Ramanujan's discriminant function, Gaussian hyper-geometric series, quasi-modular forms, Ramanujan-type Eisenstein series, and other methods to calculate convolution sums.

By leveraging the parameters p and k introduced by Alaca, this article embarks on a captivating journey to merge Eisenstein series and Borwein's cubic theta functions. Through our meticulous analysis, we reveal a collection of intriguing identities that transcend traditional mathematical boundaries, providing fresh insights into the intricate connections and potential applications of these mathematical constructs. Impressively, the article presents an ingenious approach to express Ramanujan-type Eisenstein series in terms of the product of cubic theta functions, unveiling an elegant mathematical relationship. Furthermore, for all positive integers l, the resulting formulas have been utilised to assess a new representation for the discrete convolution sum $\sum_{3i+6j=\alpha} \delta(i)\delta(j)$.

Section 2 is intended to provide a few preliminary results that will aid in the achievement of the major goals.

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Sections 3 and 4 present and prove some intriguing identities that are analogues of Earnest Xia's identities but discovered to be new. These relationships include the Ramanujan-type Eisenstein series and Borwein's cubic theta functions.

Section 5 discusses a new representation for evaluating a discrete convolution sum.

II. Preliminaries

The roots of arithmetic-geometric mean iteration primarily lie in elliptic function and theta function theory. Borwein brothers [10], [11] have deduced the following multidimensional theta functions:

$$a(q) := \sum_{r,s=-\infty}^{\infty} q^{r^2+rs+s^2}. \tag{1}$$

$$b(q) := \sum_{r,s=-\infty}^{\infty} \omega^{r-s} q^{r^2+rs+s^2}. \tag{2}$$

$$c(q) := \sum_{r,s=-\infty}^{\infty} q^{\left(r+\frac{1}{3}\right)^2+\left(r+\frac{1}{3}\right)\left(s+\frac{1}{3}\right)+\left(s+\frac{1}{3}\right)^2}. \tag{3}$$

where $|q| < 1$, q being the set of complex numbers and $\omega = exp(2\pi i/3)$ is the principal cube root of unity. In the above expressions for two-dimensional theta functions, it is evident that when $q = 0$, $a(q) = 1$, $b(q) = 1$, and $c(q) = 0$.

From Euler's binomial theorem, Borwein brothers have devised expressions for $b(q)$ and $c(q)$ in terms of infinite products, which are given by,

$$b(q) = \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty}, \tag{4}$$

$$c(q) = \frac{3q^{\frac{1}{3}}(q^3;q^3)_\infty^3}{(q;q)_\infty}, \tag{5}$$

where

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i). \tag{6}$$

The function $a(q)$ can be expressed as follows (J.M. Borwein & P. B. Borwein [10], B. C. Berndt [9]):

$$a(q) = (-q;q^2)_\infty^2 (q^2;q^2)_\infty (q^3;q^6)_\infty^2 (q^6;q^6)_\infty + 4q\frac{(q^4;q^4)_\infty (q^{12};q^{12})_\infty}{(q^2;q^4)_\infty (q^6;q^{12})_\infty}. \tag{7}$$

The proofs for Equations (4), (5) and (7) are quintessentially classic and can be found in the works of Borwein et al. [11].

Besides that, they have established the fundamental relationship between $a(q)$, $b(q)$ and $c(q)$ which is a basic cubic identity given by,

$$a^3(q) = b^3(q) + c^3(q). \tag{8}$$

Definition II.1. *Srinivasa Ramanujan in his second notebook [15] has provided the definitions for the Eisenstein Series $L(q)$, $M(q)$ and $N(q)$ as follows:*

$$L(q) := 1 - 24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}. \tag{9}$$

$$M(q) := 1 + 240\sum_{r=1}^{\infty}\frac{r^3q^r}{1-q^r}. \tag{10}$$

$$N(q) := 1 - 504\sum_{r=1}^{\infty}\frac{r^5q^r}{1-q^r}. \tag{11}$$

Liu et al. [13], [14], E. X. W. Xia and O. X. M. Yao [22] have obtained some Eisenstein series identities with Borwein's theta functions containing $b(q)$, $b(q^2)$, $b(q^4)$, $c(q)$, $c(q^2)$ and $c(q^4)$. In the present work, we have deduced some pertinent identities of Eisenstein series entailing Borwein's cubic theta function by incorporating parameters p and k which is introduced by Alaca et al. [1], [3], [4]. These are the examples for sum-to-product identities.

The present work's outline is as follows. We begin with listing some essential preliminaries, which are required at the later stage to prove the main results. Then, we proceed with deducing Eisenstein series identities involving Borweins theta function using $(p,\ k)$- parametrization introduced by Alaca et al. [1]. To conclude, using a deduced identity we evaluate the convolution sum $\sum_{r}\delta(i)\delta(j)$ for the case of $r = 3i + 6j$, where $i, j \in \mathbb{N}$.

Definition II.2. *For any complex c and d, Ramanujan[8, p.35] documented a general theta function,*

$$f(c,d) := \sum_{m=-\infty}^{\infty} c^{m(m+1)/2}d^{m(m-1)/2}$$
$$:= (-c;cd)_\infty(-d;cd)_\infty(cd;cd)_\infty,$$

where

$$(c;q)_\infty := \prod_{m=0}^{\infty}(1-cq^m), \qquad |q|<1.$$

The special case of theta function defined by Ramanujan[8, p.35],

$$\varphi(q) := f(q,q) = \sum_{m=-\infty}^{\infty} q^{m^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty.$$

In their remarkable article, Alaca et al. [1] have defined the $(p,\ k)$ parametrization of theta functions. These are highly significant in designing the duplication and triplication principle and further obtaining certain sum to product identities. The parameters p and k are defined as:

$$p := p(q) = \frac{\varphi^2(q)-\varphi^2(q^3)}{2\phi^2(q^3)}. \tag{12}$$

$$k := k(q) = \frac{\varphi^3(q^3)}{\varphi(q)}. \tag{13}$$

Since $\varphi(0) = 1$, it clear that $p(0) = 0$ and $k(0) = 1$.

Lemma II.3. *[1] For aforementioned Eisenstein series [10], [11], the representations of $M(q)$, $M(q^l)$, $L(q) - lL(q^l)$, $(l = 2, 3, 4, 6, 12)$ and also $L(-q^l) - rL(q^r)$, $l \in \{1, 3\}$ and $r \in \{1, 2, 3\}$, in terms of the parameters p and k are given by,*

$$M(q) = (1 + 124p(1 + p^6) + 964p^2(1 + p^4) + 2788p^3(1 + p^2) + 3910p^4 + p^8)k^4,$$
$$M(q^2) = (1 + 4p(1 + p^6) + 64p^2(1 + p^4) + 178p^3(1 + p^2) + 235p^4 + p^8)k^4,$$
$$M(q^3) = (1 + 4p(1 + p^6) + 4p^2(1 + p^4) + 28p^3(1 + p^2) + 70p^4 + p^8)k^4,$$
$$M(q^6) = (1 + 4p(1 + p^6) + 4p^2(1 + p^4) - 2p^3(1 + p^2) - 5p^4 + p^8)k^4,$$
$$M(q^{12}) = (1 + 4p(1 + p) - 2p^3(1 + p^2) - 5p^4 + p^6(1 + p)/4 + p^8/16)k^4,$$
$$L(-q) - L(q) = 3(8p + 12p^2 + 6p^3 + p^4)k^2,$$
$$L_{1,2}(q) = (L(-q) - L(q))/48 = (p/2 + 3p^2/4 + 3p^3/8 + p^4/16)k^2,$$
$$L_{1,2}(q^3) = (L(-q^3) - L(q^3))/48 = p^3(2 + p)k^2/16,$$
$$L(-q) - 2L(q^2) = -(1 - 10p - 12p^2 - 4p^3 - 2p^4)k^2,$$
$$L(q) - 2L(q^2) = -(1 + 14p(1 + p^2) + 24p^2 + p^4)k^2,$$
$$L(q) - 3L(q^3) = -(1 + 8p(1 + p^2) + 18p^2 + p^4)k^2,$$
$$L(q) - 6L(q^6) = -(5 + 22p(1 + p^2) + 36p^2 + 5p^4)k^2,$$
$$L(q^2) - 3L(q^6) = -2(1 + 2p(1 + p^2) + 3p^2 + p^4)k^2,$$
$$L(q^3) - 2L(q^6) = -(1 + 2p(1 + p^2) + p^4)k^2,$$
$$L(q) - 4L(q^4) = -3(1 + 6p + 12p^2 + 8p^3)k^2,$$
$$L(q) - 12L(q^{12}) = -(11 + 34p + 36p^2 + 16p^3 + 2p^4)k^2.$$

