A Novel Portfolio Based on Interval-Valued Intuitionistic Fuzzy AHP with Improved Combination Weight Method and New Score Function

Xue Deng, Fengting Geng, Jianxin Yang

Abstract—**The classical Analytic Hierarchy Process (AHP) requires an exact value to compare the relative importance of two attributes, but experts often cannot obtain an accurate assessment of every attribute in the decision-making process, since there are always some uncertainty and hesitation. Compared with classical AHP, our newly defined interval-valued intuitionistic fuzzy AHP has accurately described the vagueness and uncertainty. In the decision matrix, the real numbers are substituted by fuzzy numbers. In addition, each expert will make different evaluations according to different experiences for each attribute in the subjective weighting method, which neglects objective factors and then generates some deviations in some cases. This paper provides two ways to make up for this disadvantage. On the one hand, by combining the interval-valued intuitionistic fuzzy AHP with entropy weight, an improved combination weighting method is proposed, which can overcome the limitations of the unilateral weighted method only considering the objective or subjective factors. On the other hand, a new score function is presented by adjusting the parameters, which can overcome the invalidity of some existing score functions. In theory, some theorems and properties for the new score functions are given with strictly mathematical proofs to validate their rationality and effectiveness. In application, a novel fuzzy portfolio is proposed based on the improved combination weighted method and new score function. A numerical example shows that the results of our new score function are consistent with those of most existing score functions, which verifies that our model is feasible and effective.**

Index Terms — **Interval-valued intuitionistic fuzzy AHP, Portfolio, Score function, Combination weighted method.**

I. Introduction

THE portfolio theory was first proposed by Markowitz [1]. He used the average rate of historical return to measure the expected return level of investments and the variance of the rate of return to measure the investment risk. On this basis, the mean-variance model was established to explain how to achieve the best balance between returns and risks through the selection of securities portfolios. The central problem of portfolio theory is how to choose the combination of return and risk in the decision of securities investment. In this way, the expected benefits can be maximized at a given level of expected risk, or the expected risks can be minimized at a given level of expected return. Makovitz assumed that the income distribution was symmetric and used the variance to depict the risk. But in actual cases, variance is not always accurate to describe the risk, and the distribution of income is not necessarily symmetric. Markowitz [2] pointed out that there are some limitations to measuring risk with variance and then he proposed semi-variance to measure risk. Konno and Suzuki [3] studied the mean variance-skewness model which is valuable in the case of asymmetric return distribution. Konno and Yamazaki [4] used expected absolute deviation to describe risks and established a linear programming model for portfolio selection, which is called the mean absolute deviation model. Athayde and Flores [5] considered asset allocation under the condition of asymmetric distribution. Jondeau and Rockinger [6] considered the non-normal distribution and time-varying characteristics of the rate of return. In their paper, Taylor expansion was performed on the final expected income and the first four high-order moments are taken, and the first-order condition was used to optimize the asset allocation. Li [7] constructed an asymmetry robust mean absolute deviation (ARMAD) model that takes the asymmetry distribution of returns into consideration. Deng and Yuan [8] constructed a hybrid multi-objective portfolio model which considers fuzzy return status, systematic risk, non-systematic risk and entropy. Min et al. [9] developed the less conservative robust Omega portfolio and designed a two-stage portfolio structure.

The concept of entropy originating from thermodynamics reflects the degree of chaos in a system. The smaller the corresponding entropy value of the system, the more stable the system. Zadeh [10] put forward the concept of fuzzy entropy for the first time in 1965. Then many scholars offered different definitions of interval intuitionistic fuzzy entropy. In case attributes were completely unknown, most scholars used the entropy weight method to determine weights. Burillo and Bustince [11] proposed the notions of entropy to measure the degree of intuitionism of interval-valued intuitionistic fuzzy sets and intuitionistic fuzzy sets.

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Szimidt and Kacprzyk [12] proposed a non-probabilistic-type entropy measure with a geometric interpretation of intuitionistic fuzzy sets. Deng and Liu [13] used an improved entropy method to calculate the weight of each indicator in order to conduct a quantitative analysis of 20 indicator variables which can be divided into four digital economic types in Guangdong Province from 2015 to 2018. Deng and Zhao [14] firstly parametrized the fuzzy entropy for triangular fuzzy numbers based on credibility theory.

The analytic hierarchy process [15] is a multi-criteria decision-making method combining qualitative and quantitative analysis, which is practical in the case of the complex target structure and lack of necessary data. Because the fuzziness of expert judgment is not considered when evaluating the weight distribution of various factors, Atanassov [16][17] successively proposed the intuitionistic fuzzy set and the interval-valued intuitionistic fuzzy set to effectively solve the double fuzzy situation in life. Sadiq [18] applied the intuitionistic fuzzy set to the AHP and then construct the intuitionistic fuzzy AHP method. Zheng et al. [19] studied the mail author identification system based on AHP.