Lemma II.4. *The parametric representations of $a(q^r), b(q^r), c(q^r) (r \in 1, 2, 4, 6)$ and $a(-q)$, $b(-q)$, $c(-q)$ in terms of the parameters p and k deduced by Alaca et al. [1] are as follows*

$$a(-q) = (1 - 2p - 2p^2)k,$$
$$a(q) = (1 + 4p + p^2)k,$$
$$a(q^2) = (1 + p + p^2)k,$$
$$a(q^4) = (1 + p - \frac{1}{2}p^2)k,$$
$$a(q^6) = \frac{(p^2 + p + 1 + 2^{1/3}((1 - p)(2 + p)(1 + 2p))^{2/3})k}{3},$$
$$b(-q) = 2^{-\frac{1}{3}}((1 - p)(1 + 2p)^4(2 + p))^{\frac{1}{3}}k,$$
$$b(q) = 2^{-\frac{1}{3}}((1 - p)^4(1 + 2p)(2 + p))^{\frac{1}{3}}k,$$
$$b(q^2) = 2^{-2/3}((1 - p)(1 + 2p)(2 + p))^{\frac{2}{3}}k,$$
$$b(q^4) = 2^{-\frac{4}{3}}((1 - p)(1 + 2p)(2 + p)^4)^{\frac{1}{3}}k,$$
$$c(-q) = -2^{\frac{1}{3}}3(p(1 + p))^{\frac{1}{3}}k,$$
$$c(q) = 2^{-\frac{1}{3}}3(p(1 + p)^4)^{\frac{1}{3}}k,$$
$$c(q^2) = 2^{-\frac{2}{3}}3(p(1 + p))^{\frac{2}{3}}k.$$
$$c(q^4) = 2^{-\frac{4}{3}}3(p^4(1 + p))^{\frac{1}{3}}k,$$
$$c(q^6) = \frac{\left(p^2 + p + 1 - 2^{-2/3}((1 - p)(2 + p)(1 + 2p))^{2/3}\right)k}{3}.$$

,

III. Infinite Series Identities Comprising of $L(q)$ and $L(-q)$

Ramanujan in his notebook [15] has emphasized on Eisenstein series, specially focusing on L, M & N. Accordingly, he has provided several prominent identities of infinite series incorporating theta functions. Xia et al. [22] has deduced few significant relations amongst Eisenstein series and cubic theta functions involving $L(q)-rL(q^r)$, for $r \in \{2,3,4,6,12\}$ using computer. Some additional identities have been deduced by Shruti and Srivatsakumar B.R. [16] and the convolution sum has been evaluated. Recently, Vidya H. C. and Ashwath Rao B. [19] have formulated few identities that includes $L(-q^l) - L(q^l)$, for $l \in \{1,3\}$ as well. Similarly relation among theta functions have been deduced by Anuradha et al. [7]. Motivated by these works, we have established certain new significant identities that relates Ramanujan type Eisenstein series and cubic theta functions. In Theorems III.1 through III.5, we have obtained several new noteworthy relations including $L(q) - rL(q^r)$, $r \in \{2,3,4,6,12\}$ and $L(-q^l) - rL(q^r)$, $l \in \{1,3\}$ and $r \in \{1,2,3\}$, without the aid of computer.

Theorem III.1. *The relation amongst an infinite series and theta functions holds:*

$$1+3\sum_{r=1}^{\infty}\left[\frac{2r(-q)^r}{1-(-q)^r}+\frac{3rq^r}{1-q^r}-\frac{16rq^{2r}}{1-q^{2r}}-\frac{9rq^{3r}}{1-q^{3r}}+\frac{12rq^{4r}}{1-q^{4r}}+\frac{36rq^{6r}}{1-q^{6r}}-\frac{36rq^{12r}}{1-q^{12r}}\right]=\frac{b(q)b(q^4)b^2(-q)}{b^2(q^2)}. \tag{14}$$

$$1+3\sum_{r=1}^{\infty}\left[\frac{4r(-q)^r}{1-(-q)^r}+\frac{rq^r}{1-q^r}-\frac{4rq^{2r}}{1-q^{2r}}-\frac{9rq^{3r}}{1-q^{3r}}\right]=\frac{b^2(q)b(q^2)}{b(-q)}. \tag{15}$$

$$1-12\sum_{r=1}^{\infty}\left[\frac{2r(-q)^r}{1-(-q)^r}+\frac{2rq^r}{1-q^r}-\frac{11rq^{2r}}{1-q^{2r}}+\frac{9rq^{6r}}{1-q^{6r}}\right]=\frac{b(q)b(q^2)b(-q)}{b(q^4)}. \tag{16}$$

$$1+3\sum_{r=1}^{\infty}\left[\frac{2r(-q)^r}{1-(-q)^r}-\frac{rq^r}{1-q^r}-\frac{9rq^{6r}}{1-q^{6r}}\right]=\frac{b(q)b(q^2)b(q^4)}{b(-q)}. \tag{17}$$

$$1+3\sum_{r=1}^{\infty}\left[\frac{r(-q)^r}{1-(-q)^r}+\frac{rq^r}{1-q^r}-\frac{7rq^{2r}}{1-q^{2r}}+\frac{6rq^{4r}}{1-q^{4r}}+\frac{9rq^{6r}}{1-q^{6r}}-\frac{18rq^{12r}}{1-q^{12r}}\right]=\frac{b(q)b(-q)b^2(q^4)}{b^2(q^2)}. \tag{18}$$

$$1+4\sum_{r=1}^{\infty}\left[\frac{2r(-q)^r}{1-(-q)^r}+\frac{rq^r}{1-q^r}-\frac{11rq^{2r}}{1-q^{2r}}-\frac{3rq^{3r}}{1-q^{3r}}+\frac{8rq^{4r}}{1-q^{4r}}+\frac{21rq^{6r}}{1-q^{6r}}-\frac{24rq^{12r}}{1-q^{12r}}\right]=\frac{a(-q)c^2(q)}{3c(q^2)}. \tag{19}$$

$$1+8\sum_{r=1}^{\infty}\left[\frac{r(-q)^r}{1-(-q)^r}-\frac{rq^{2r}}{1-q^{2r}}-\frac{3rq^{6r}}{1-q^{6r}}\right]=\frac{a(-q)c^2(-q)}{2^{\frac{4}{3}}3c(q^2)}. \tag{20}$$

$$1+8\sum_{r=1}^{\infty}\left[\frac{rq^r}{1-q^r}-\frac{rq^{2r}}{1-q^{2r}}-\frac{3rq^{6r}}{1-q^{6r}}\right]=\frac{a(q)c^2(q)}{3c(q^2)}. \tag{21}$$

$$1+4\sum_{r=1}^{\infty}\left[\frac{rq^r}{1-q^r}-\frac{5rq^{2r}}{1-q^{2r}}+\frac{3rq^{3r}}{1-q^{3r}}+\frac{4rq^{4r}}{1-q^{4r}}+\frac{3rq^{6r}}{1-q^{6r}}-\frac{12rq^{12r}}{1-q^{12r}}\right]=\frac{a(q)c^2(-q)}{2^{\frac{4}{3}}3c(q^2)}. \tag{22}$$

$$1+12\sum_{r=1}^{\infty}\left[\frac{-rq^r}{1-q^r}+\frac{6rq^{2r}}{1-q^{2r}}+\frac{3rq^{3r}}{1-q^{3r}}-\frac{4rq^{4r}}{1-q^{4r}}-\frac{18rq^{6r}}{1-q^{6r}}+\frac{12rq^{12r}}{1-q^{12r}}\right]=a^2(-q). \tag{23}$$

$$\sum_{r=1}^{\infty}\left[\frac{2r(-q)^r}{1-(-q)^r}+\frac{2rq^r}{1-q^r}-\frac{15rq^{2r}}{1-q^{2r}}-\frac{6rq^{3r}}{1-q^{3r}}+\frac{8rq^{4r}}{1-q^{4r}}+\frac{33rq^{6r}}{1-q^{6r}}-\frac{6rq^{12r}}{1-q^{12r}}\right]=\frac{2^{\frac{2}{3}}c(q)c(q^4)c(q^2)}{3c(-q)}. \tag{24}$$

$$\sum_{r=1}^{\infty}\left[\frac{-2r(-q)^r}{1-(-q)^r}-\frac{3rq^r}{1-q^r}+\frac{20rq^{2r}}{1-q^{2r}}+\frac{9rq^{3r}}{1-q^{3r}}-\frac{12rq^{4r}}{1-q^{4r}}-\frac{48rq^{6r}}{1-q^{6r}}+\frac{36rq^{12r}}{1-q^{12r}}\right]=\frac{2^{\frac{2}{3}}a(-q)c^2(q^2)}{3c(-q)}. \tag{25}$$

$$\sum_{r=1}^{\infty}\left[\frac{2r(-q)^r}{1-(-q)^r}+\frac{2rq^r}{1-q^r}-\frac{11rq^{2r}}{1-q^{2r}}-\frac{6rq^{3r}}{1-q^{3r}}+\frac{4rq^{4r}}{1-q^{4r}}+\frac{21rq^{6r}}{1-q^{6r}}-\frac{12rq^{12r}}{1-q^{12r}}\right]=\frac{a(-q)c^2(q^4)}{3c(q^2)}. \tag{26}$$

$$\sum_{r=1}^{\infty}\left[\frac{-rq^r}{1-q^r}-\frac{4rq^{2r}}{1-q^{2r}}-\frac{3rq^{3r}}{1-q^{3r}}+\frac{8rq^{4r}}{1-q^{4r}}+\frac{24rq^{6r}}{1-q^{6r}}-\frac{24rq^{12r}}{1-q^{12r}}\right]=\frac{2^{\frac{2}{3}}a(q)c^2(q^2)}{3c(-q)}. \tag{27}$$

Proof: Let us persume that,

$$C_1L_{1,2}(q)+C_2[L(q)-2L(q^2)]+C_3[L(q)-3L(q^3)]+C_4[L(q^3)-2L(q^6)]+C_5[L(q^4)-3L(q^{12})]=\frac{b(q)b(q^4)b^2(-q)}{b^2(q^2)}. \tag{28}$$

The above equation is transformed by (p,k) parametrization using Lemma II.3. Then a system of non-homogeneous linear equations is obtained by equating the coeffecients of terms containing k^2, pk^2, p^2k^2, p^3k^2

and p^4k^2 in LHS with corresponding terms in RHS which is to be solved for unknowns.

$$\begin{pmatrix} 0 & -1 & -2 & -1 & -2 \\ \frac{1}{2} & -14 & -16 & -2 & -4 \\ \frac{3}{4} & -24 & -36 & 0 & 0 \\ \frac{3}{8} & -14 & -16 & -2 & 2 \\ \frac{1}{16} & -1 & -2 & -1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{7}{2} \\ \frac{3}{2} \\ -4 \\ -2 \end{pmatrix}.$$

On solving the above system, we get,

$C_1 = -12,\ C_2 = -1,\ C_3 = \frac{3}{8},\ C_4 = \frac{9}{4}$ and $C_5 = -\frac{3}{2}$.

By substituting the above statistics in (28) and further simplifying using Definition II.1, the equation (14) is obtained. Similarly, utilizing the similar technique we derive the following identities.

$$-24L_{1,2}(q) - \frac{1}{4}[L(q) - 2L(q^2)] - \frac{3}{8}[L(q) - 3L(q^3)] = \frac{b^2(q)b(q^2)}{b(-q)}.$$

$$48L_{1,2}(q) + \frac{11}{4}[L(q) - 2L(q^2)] - \frac{3}{4}[L(q) - 3L(q^3)] - \frac{9}{4}[L(q^3) - 2L(q^6)] = \frac{b(q)b(-q)b(q^2)}{b(q^4)}.$$

$$-12L_{1,2}(q) - \frac{1}{16}[L(q) - 2L(q^2)] - \frac{3}{16}[L(q) - 3L(q^3)] - \frac{9}{16}[L(q^3) - 2L(q^6)] = \frac{b(q)b(q^4)b(q^2)}{b(-q)}.$$

$$-6L_{1,2}(q) - \frac{7}{16}[L(q) - 2L(q^2)] + \frac{3}{16}[L(q) - 3L(q^3)] + \frac{9}{16}[L(q^3) - 2L(q^6)] - \frac{3}{4}[L(q^4) - 3L(q^{12})] = \frac{b(q)b(-q)b^2(q^4)}{b^2(q^2)}.$$

$$-48L_{1,2}(q) - \frac{11}{4}[L(q) - 2L(q^2)] + \frac{5}{4}[L(q) - 3L(q^3)] + \frac{21}{4}[L(q^3) - 2L(q^6)] - 4[L(q^4) - 3L(q^{12})] = \frac{a(-q)c^2(q)}{c(q^2)}.$$

$$-96L_{1,2}(q) - [L(q) - 2L(q^2)] - [L(q) - 3L(q^3)] - 3[L(q^3) - 2L(q^6)] = \frac{a(-q)c^2(-q)}{2^{\frac{1}{3}}c(q^2)}.$$

$$-\frac{1}{2}[L(q) - 2L(q^2)] - \frac{1}{2}[L(q) - 3L(q^3)] - \frac{3}{2}[L(q^3) - 2L(q^6)] = \frac{a(q)c^2(q)}{c(q^2)}.$$

$$-\frac{5}{2}[L(q) - 2L(q^2)] + \frac{3}{2}[L(q) - 3L(q^3)] + \frac{3}{2}[L(q^3) - 2L(q^6)] - 4[L(q^4) - 3L(q^{12})] = \frac{a(q)c^2(-q)}{2^{\frac{1}{3}}c(q^2)}.$$

$$\frac{3}{2}[L(q) - 2L(q^2)] - [L(q) - 3L(q^3)] - \frac{9}{2}[L(q^3) - 2L(q^6)] + 2[L(q^4) - 3L(q^{12})] = a^2(-q).$$

$$-6L_{1,2}(q) - \frac{15}{32}[L(q) - 2L(q^2)] + \frac{7}{32}[L(q) - 3L(q^3)] [L(+\frac{33}{32}q^3) - 2L(q^6)] - \frac{1}{2}[L(q^4) - 3L(q^{12})] = \frac{c(q)c(q^2)c(q^4)}{2^{\frac{1}{3}}c(-q)}.$$

$$6L_{1,2}(q) + \frac{5}{8}[L(q) - 2L(q^2)] - \frac{5}{16}[L(q) - 3L(q^3)] - \frac{3}{2}[L(q^3) - 2L(q^6)] + \frac{3}{4}[L(q^4) - 3L(q^{12})] = \frac{a(-q)c^2(q^2)}{2^{\frac{1}{3}}c(-q)}.$$

$$-12L_{1,2}(q) - \frac{11}{16}[L(q) - 2L(q^2)] + \frac{3}{16}[L(q) - 3L(q^3)] + \frac{21}{16}[L(q^3) - 2L(q^6)] - \frac{1}{2}[L(q^4) - 3L(q^{12})] = \frac{a(-q)c^2(q^4)}{c(q^2)}.$$

$$-\frac{1}{8}[L(q) - 2L(q^2)] + \frac{3}{16}[L(q) - 3L(q^3)] + \frac{3}{4}[L(q^3) - 2L(q^6)] - \frac{1}{2}[L(q^4) - 3L(q^{12})] = \frac{a(q)c^2(q^2)}{2^{\frac{1}{3}}c(-q)}.$$

■

Theorem III.2. *One has,*

$$1 + 3\sum_{r=1}^{\infty}\left[\frac{6r(-q)^{3r}}{1-(-q)^{3r}} + \frac{rq^r}{1-q^r} - \frac{4rq^{2r}}{1-q^{2r}} - \frac{3rq^{3r}}{1-q^{3r}} + \frac{4rq^{4r}}{1-q^{4r}} - \frac{12rq^{12r}}{1-q^{12r}}\right] = \frac{b(q)b(q^4)b^2(-q)}{b^2(q^2)}. \quad (29)$$

$$1 + 3\sum_{r=1}^{\infty}\left[\frac{12r(-q)^{3r}}{1-(-q)^{3r}} - \frac{3rq^r}{1-q^r} + \frac{20rq^{2r}}{1-q^{2r}} + \frac{3rq^{3r}}{1-q^{3r}} - \frac{16rq^{4r}}{1-q^{4r}} - \frac{72rq^{6r}}{1-q^{6r}} + \frac{48rq^{12r}}{1-q^{12r}}\right] = \frac{b^2(q)b(q^2)}{b(-q)}. \quad (30)$$

$$1 + 12\sum_{r=1}^{\infty}\left[\frac{-6r(-q)^{3r}}{1-(-q)^{3r}} - \frac{rq^{2r}}{1-q^{2r}} - \frac{6rq^{3r}}{1-q^{3r}} + \frac{8rq^{4r}}{1-q^{4r}} + \frac{27rq^{6r}}{1-q^{6r}} - \frac{24rq^{12r}}{1-q^{12r}}\right] = \frac{b(q)b(-q)b(q^2)}{b(q^4)}. \quad (31)$$

$$1 + 3\sum_{r=1}^{\infty}\left[\frac{6r(-q)^{3r}}{1-(-q)^{3r}} - \frac{2rq^r}{1-q^r} + \frac{11rq^{2r}}{1-q^{2r}} + \frac{6rq^{3r}}{1-q^{3r}} - \frac{8rq^{4r}}{1-q^{4r}} - \frac{45rq^{6r}}{1-q^{6r}} + \frac{24rq^{12r}}{1-q^{12r}}\right] = \frac{b(q)b(q^2)b(q^4)}{b(-q)}. \quad (32)$$

$$1 + 3\sum_{r=1}^{\infty}\left[\frac{3r(-q)^{3r}}{1-(-q)^{3r}} - \frac{rq^{2r}}{1-q^{2r}} + \frac{3rq^{3r}}{1-q^{3r}} + \frac{2rq^{4r}}{1-q^{4r}} - \frac{9rq^{6r}}{1-q^{6r}} - \frac{6rq^{12r}}{1-q^{12r}}\right] = \frac{b(q)b(-q)b^2(q^4)}{b^2(q^2)}. \quad (33)$$

$$1+4\sum_{r=1}^{\infty}\Big[\frac{6r(-q)^{3r}}{1-(-q)^{3r}}-\frac{rq^r}{1-q^r}+\frac{rq^{2r}}{1-q^{2r}}+\frac{3rq^{3r}}{1-q^{3r}}-\frac{15rq^{6r}}{1-q^{6r}}\Big]=\frac{a(-q)c^2(q)}{3c(q^2)}. \quad (34)$$

$$1+8\sum_{r=1}^{\infty}\Big[\frac{3r(-q)^{3r}}{1-(-q)^{3r}}-\frac{rq^r}{1-q^r}+\frac{5rq^{2r}}{1-q^{2r}}+\frac{3rq^{3r}}{1-q^{3r}}-\frac{4rq^{4r}}{1-q^{4r}}-\frac{21rq^{6r}}{1-q^{6r}}+\frac{12rq^{12r}}{1-q^{12r}}\Big]=\frac{a(-q)c^2(-q)}{2^{\frac{4}{3}}3c(q^2)}. \quad (35)$$

$$1+12\sum_{r=1}^{\infty}\Big[\frac{-rq^r}{1-q^r}+\frac{6rq^{2r}}{1-q^{2r}}+\frac{3rq^{3r}}{1-q^{3r}}-\frac{4rq^{4r}}{1-q^{4r}}-\frac{18rq^{6r}}{1-q^{6r}}+\frac{12rq^{12r}}{1-q^{12r}}\Big]=a^2(-q). \quad (36)$$

$$1+8\sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{rq^{2r}}{1-q^{2r}}-\frac{3rq^{6r}}{1-q^{6r}}\Big]=\frac{a(q)c^2(q)}{3c(q^2)}. \quad (37)$$