The sort function is a means to measure the intuitionistic fuzzy number. In order to compare the advantages and disadvantages of two interval-valued intuitionistic fuzzy numbers, Xu [20] defined the score function and the accurate function of interval-valued intuitionistic fuzzy numbers as the sort function, and gave the corresponding sorting rules. However, Xu's ordering rules are invalid for some interval-valued intuitionistic fuzzy numbers. Therefore, many scholars put forward new sort functions from different perspectives. Ye [21] proposed the accurate function from the perspective of hesitation. Nayagam and Sivaraman et al. [22][23], Gao [24], Kang [25] and Wang [26][27] proposed some different sort functions. But there are still cases of sorting failure. Deng and Chen [28] constructed the score-variance models based on methods for ranking fuzzy numbers.

The inherent statistical rules and authoritative values among index data should be considered when assigning weights to indexes. Many scholars have studied different combination weighting methods to make up for the limitations of a single weighting method. Wang [29] used the combination weighting method and the fuzzy multi-criteria model to select the optimal cool storage system for air conditioning. In the evaluation process, the optimal weighting method combined the subjective knowledge of the decision-maker and the objectivity of numerical data to obtain the comprehensive weights of criteria and avoided the subjective one-sidedness of weights. To get the subjective and objective weights, Yi [30] used the fuzzy analytic hierarchy process method and improved criteria importance through inter-criteria correlation, and he applied the least square method to obtain the combined weights, which reduced the influence of artificial experience. Hu [31] established a credit evaluation model based on the combination weighting method, considering the information volume, volatility, and difference of the road transportation enterprises' data and using normalized constraints of maximum variance to determine the combination weights. The model fully considered the degree of difference between the indicators and made up for the deviation of the single weighting method. Tan [32] used an improved analytic hierarchy process AHP and the entropy method to make the suitability evaluation of underground space. The method ensured the rationality of the evaluation results to the greatest extent, thereby providing a certain guiding significance for the development of underground space. Wu [33] used the coefficient of variation method and entropy weight method to determine the combined weight of the evaluation indicators, and realized the optimization of the green building programs in the South Sichuan Economic Zone. Genett et al. [34] combined the fuzzy AHP, and the TOPSIS method to evaluate the performance of environmental management and reverse logistics in plastic industry enterprises.

As one kind of decision problem, portfolio selection also needs to take the subjective knowledge of investors into account. In addition, to better describe the uncertainty of the financial market, interval-valued fuzzy portfolios are to be used to obtain more detailed information on securities. For this purpose, we study the interval-valued fuzzy portfolio model based on the combination weighting method and the score function. The main contributions of this paper are as follows. (1) Combining the interval-valued intuitionistic fuzzy AHP with the entropy weight method, we get the combination weighting method. It overcomes the defects of unilateral empowerment law; (2) A new score function is obtained by adjusting the parameters. It can overcome the invalidity of the previous score function for some special interval-valued intuitionistic fuzzy numbers; (3) We propose an interval fuzzy portfolio model based on the combination weighting method and the new score function, and the theoretical theorem and proof are given. In practical application, a numerical example is given to verify the feasibility and effectiveness of the model.

The rest of this paper is arranged as follows. In Section II, the basic theory is introduced. In Section III, the interval-valued intuitionistic fuzzy analytic hierarchy process is introduced. In Section IV, we introduce ten kinds of score functions and accurate functions and their limitations, and construct a new score function. In Section V, a novel portfolio model with improved interval-valued intuitionistic AHP and score function is constructed, and the feasibility of the model is verified by a numerical example. In Section VI, we summarize the work of this paper.

II. Some Existing Definitions and Properties

A. Interval-valued intuitionistic fuzzy set

Definition 1 [17] Suppose $\operatorname{int}[0,1]$ is the collection of closed subsets of the interval-valued number $[0,1]$. X is a given theoretical field, $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle \mid x\in X\}$ is called an interval-valued intuitionistic fuzzy set on the domain X. $\mu_A(x): X\to \operatorname{int}[0,1]$ and $\nu_A(x): X\to \operatorname{int}[0,1]$ satisfy $0\le \sup\mu_A(x)+\sup\nu_A(x)\le 1$, for $\forall x\in X$. The interval-valued number $\mu_A(x)$ is the membership of the element x. $\nu_A(x)$ is the non-membership of the element x:

$$\begin{cases}\mu_A(x)=\left[\mu_A^L(x),\mu_A^U(x)\right],\\ \nu_A(x)=\left[\nu_A^L(x),\nu_A^U(x)\right].\end{cases} \tag{1}$$