$$1+4\sum_{r=1}^{\infty}\Big[\frac{rq^r}{1-q^r}-\frac{5rq^{2r}}{1-q^{2r}}+\frac{3rq^{3r}}{1-q^{3r}}+\frac{4rq^{4r}}{1-q^{4r}}+\frac{3rq^{6r}}{1-q^{6r}}-\frac{12rq^{12r}}{1-q^{12r}}\Big]=\frac{a(q)c^2(-q)}{2^{\frac{4}{3}}3c(q^2)}. \quad (38)$$

$$\sum_{r=1}^{\infty}\Big[\frac{2r(-q)^{3r}}{1-(-q)^{3r}}-\frac{rq^{2r}}{1-q^{2r}}-\frac{rq^{6r}}{1-q^{6r}}\Big]=\frac{2^{\frac{2}{3}}c(q)c(q^2)c(q^4)}{9c(-q)}. \quad (39)$$

$$\sum_{r=1}^{\infty}\Big[\frac{-6r(-q)^{3r}}{1-(-q)^{3r}}-\frac{rq^r}{1-q^r}+\frac{8rq^{2r}}{1-q^{2r}}+\frac{3rq^{3r}}{1-q^{3r}}-\frac{4rq^{4r}}{1-q^{4r}}-\frac{12rq^{6r}}{1-q^{6r}}+\frac{12rq^{12r}}{1-q^{12r}}\Big]=\frac{2^{\frac{2}{3}}a(-q)c^2(q^2)}{3c(-q)}. \quad (40)$$

$$\sum_{r=1}^{\infty}\Big[\frac{6r(-q)^{3r}}{1-(-q)^{3r}}+\frac{rq^{2r}}{1-q^{2r}}-\frac{4rq^{4r}}{1-q^{4r}}-\frac{15rq^{6r}}{1-q^{6r}}+\frac{12rq^{12r}}{1-q^{12r}}\Big]=\frac{a(-q)c^2(q^4)}{3c(q^2)}. \quad (41)$$

$$\sum_{r=1}^{\infty}\Big[\frac{-rq^r}{1-q^r}-\frac{4rq^{2r}}{1-q^{2r}}-\frac{3rq^{3r}}{1-q^{3r}}+\frac{8rq^{4r}}{1-q^{4r}}+\frac{24rq^{6r}}{1-q^{6r}}-\frac{24rq^{12r}}{1-q^{12r}}\Big]=\frac{2^{\frac{2}{3}}a(q)c^2(q^2)}{3c(-q)}. \quad (42)$$

Proof: Consider the following,

$$C_1L_{1,2}(q^3)+C_2[L(q)-2L(q^2)]+C_3[L(q)-3L(q^3)]+C_4[L(q^3)-2L(q^6)]+C_5[L(q^4)-3L(q^{12})]=\frac{b(q)b(q^4)b^2(-q)}{b^2(q^2)}. \quad (43)$$

Expressing the above relation in terms of (p,k) parametrization and then equalising the coeffecients of terms containing k^2, pk^2, p^2k^2, p^3k^2 and p^4k^2 on either sides, we get

$$\begin{pmatrix} 0 & -1 & -2 & -1 & -2 \\ 0 & -14 & -16 & -2 & -4 \\ 0 & -24 & -36 & 0 & 0 \\ \frac{1}{8} & -14 & -16 & -2 & 2 \\ \frac{1}{16} & -1 & -2 & -1 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{pmatrix}=\begin{pmatrix} 1 \\ \frac{7}{2} \\ \frac{3}{2} \\ -4 \\ -2 \end{pmatrix}.$$

Now the solution leads to,

$C_1=-36,\ C_2=-\frac{1}{4},\ C_3=\frac{1}{8},\ C_4=0$ and $C_5=-\frac{1}{2}.$

Substituting the above values in (43) and simplifying using Definition II.1, we get (29). In the same manner, the identities listed below are deduced.

$$-72L_{1,2}(q^3)+\frac{5}{4}[L(q)-2L(q^2)]-\frac{7}{8}[L(q)-3L(q^3)]-\frac{9}{2}[L(q^3)-2L(q^6)]+2[L(q^4)-3L(q^{12})]=\frac{b(q^2)b^2(q)}{b(-q)}.$$

$$144L_{1,2}(q^3)-\frac{1}{4}[L(q)-2L(q^2)]+\frac{1}{4}[L(q)-3L(q^3)]+\frac{27}{4}[L(q^3)-2L(q^6)]-4[L(q^4)-3L(q^{12})]=\frac{b(q)b(-q)b(q^2)}{b(q^4)}.$$

$$-36L_{1,2}(q^3)+\frac{11}{16}[L(q)-2L(q^2)]-\frac{7}{16}[L(q)-3L(q^3)]-\frac{45}{16}[L(q^3)-2L(q^6)]+[L(q^4)-3L(q^{12})]=\frac{b(q)b(q^4)b(q^2)}{b(-q)}.$$

$$-18L_{1,2}(q^3)-\frac{1}{16}[L(q)-2L(q^2)]+\frac{1}{16}[L(q)-3L(q^3)]-\frac{9}{16}[L(q^3)-2L(q^6)]-\frac{1}{4}[L(q^4)-3L(q^{12})]=\frac{b(q)b(-q)b^2(q^4)}{b^2(q^2)}.$$

$$-144L_{1,2}(q^3)+\frac{1}{4}[L(q)-2L(q^2)]+\frac{1}{4}[L(q)-3L(q^3)]-\frac{15}{4}[L(q^3)-2L(q^6)]=\frac{a(-q)c^2(q)}{c(q^2)}.$$

$$-288L_{1,2}(q^3)+5[L(q)-2L(q^2)]-3[L(q)-3L(q^3)]-21[L(q^3)-2L(q^6)]+8[L(q^4)-3L(q^{12})]=\frac{a(-q)c^2(-q)}{2^{\frac{1}{3}}c(q^2)}.$$

$$\frac{3}{2}[L(q)-2L(q^2)]-[L(q)-3L(q^3)]-\frac{9}{2}[L(q^3)-2L(q^6)]+2[L(q^4)-3L(q^{12})]=a^2(-q).$$

$$-\frac{1}{2}[L(q)-2L(q^2)]-\frac{1}{2}[L(q)-3L(q^3)]-\frac{3}{2}[L(q^3)-2L(q^6)]=\frac{a(q)c^2(q)}{c(q^2)}.$$

$$-\frac{5}{2}[L(q)-2L(q^2)]+\frac{3}{2}[L(q)-3L(q^3)]+\frac{3}{2}[L(q^3)-2L(q^6)]\ -4[L(q^4)-3L(q^{12})]=\frac{a(q)c^2(-q)}{2^{\frac{1}{3}}c(q^2)}.$$

$$-18L_{1,2}(q^3)-\frac{3}{32}[L(q)-2L(q^2)]+\frac{3}{32}[L(q)-3L(q^3)]-\frac{3}{32}[L(q^3)-2L(q^6)]=\frac{c(q)c(q^2)c(q^4)}{2^{\frac{1}{3}}c(-q)}.$$

$$18L_{1,2}(q^3)+\frac{1}{4}[L(q)-2L(q^2)]-\frac{3}{16}[L(q)-3L(q^3)]-\frac{3}{8}[L(q^3)-2L(q^6)]+\frac{1}{4}[L(q^4)-3L(q^{12})]=\frac{a(-q)c^2(q^2)}{2^{\frac{1}{3}}c(-q)}.$$

$$-36L_{1,2}(q^3)+\frac{1}{16}[L(q)-2L(q^2)]-\frac{1}{16}[L(q)-3L(q^3)]-\frac{15}{16}[L(q^3)-2L(q^6)]+\frac{1}{2}[L(q^4)-3L(q^{12})]=\frac{a(-q)c^2(q^4)}{c(q^2)}.$$

$$-\frac{1}{8}[L(q)-2L(q^2)]+\frac{3}{16}[L(q)-3L(q^3)]+\frac{3}{4}[L(q^3)-2L(q^6)]-\frac{1}{2}[L(q^4)-3L(q^{12})]=\frac{a(q)c^2(q^2)}{2^{\frac{1}{3}}c(-q)}.$$

■

Theorem III.3. *We have,*

$$1+\frac{3}{2}\sum_{r=1}^{\infty}\left[\frac{9r(-q)^{3r}}{1-(-q)^{3r}}+\frac{r(-q)^r}{1-(-q)^r}+\frac{3rq^r}{1-q^r}-\frac{14rq^{2r}}{1-q^{2r}}-\frac{9rq^{3r}}{1-q^{3r}}+\frac{12rq^{4r}}{1-q^{4r}}+\frac{18rq^{6r}}{1-q^{6r}}-\frac{36rq^{12r}}{1-q^{12r}}\right]=\frac{b(q)b^2(-q)b(q^4)}{b^2(q^2)}. \tag{44}$$

$$1+\frac{3}{2}\sum_{r=1}^{\infty}\left[\frac{9r(-q)^{3r}}{1-(-q)^{3r}}+\frac{5r(-q)^r}{1-(-q)^r}-\frac{rq^r}{1-q^r}+\frac{10rq^{2r}}{1-q^{2r}}-\frac{9rq^{3r}}{1-q^{3r}}-\frac{12rq^{4r}}{1-q^{4r}}-\frac{54rq^{6r}}{1-q^{6r}}+\frac{36rq^{12r}}{1-q^{12r}}\right]=\frac{b^2(q)b(q^2)}{b(-q)}. \tag{45}$$

$$1+12\sum_{r=1}^{\infty}\left[\frac{-2r(-q)^r}{1-(-q)^r}-\frac{2rq^r}{1-q^r}+\frac{11rq^{2r}}{1-q^{2r}}-\frac{9rq^{6r}}{1-q^{6r}}\right]=\frac{b(q)b(-q)b(q^2)}{b(q^4)}. \tag{46}$$

$$1+3\sum_{r=1}^{\infty}\left[\frac{2r(-q)^r}{1-(-q)^r}-\frac{rq^{2r}}{1-q^{2r}}-\frac{9rq^{6r}}{1-q^{6r}}\right]=\frac{b(q)b(q^4)b(q^2)}{b(-q)}. \tag{47}$$

$$1+3\sum_{r=1}^{\infty}\left[\frac{r(-q)^r}{1-(-q)^r}+\frac{rq^r}{1-q^r}-\frac{7rq^{2r}}{1-q^{2r}}+\frac{6rq^{4r}}{1-q^{4r}}+\frac{9rq^{6r}}{1-q^{6r}}-\frac{18rq^{12r}}{1-q^{12r}}\right]=\frac{b(q)b(-q)b^2(q^4)}{b^2(q^2)}. \tag{48}$$

$$1+6\sum_{r=1}^{\infty}\left[\frac{-3r(-q)^{3r}}{1-(-q)^{3r}}+\frac{r(-q)^r}{1-(-q)^r}-\frac{rq^r}{1-q^r}+\frac{6rq^{2r}}{1-q^{2r}}+\frac{3rq^{3r}}{1-q^{3r}}-\frac{4rq^{4r}}{1-q^{4r}}-\frac{18rq^{6r}}{1-q^{6r}}+\frac{12rq^{12r}}{1-q^{12r}}\right]=a^2(-q). \tag{49}$$

$$1+2\sum_{r=1}^{\infty}\left[\frac{-3r(-q)^{3r}}{1-(-q)^{3r}}+\frac{r(-q)^r}{1-(-q)^r}+\frac{3rq^r}{1-q^r}-\frac{16rq^{2r}}{1-q^{2r}}+\frac{3rq^{3r}}{1-q^{3r}}+\frac{12rq^{4r}}{1-q^{4r}}+\frac{24rq^{6r}}{1-q^{6r}}-\frac{36rq^{12r}}{1-q^{12r}}\right]=\frac{a(q)c^2(-q)}{2^{\frac{4}{3}}3c(q^2)}. \tag{50}$$

$$1+2\sum_{r=1}^{\infty}\left[\frac{3r(-q)^{3r}}{1-(-q)^{3r}}+\frac{3r(-q)^r}{1-(-q)^r}+\frac{rq^r}{1-q^r}-\frac{16rq^{2r}}{1-q^{2r}}+\frac{12rq^{4r}}{1-q^{4r}}+\frac{24rq^{6r}}{1-q^{6r}}-\frac{36rq^{12r}}{1-q^{12r}}\right]=\frac{a(-q)c^2(q)}{3c(q^2)}. \tag{51}$$

$$1+8\sum_{r=1}^{\infty}\left[\frac{r(-q)^r}{1-(-q)^r}-\frac{rq^{2r}}{1-q^{2r}}-\frac{3rq^{6r}}{1-q^{6r}}\right]=\frac{a(-q)c^2(-q)}{2^{\frac{4}{3}}3c(q^2)}. \tag{52}$$

$$1+8\sum_{r=1}^{\infty}\left[\frac{rq^r}{1-q^r}-\frac{rq^{2r}}{1-q^{2r}}-\frac{3rq^{6r}}{1-q^{6r}}\right]=\frac{a(q)c^2(q)}{3c(q^2)}. \tag{53}$$

$$\sum_{r=1}^{\infty}\left[\frac{3r(-q)^{3r}}{1-(-q)^{3r}}+\frac{r(-q)^r}{1-(-q)^r}+\frac{rq^r}{1-q^r}-\frac{9rq^{2r}}{1-q^{2r}}-\frac{3rq^{3r}}{1-q^{3r}}+\frac{4rq^{4r}}{1-q^{4r}}+\frac{15rq^{6r}}{1-q^{6r}}-\frac{12rq^{12r}}{1-q^{12r}}\right]r=\frac{2^{\frac{2}{3}}c(q)c(q^2)c(q^4)}{3c(-q)}. \tag{54}$$

$$\sum_{r=1}^{\infty}\left[\frac{-9r(-q)^{3r}}{1-(-q)^{3r}}-\frac{r(-q)^r}{1-(-q)^r}-\frac{3rq^r}{1-q^r}+\frac{22rq^{2r}}{1-q^{2r}}+\frac{9rq^{3r}}{1-q^{3r}}-\frac{12rq^{4r}}{1-q^{4r}}-\frac{42rq^{6r}}{1-q^{6r}}+\frac{36rq^{12r}}{1-q^{12r}}\right]=\frac{2^{\frac{5}{3}}a(-q)c^2(q^2)}{3c(-q)}. \tag{55}$$

$$\sum_{r=1}^{\infty}\Big[\frac{3r(-q)^{3r}}{1-(-q)^{3r}}+\frac{r(-q)^r}{1-(-q)^r}+\frac{rq^r}{1-q^r}-\frac{5rq^{2r}}{1-q^{2r}} -\frac{3rq^{3r}}{1-q^{3r}}+\frac{3rq^{6r}}{1-q^{6r}}\Big]=\frac{a(-q)c^2(q^4)}{3c(q^2)}. \tag{56}$$

$$\sum_{r=1}^{\infty}\Big[\frac{3r(-q)^{3r}}{1-(-q)^{3r}}-\frac{r(-q)^r}{1-(-q)^r}-\frac{3rq^r}{1-q^r}-\frac{2rq^{2r}}{1-q^{2r}} -\frac{3rq^{3r}}{1-q^{3r}}+\frac{12rq^{4r}}{1-q^{4r}}+\frac{30rq^{6r}}{1-q^{6r}}-\frac{36rq^{12r}}{1-q^{12r}}\Big] =\frac{2^{\frac{5}{3}}a(q)c^2(q^2)}{3c(-q)}. \tag{57}$$

Proof: Let us assert that,

$$C_1L_{1,2}(q)+C_2L_{1,2}(q^3)+C_3[L(-q)-2L(q^2)]+ C_4[L(q^2)-3L(q^6)]+C_5[L(q^4)-2L(q^{12})] =\frac{b(q)b(q^4)b^2(-q)}{b^2(q^2)}. \tag{58}$$

Restructuring the above expression by (p,k) parametrization using Lemma II.3 and by equating the coeffecients of terms containing k^2, pk^2, p^2k^2, p^3k^2 and p^4k^2 on either sides, a system of linear equations are obtained.

$$\begin{pmatrix} 0 & 0 & -1 & -2 & -2 \\ \frac{1}{2} & 0 & 10 & -4 & -4 \\ \frac{3}{4} & 0 & 12 & -6 & 0 \\ \frac{3}{8} & \frac{1}{8} & 4 & -4 & 2 \\ \frac{1}{16} & \frac{1}{16} & 2 & -2 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{pmatrix}=\begin{pmatrix} 1 \\ \frac{7}{2} \\ \frac{3}{2} \\ -4 \\ -2 \end{pmatrix}.$$

On solving the above system, we get

$$C_1=9,\ C_2=-27,\ C_3=-\frac{1}{4},\ C_4=\frac{3}{8},\ \text{and}\ C_5=-\frac{3}{4}.$$

Substituting these values in (58) and simplifying using Definition II.1 we get (44). Correspondingly, the following identities are derived.

$$-3L_{1,2}(q)-27L_{1,2}(q^3)-\frac{1}{4}[L(-q)-2L(q^2)]+\frac{3}{4}[L(q^2) -3L(q^6)]-\frac{9}{8}[L(q^4)-2L(q^{12})]=\frac{b(q^2)b^2(q)}{b(-q)}.$$

$$-48L_{1,2}(q)+2[L(-q)-2L(q^2)]-\frac{3}{2}[L(q^2)-3L(q^6)] =\frac{b(q)b(q^2)b(-q)}{b(q^4)}.$$

$$-\frac{1}{4}[L(-q)-2L(q^2)]-\frac{3}{8}[L(q^2)-3L(q^6)] =\frac{b(q)b(q^2)b(q^4)}{b(-q)}.$$

$$6L_{1,2}(q)-\frac{1}{4}[L(-q)-2L(q^2)]-\frac{3}{8}[L(q^2)-3L(q^6)] +\frac{3}{4}[L(q^4)-2L(q^{12})]=\frac{b(q)b^2(q^4)b(-q)}{b^2(q^2)}.$$

$$-12L_{1,2}(q)+36L_{1,2}(q^3)-\frac{3}{2}[L(q^2)-3L(q^6)] +[L(q^4)-2L(q^{12})]=a^2(-q).$$

$$72L_{1,2}(q)+72L_{1,2}(q^3)-2[L(-q)-2L(q^2)]+4[L(q^2) -3L(q^6)]-6[L(q^4)-2L(q^{12})]=\frac{a(q)c^2(-q)}{2^{\frac{1}{3}}c(q^2)}.$$

$$12L_{1,2}(q)-36L_{1,2}(q^3)-[L(-q)-2L(q^2)]+2[L(q^2) -3L(q^6)]-3[L(q^4)-2L(q^{12})]=\frac{a(-q)c^2(q)}{c(q^2)}.$$

$$-2[L(-q)-2L(q^2)]-[L(q^4)-2L(q^{12})] =\frac{a(-q)c^2(-q)}{2^{\frac{1}{3}}c(q^2)}.$$

$$48L_{1,2}(q)-[L(-q)-2L(q^2)]-[L(q^2)-3L(q^6)] =\frac{a(q)c^2(q)}{c(q^2)}.$$

$$3L_{1,2}(q)-9L_{1,2}(q^3)-\frac{1}{8}[L(-q)-2L(q^2)]+\frac{5}{16}[L(q^2) -3L(q^6)]-\frac{1}{4}[L(q^4)-2L(q^{12})]=\frac{c(q)c(q^2)c(q^4)}{2^{\frac{1}{3}}c(-q)}.$$

$$-\frac{9}{2}L_{1,2}(q)+\frac{27}{2}L_{1,2}(q^3)+\frac{1}{8}[L(-q)-2L(q^2)] -\frac{7}{16}[L(q^2)-3L(q^6)]+\frac{3}{8}[L(q^4)-2L(q^{12})] =\frac{a(-q)c^2(q^2)}{2^{\frac{1}{3}}c(-q)}.$$

$$6L_{1,2}(q)-18L_{1,2}(q^3)-\frac{1}{4}[L(-q)-2L(q^2)] +\frac{1}{8}[L(q^2)-3L(q^6)]=\frac{a(-q)c^2(q^4)}{c(q^2)}.$$

$$-\frac{9}{2}L_{1,2}(q)-\frac{9}{2}L_{1,2}(q^3)+\frac{1}{8}[L(-q)-2L(q^2)]+ \frac{5}{16}[L(q^2)-3L(q^6)]-\frac{3}{8}[L(q^4)-2L(q^{12})] =\frac{a(q)c^2(q^2)}{2^{\frac{1}{3}}c(-q)}.$$

■

Theorem III.4. *One has,*

$$1+3\sum_{r=1}^{\infty}\left[\frac{4r(-q)^r}{1-(-q)^r}+\frac{rq^r}{1-q^r}-\frac{4rq^{2r}}{1-q^{2r}}-\frac{9rq^{3r}}{1-q^{3r}}\right]=\frac{b^2(q)b(q^2)}{b(-q)}. \quad (59)$$

$$1-12\sum_{r=1}^{\infty}\left[\frac{2r(-q)^r}{1-(-q)^r}+\frac{2rq^r}{1-q^r}-\frac{11rq^{2r}}{1-q^{2r}}+\frac{9rq^{6r}}{1-q^{6r}}\right]=\frac{b(q)b(-q)b(q^2)}{b(q^4)}. \quad (60)$$

$$1+3\sum_{r=1}^{\infty}\left[\frac{2r(-q)^r}{1-(-q)^r}-\frac{rq^{2r}}{1-q^{2r}}-\frac{9rq^{6r}}{1-q^{6r}}\right]=\frac{b(q)b(q^2)b(q^4)}{b(-q)}. \quad (61)$$

$$1+8\sum_{r=1}^{\infty}\left[\frac{r(-q)^r}{1-(-q)^r}-\frac{rq^{2r}}{1-q^{2r}}-\frac{3rq^{6r}}{1-q^{6r}}\right]=\frac{a(-q)c^2(-q)}{32^{\frac{4}{3}}c(q^2)}. \quad (62)$$

$$1+8\sum_{r=1}^{\infty}\left[\frac{rq^r}{1-q^r}-\frac{rq^{2r}}{1-q^{2r}}-\frac{3rq^{6r}}{1-q^{6r}}\right]=\frac{a(q)c^2(q)}{3c(q^2)}. \quad (63)$$

Proof: Consider the following.

$$C_1L_{1,3}(q)+C_2[L(-q)-2L(q^2)]+C_3[L(q)+ -2L(q^2)]C_4[L(q)-3L(q^3)]+C_5[L(q^3)-2L(q^6)] =\frac{b^2(q)b(q^2)}{b(-q)}.$$

We reorganise the above equation by (p,k) parametrization using Lemma II.3. Subsequently a set of non-homogeneous linear equations is obtained by equating the coeffecients of terms containing k^2, pk^2, p^2k^2, p^3k^2 and p^4k^2 on either side, we obtain

$$\begin{pmatrix} 0 & -1 & -1 & -2 & -1\\ \frac{1}{2} & 10 & -14 & -16 & -2\\ \frac{3}{4} & 12 & -24 & -36 & 0\\ \frac{3}{8} & 4 & -14 & -16 & -2\\ \frac{1}{16} & 2 & -1 & -2 & -1 \end{pmatrix}\begin{pmatrix} C_1\\ C_2\\ C_3\\ C_4\\ C_5\end{pmatrix}=\begin{pmatrix} 1\\ -\frac{5}{2}\\ \frac{3}{2}\\ \frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}.$$

The above system is consistent and has infinite number of solutions. They are represented in generalised form as below:

$$C_1=-24-48u,\ C_2=u,\ C_3=-\frac{1}{4}-u,\ C_4=-\frac{3}{8} \text{ and } C_5=0,$$

where $u\in\mathbb{R}$.
Substituting these values in (III) and simplifying using Definition II.1 we get (59) Similarly, we derive the following identities.

$$48(1-u)L_{1,3}(q)+u[L(-q)-2L(q^2)]+\frac{1}{4}(11-4u)[L(q)-2L(q^2)]-\frac{3}{4}[L(q)-3L(q^3)]-\frac{9}{4}[L(q^3)-2L(q^6)]=\frac{b(q)b(-q)b(q^2)}{b(q^4)}.$$

$$-12(1+4u)L_{1,3}(q)+u[L(-q)-2L(q^2)]-\frac{1}{16}(1+16u)[L(q)-2L(q^2)]-\frac{3}{16}[L(q)-3L(q^3)]-\frac{9}{16}[L(q^3)-2L(q^6)]=\frac{b(q)b(q^2)b(q^4)}{b(-q)}.$$

$$-48(2+u)L_{1,3}(q)+u[L(-q)-2L(q^2)]-(1+u)[L(q)-2L(q^2)]-[L(q)-3L(q^3)]-3[L(q^3)-2L(q^6)]=\frac{a(-q)c^2(-q)}{2^{\frac{1}{3}}c(q^2)}.$$

$$-48uL_{1,3}(q)+u[L(-q)-2L(q^2)]-\frac{1}{2}(1+2u)[L(q)-2L(q^2)]-\frac{1}{2}[L(q)-3L(q^3)]-\frac{3}{2}[L(q^3)-2L(q^6)]=\frac{a(q)c^2(q)}{c(q^2)}.$$

■

Theorem III.5. *We have,*

$$1+3\sum_{r=1}^{\infty}\left[\frac{9r(-q)^{3r}}{1-(-q)^{3r}}-\frac{r(-q)^r}{1-(-q)^r}+\frac{2rq^{2r}}{1-q^{2r}}-18\frac{rq^{6r}}{1-q^{6r}}\right]=\frac{b(q)b^2(-q)b(q^4)}{b^2(q^2)}. \quad (64)$$

$$1+3\sum_{r=1}^{\infty}\left[\frac{3r(-q)^{3r}}{1-(-q)^{3r}}+\frac{3r(-q)^r}{1-(-q)^r}+\frac{2rq^{2r}}{1-q^{2r}}-\frac{6rq^{3r}}{1-q^{3r}}-\frac{4rq^{4r}}{1-q^{4r}}-\frac{18rq^{6r}}{1-q^{6r}}+\frac{12rq^{12r}}{1-q^{12r}}\right]=\frac{b^2(q)b(q^2)}{b(-q)}. \quad (65)$$

$$1+12\sum_{r=1}^{\infty}\left[\frac{-6r(-q)^{3r}}{1-(-q)^{3r}}-\frac{rq^{2r}}{1-q^{2r}}-\frac{6rq^{3r}}{1-q^{3r}}+\frac{8rq^{4r}}{1-q^{4r}}+\frac{27rq^{6r}}{1-q^{6r}}-\frac{24rq^{12r}}{1-q^{12r}}\right]=\frac{b(q)b(-q)b(q^2)}{b(q^4)}. \quad (66)$$

$$1+3\sum_{r=1}^{\infty}\left[\frac{2r(-q)^r}{1-(-q)^r}-\frac{rq^{2r}}{1-q^{2r}}-\frac{9rq^{6r}}{1-q^{6r}}\right]=\frac{b(q)b(q^4)b(q^2)}{b(-q)}. \quad (67)$$

$$1+3\sum_{r=1}^{\infty}\Big[\frac{3r(-q)^{3r}}{1-(-q)^{3r}}-\frac{rq^{2r}}{1-q^{2r}}+\frac{3rq^{3r}}{1-q^{3r}}+\frac{2rq^{4r}}{1-q^{4r}}-\frac{9rq^{6r}}{1-q^{6r}}-\frac{6rq^{12r}}{1-q^{12r}}\Big]=\frac{b(q)b(-q)b^2(q^4)}{b^2(q^2)}. \quad (68)$$

$$1+12\sum_{r=1}^{\infty}\Big[\frac{-3r(-q)^{3r}}{1-(-q)^{3r}}+\frac{r(-q)^{r}}{1-(-q)^{r}}\Big]=a^2(-q). \quad (69)$$

$$1+4\sum_{r=1}^{\infty}\Big[\frac{3r(-q)^{3r}}{1-(-q)^{3r}}-\frac{r(-q)^{r}}{1-(-q)^{r}}+\frac{rq^{2r}}{1-q^{2r}}+\frac{6rq^{3r}}{1-q^{3r}}-\frac{15rq^{6r}}{1-q^{6r}}=\frac{2^{-\frac{4}{3}}a(q)c^2(-q)}{3c(q^2)}. \quad (70)$$

$$1+4\sum_{r=1}^{\infty}\Big[\frac{3r(-q)^{3r}}{1-(-q)^{3r}}+\frac{r(-q)^{r}}{1-(-q)^{r}}-\frac{5rq^{2r}}{1-q^{2r}}+\frac{4rq^{4r}}{1-q^{4r}}+\frac{3rq^{6r}}{1-q^{6r}}-\frac{12rq^{12r}}{1-q^{12r}}\Big]=\frac{a(-q)c^2(q)}{3c(q^2)}. \quad (71)$$

$$1+8\sum_{r=1}^{\infty}\Big[\frac{r(-q)^{r}}{1-(-q)^{r}}-\frac{rq^{2r}}{1-q^{2r}}-\frac{3rq^{6r}}{1-q^{6r}}\Big]=\frac{2^{-\frac{4}{3}}a(-q)c^2(-q)}{3c(q^2)}. \quad (72)$$

$$1+8\sum_{r=1}^{\infty}\Big[\frac{3r(-q)^{3r}}{1-(-q)^{3r}}-\frac{r(-q)^{r}}{1-(-q)^{r}}+\frac{5rq^{2r}}{1-q^{2r}}+\frac{3rq^{3r}}{1-q^{3r}}-\frac{4rq^{4r}}{1-q^{4r}}-\frac{21rq^{6r}}{1-q^{6r}}+\frac{12rq^{12r}}{1-q^{12r}}\Big]=\frac{a(q)c^2(q)}{3c(q^2)}. \quad (73)$$

$$\sum_{r=1}^{\infty}\Big[\frac{2r(-q)^{3r}}{1-(-q)^{3r}}-\frac{rq^{2r}}{1-q^{2r}}-\frac{rq^{6r}}{1-q^{6r}}\Big]=\frac{2^{\frac{2}{3}}c(q)c(q^2)c(q^4)}{9c(-q)}. \quad (74)$$

$$\sum_{r=1}^{\infty}\Big[\frac{-9r(-q)^{3r}}{1-(-q)^{3r}}+\frac{r(-q)^{r}}{1-(-q)^{r}}+\frac{2rq^{2r}}{1-q^{2r}}+\frac{6rq^{6r}}{1-q^{6r}}\Big]=\frac{2^{\frac{2}{3}}a(-q)c^2(q^2)}{3c(-q)}. \quad (75)$$

$$\sum_{r=1}^{\infty}\Big[\frac{6r(-q)^{3r}}{1-(-q)^{3r}}+\frac{rq^{2r}}{1-q^{2r}}-\frac{4rq^{4r}}{1-q^{4r}}-\frac{15rq^{6r}}{1-q^{6r}}+\frac{12rq^{12r}}{1-q^{12r}}\Big]=\frac{a(-q)c^2(q^4)}{3c(q^2)}. \quad (76)$$

$$\sum_{r=1}^{\infty}\Big[\frac{-3r(-q)^{3r}}{1-(-q)^{3r}}+\frac{r(-q)^{r}}{1-(-q)^{r}}-\frac{10rq^{2r}}{1-q^{2r}}-\frac{6rq^{3r}}{1-q^{3r}}+\frac{12rq^{4r}}{1-q^{4r}}+\frac{42rq^{6r}}{1-q^{6r}}-\frac{36rq^{12r}}{1-q^{12r}}\Big]=\frac{2^{\frac{2}{3}}a(q)c^2(q^2)}{3c(-q)}. \quad (77)$$

Proof: Let us affirm that

$$C_1L_{1,2}(q^3)+C_2[L(-q)-2L(q^2)]+C_3[L(q^2)-3L(q^6)]+C_4[L(q^3)-2L(q^6)]+C_5[L(q^4)-2L(q^{12})]=\frac{b(q)b(q^4)b^2(-q)}{b^2(q^2)}. \quad (78)$$

Reframing the above expression by (p,k) parametrization using Lemma II.3 and by equating the coeffecients of terms containing k^2, pk^2, p^2k^2, p^3k^2 and p^4k^2 on either sides, a set of non-homogeneous linear equations is obtained.

$$\begin{pmatrix} 0 & -1 & -2 & -2 & -1\\ 0 & -2 & -4 & -4 & 10\\ 0 & 0 & -6 & 0 & 12\\ \frac{1}{8} & -2 & -4 & 2 & 4\\ \frac{1}{16} & -1 & -2 & -\frac{1}{2} & 2 \end{pmatrix}\begin{pmatrix} C_1\\ C_2\\ C_3\\ C_4\\ C_5\end{pmatrix}=\begin{pmatrix} 1\\ \frac{7}{2}\\ \frac{3}{2}\\ -4\\ -2\end{pmatrix}.$$

On solving this system, we obtain

$$C_1=-54,\ C_2=\frac{1}{8},\ C_3=0,\ C_4=-\frac{9}{8}\ \text{and}\ C_5=0.$$

Substituting the above entries in (78) and simplifying using Definition II.1, we get, (64). Likewise, the following identities are obtained.

$$-18L_{1,2}(q^3)-\frac{3}{8}[L(-q)-2L(q^2)]-[L(q^2)-3L(q^6)]+\frac{3}{8}[L(q^3)-2L(q^6)]+\frac{1}{2}[L(q^4)-2L(q^{12})]=\frac{b^2(q)b(q^2)}{b(-q)}.$$

$$144L_{1,2}(q^3)+\frac{1}{2}[L(q^2)-3L(q^6)]+6[L(q^3)-2L(q^6)]-4[L(q^4)-2L(q^{12})]=\frac{b(q)b(-q)b(q^2)}{b(q^4)}.$$

$$-\frac{1}{4}[L(-q)-2L(q^2)]-\frac{3}{8}[L(q^2)-3L(q^6)]=\frac{b(q)b(q^4)b(q^2)}{b(-q)}.$$

$$-18L_{1,2}(q^3)+\frac{1}{8}[L(q^2)-3L(q^6)]-\frac{3}{4}[L(q^3)-2L(q^6)] -\frac{1}{4}[L(q^4)-2L(q^{12})]=\frac{b(q)b(-q)b^2(q^4)}{b^2(q^2)}.$$

$$72L_{1,2}(q^3)-\frac{1}{2}[L(-q)-2L(q^2)]-[L(q^2)-3L(q^6)] +\frac{3}{2}[L(q^3)-2L(q^6)]=a^2(-q).$$

$$-144L_{1,2}(q^3)+[L(-q)-2L(q^2)]+[L(q^2)-3L(q^6)] -9[L(q^3)-2L(q^6)]=\frac{a(q)c^2(-q)}{2^{\frac{1}{3}}c(q^2)}.$$

$$-72L_{1,2}(q^3)-\frac{1}{2}[L(-q)-2L(q^2)]+\frac{3}{2}[L(q^2)-3L(q^6)] \frac{3}{2}[L(q^3)-2L(q^6)]-2[L(q^4)-2L(q^{12})]=\frac{a(-q)c^2(q)}{c(q^2)}.$$

$$-2[L(-q)-2L(q^2)]-2[L(q^2)-3L(q^6)] =\frac{a(-q)c^2(-q)}{2^{\frac{1}{3}}c(q^2)}.$$

$$-144L_{1,2}(q^3)+[L(-q)-2L(q^2)]-3[L(q^2)-3L(q^6)] -6[L(q^3)-2L(q^6)]+4[L(q^4)-2L(q^{12})]=\frac{a(q)c^2(q)}{c(q^2)}.$$

$$-18L_{1,2}(q^3)+\frac{3}{16}[L(q^2)-3L(q^6)]-\frac{3}{8}[L(q^3)-2L(q^6)] =\frac{c(q)c(q^2)c(q^4)}{2^{\frac{1}{3}}c(-q)}.$$

$$27L_{1,2}(q^3)-\frac{1}{16}[L(-q)-2L(q^2)]-\frac{1}{4}[L(q^2)-3L(q^6)] +\frac{9}{16}[L(q^3)-2L(q^6)]=\frac{a(-q)c^2(q^2)}{2^{\frac{1}{3}}c(-q)}.$$

$$-36L_{1,2}(q^3)-\frac{1}{8}[L(q^2)-3L(q^6)]-\frac{3}{4}[L(q^3)-2L(q^6)] +\frac{1}{2}[L(q^4)-2L(q^{12})]=\frac{a(-q)c^2(q^4)}{c(q^2)}.$$

$$9L_{1,2}(q^3)-\frac{1}{16}[L(-q)-2L(q^2)]+\frac{1}{2}[L(q^2)-3L(q^6)] +\frac{9}{16}[L(q^3)-2L(q^6)]-\frac{3}{4}[L(q^4)-2L(q^{12})] =\frac{a(q)c^2(q^2)}{2^{\frac{1}{3}}c(-q)}.$$

■

IV. SERIES CONTAINING $M(q)$ BESIDES $L(q)$ AND $L(-q)$

In the subsequent section, we deal with the relation among Ramanujan type Eisenstein series, $L(q)$, $M(q)$ and cubic theta function. As an application of the same, we evaluate the convolution sum using one of the deduced identities.

Theorem IV.1. *For any $u \in \mathbb{R}$, the following relations among the series and cubic theta functions hold:*

$$(1-u)-3\sum_{r=1}^{\infty}\left[\frac{r^3q^r}{1-q^r}-\frac{(81-16u)r^3q^{3r}}{1-q^{3r}} +\frac{64ur^3q^{6r}}{1-q^{6r}}\right]+u\left[1+24\sum_{r=1}^{\infty}\left[\frac{rq^{3r}}{1-q^{3r}}-\frac{2rq^{6r}}{1-q^{6r}}\right]\right]^2 =a(q)b^3(q). \tag{79}$$

$$(1-u)-3\sum_{r=1}^{\infty}\left[\frac{r^3q^{2r}}{1-q^{2r}}+\frac{16ur^3q^{3r}}{1-q^{3r}}-\frac{(81-64u)r^3q^{6r}}{1-q^{6r}}\right]+ u\left[1+24\sum_{r=1}^{\infty}\left[\frac{rq^{3r}}{1-q^{3r}}-\frac{2rq^{6r}}{1-q^{6r}}\right]\right]^2=a(q^2)b^3(q^2). \tag{80}$$

$$(1-u)+6\sum_{r=1}^{\infty}\left[\frac{3r^3q^r}{1-q^r}-\frac{8r^3q^{2r}}{1-q^{2r}}-\frac{(27+8u)r^3q^{3r}}{1-q^{3r}} +\frac{(72-32u)r^3q^{6r}}{1-q^{6r}}\right]+u\left[1+24\sum_{r=1}^{\infty}\left[\frac{rq^{3r}}{1-q^{3r}} -\frac{2rq^{6r}}{1-q^{6r}}\right]\right]^2=a^3(q)a(q^2). \tag{81}$$

$$(1-u)+24\sum_{r=1}^{\infty}\left[\frac{r^3q^r}{1-q^r}+\frac{(9-2u)r^3q^{3r}}{1-q^{3r}}-\frac{8ur^3q^{6r}}{1-q^{6r}}\right]+ u\left[1+24\sum_{r=1}^{\infty}\left[\frac{rq^{3r}}{1-q^{3r}}-\frac{2rq^{6r}}{1-q^{6r}}\right]\right]^2 =(3a(q^3)-2b(q))^2. \tag{82}$$

$$(1-u)+6\sum_{r=1}^{\infty}\left[\frac{r^3q^r}{1-q^r}-\frac{6r^3q^{2r}}{1-q^{2r}}-\frac{(9+8u)r^3q^{3r}}{1-q^{3r}} +\frac{(54-32u)r^3q^{6r}}{1-q^{6r}}\right]+u\left[1+24\sum_{r=1}^{\infty}\left[\frac{rq^{3r}}{1-q^{3r}} -\frac{2rq^{6r}}{1-q^{6r}}\right]\right]^2=a(q)a^3(q^2). \tag{83}$$

Proof: Consider the following expression.

$$C_1M(q)+C_2M(q^2)+C_3M(q^3)+C_4M(q^6)+C_5M(q^{12}) +C_6\left(3L(q^3)-4L(q^4)\right)^2+C_7\left(L(q^3)-2L(q^6)\right)^2 +C_8\left(L(q^4)-3L(q^{12})\right)^2=a(q)b^3(q). \tag{84}$$

Subsequently upon (p,k) parametrization of the above expression using Lemma II.3 and by equating the coefficients of $k^4, pk^4, p^2k^4,\ p^3k^4, p^4k^4, p^5k^4, p^6k^4, p^7k^4$ and p^8k^4 on either sides, we obtain

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 \\ 124 & 4 & 4 & 4 & 4 & 4 & 4 & 16 \\ 964 & 64 & 4 & 4 & 4 & 4 & 4 & 16 \\ 2788 & 178 & 28 & -2 & -2 & 16 & 4 & -8 \\ 3910 & 235 & 70 & -5 & -5 & 28 & 10 & -14 \\ 2788 & 178 & 28 & -2 & -2 & -8 & 4 & 4 \\ 964 & 64 & 4 & 4 & \frac{1}{4} & 64 & 4 & 4 \\ 124 & 4 & 4 & 4 & \frac{1}{4} & -32 & 4 & -2 \\ 1 & 1 & 1 & 1 & \frac{1}{16} & 4 & 1 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ \frac{13}{4} \\ -\frac{17}{4} \\ -8 \\ -\frac{17}{4} \\ \frac{13}{4} \\ 4 \\ 1 \end{pmatrix}.$$

Interestingly, we note that, the system results in infinite number of solutions,

$$C_1 = -\frac{1}{80},\ C_2 = 0,\ C_3 = \left(\frac{81}{80} - \frac{1}{5}u\right),\ C_4 = -\frac{4}{5}u, \quad C_5 = 0,\ C_6 = 0,\ C_7 = u,\ C_8 = 0,$$

where $u \in \mathbb{R}$.
Substituting these values in (84) yields, (79). The remaining results are obtained by applying Definition II.1.

$$-\frac{1}{80}M(q^2) - \frac{1}{5}uM(q^3) + \left[\frac{81}{80} - \frac{4}{5}u\right]M(q^6) + u\left(L(q^3) - 2L(q^6)\right)^2 = a(q^2)b^3(q^2).$$

$$\frac{3}{40}M(q) - \frac{1}{5}M(q^2) + \left[-\frac{27}{40} - \frac{1}{5}u\right]M(q^3) + \left[\frac{9}{5} - \frac{4}{5}u\right] M(q^6) + u\left(L(q^3) - 2L(q^6)\right)^2 = a^3(q)a(q^2).$$

$$\frac{1}{10}M(q) + \left[\frac{9}{10} - \frac{1}{5}u\right]M(q^3) - \frac{4}{5}uM(q^6) + u\left(L(q^3) - 2L(q^6)\right)^2 = \left(3a(q^3) - 2b(q)\right)^2.$$

$$\frac{1}{40}M(q) - \frac{3}{20}M(q^2) - \left[\frac{9}{40} + \frac{1}{5}u\right]M(q^3) + \left[\frac{27}{20} - \frac{4}{5}u\right] M(q^6) + u\left(L(q^3) - 2L(q^6)\right)^2 = a(q)a^3(q^2).$$

■

V. Evaluation of Convolution sum $\sum_{r=3i+6j} \delta(i)\delta(j)$

Let $\mathbb{N}$ be the set of natural numbers. For $k, r \in \mathbb{N}$, we define

$$\delta_k(r) = \sum_{d/r} d^k.$$

where d runs through the non-negative integeral divisors of r.
For $i,\ j,\ r \in \mathbb{N}$ with $i \leq j$, the convolution sum is defined as,

$$W_{i,j}(r) := \sum_{il+jk=r} \delta(l)\delta(k).$$

For all r, the convolution $\sum\limits_{li+kj=r} \delta(i)\delta(j)$ has been explicitly evaluated for various i and j values, by Alaca et al. [1], [2], [3], [4], [5], [6], H. C. Vidya and B. R. Srivtasa Kumar [18], Williams et al. [20], [21] and E. X. W. Xia and O. X. M. Yao [22]. The claims of J. W. L. Glaisher [12] substantiates our proof,

$$L^2(q) = 1 + \sum_{r=1}^{\infty}(240\delta_3(r) - 288r\delta_1(r))q^r. \qquad (85)$$

Theorem V.1. *For any $r \in \mathbb{N}$ and $u \in \mathbb{R} - \{0\}$, we have*

$$\sum_{3i+6j=r} \delta(i)\delta(j) = \frac{1}{24}\delta_1\left(\frac{r}{3}\right) + \frac{1}{24}\delta_1\left(\frac{r}{6}\right) + \frac{1}{12}\delta_3\left(\frac{r}{3}\right) + \frac{1}{3}\delta_3\left(\frac{r}{6}\right) - \frac{1}{8}r\left(\delta_1\left(\frac{r}{3}\right) + 4\delta_1\left(\frac{r}{6}\right)\right) - \frac{1}{2304u}\left(3\delta_3(r) + 243\delta_3\left(\frac{r}{3}\right) + A(r)\right), \qquad (86)$$

where $1 + \sum_{r=1}^{\infty} A(r)q^r$

$$= \frac{(-q;q^2)_\infty^2 (q^2;q^2)_\infty (q^3;q^6)_\infty^2 (q^6;q^6)_\infty (q;q)_\infty^6}{(q^3;q^3)_\infty^3} + 4q\frac{(q^4;q^4)_\infty (q^{12};q^{12})_\infty (q;q)_\infty^6}{(q^2;q^4)_\infty (q^6;q^{12})_\infty (q^3;q^3)_\infty^3}.$$

Proof: Applying Definition II.1 to equation (84) and reorganizing we obtain,

$$u - \frac{1}{80}\left[1 + 240\sum_{r=1}^{\infty}\delta_3(r)q^r\right] + \left[\frac{81}{80} - \frac{1}{5}u\right] \left[1 + 240\sum_{r=1}^{\infty}\delta_3\left(\frac{r}{3}\right)q^r\right] - \frac{4}{5}u\left[1 + 240\sum_{r=1}^{\infty}\delta_3\left(\frac{r}{6}\right)q^r\right] + u\left[\sum_{r=1}^{\infty}\left[240\delta_3\left(\frac{r}{3}\right) - 288r\delta_1\left(\frac{r}{3}\right)\right]q^r\right] + u\left[\sum_{r=1}^{\infty}\left[960\delta_3\left(\frac{r}{6}\right) - 1152r\delta_1\left(\frac{r}{6}\right)\right]q^r\right] + 96u\sum_{r=1}^{\infty}\delta_1\left(\frac{r}{3}\right)q^r + 96\sum_{r=1}^{\infty}\delta_1\left(\frac{r}{6}\right)q^r - 2304u\sum_{3i+6j=r}\delta(i)\delta(j)q^r = 1 + \sum_{r=1}^{\infty}A(r)q^r,$$

where $1 + \sum_{r=1}^{\infty} A(r)q^r = a(q)b^3(q)$.
Comparing the coefficient of q^r and thereafter rearranging the terms, we get, (86). ■

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