The hesitating degree of the element is denoted by

$$\pi_A(x)=\left[\pi_A^L(x),\pi_A^U(x)\right], \quad (2)$$

where

$$\begin{cases}\pi_A^L(x)=1-\mu_A^U(x)-\nu_A^U(x),\\ \pi_A^U(x)=1-\mu_A^L(x)-\nu_A^L(x).\end{cases} \quad (3)$$

Particularly, when $\mu_A^L(x)=\mu_A^U(x)$, $\nu_A^L(x)=\nu_A^U(x)$, the interval-valued intuitionistic fuzzy set degenerates into the intuitionistic fuzzy set. The order interval pairs $\langle\mu_A(x),\nu_A(x)\rangle$ composed of the membership interval $\mu_A(x)$ and the non-membership interval $\nu_A(x)$ is called an interval-valued intuitionistic fuzzy number.

Definition 2 [17] For two interval-valued intuitionistic fuzzy sets

$A=\left\{\left\langle x,\left[\mu_A^L(x),\mu_A^U(x)\right],\left[\nu_A^L(x),\nu_A^U(x)\right]\right\rangle \mid x\in X\right\}$,

$B=\left\{\left\langle x,\left[\mu_B^L(x),\mu_B^U(x)\right],\left[\nu_B^L(x),\nu_B^U(x)\right]\right\rangle \mid x\in X\right\}$,

we have the following relationships

(1) $A\subseteq B\Leftrightarrow\begin{cases}\mu_A^L(x)\le\mu_B^L(x),\mu_A^U(x)\le\mu_B^U(x),\\ \nu_A^L(x)\ge\nu_B^L(x),\nu_A^U(x)\ge\nu_B^U(x);\end{cases}$

(2) $A=B\Leftrightarrow A\subseteq B, B\subseteq A;$

(3) $A^C=\{\left\langle x,\left[\nu_A^L(x),\nu_A^U(x)\right],\left[\mu_A^L(x),\mu_A^U(x)\right]\right\rangle \mid x\in X\}.$

Definition 3 [17] For any two interval-valued intuitionistic fuzzy numbers

$\alpha_1=([a_1,b_1],[c_1,d_1]),\alpha_2=([a_2,b_2],[c_2,d_2])$,

we have the following operational rules.

(1) $\alpha_1\cap\alpha_2=\begin{pmatrix}\left[\min(a_1,a_2),\min(b_1,b_2)\right],\\ \left[\max(c_1,c_2),\max(d_1,d_2)\right]\end{pmatrix};$

(2) $\alpha_1\cup\alpha_2=\begin{pmatrix}\left[\max(a_1,a_2),\max(b_1,b_2)\right],\\ \left[\min(c_1,c_2),\min(d_1,d_2)\right]\end{pmatrix};$

(3) $\alpha_1+\alpha_2=([a_1+a_2-a_1a_2,b_1+b_2-b_1b_2],[c_1c_2,d_1d_2]);$

(4) $\alpha_1\cdot\alpha_2=([a_1a_2,b_1b_2],[c_1+c_2-c_1c_2,d_1+d_2-d_1d_2]);$

(5) $\lambda\alpha_1=\left(\left[1-(1-a_1)^\lambda,1-(1-b_1)^\lambda\right],\left[c_1^\lambda,d_1^\lambda\right]\right),\ \lambda>0;$

(6) $\alpha_1^\lambda=\left(\left[a_1^\lambda,b_1^\lambda\right],\left[1-(1-c_1)^\lambda,1-(1-d_1)^\lambda\right]\right),\ \lambda>0.$

Definition4 [17] Suppose $\alpha_j=\left(\left[a_j,b_j\right],\left[c_j,d_j\right]\right)$, $(j=1,2,\ldots,n)$ is a set of interval-valued intuitionistic fuzzy numbers, then we have Formulas (4) and (5).where Formula (4) is called interval-valued intuitionistic fuzzy weighted arithmetic average operator; Formula (5) is called interval-valued intuitionistic fuzzy weighted geometric average operator. $\omega_j(j=1,2,\ldots,n)$ is the weight vector of $\alpha_j(j=1,2,\ldots,n)$, where $0<\omega_j<1$ and $\sum_{j=1}^n\omega_j=1$.

Specially, when $\omega_j=\frac{1}{n}(j=1,2,\ldots,n)$, then $IIFWA_\omega(\alpha_1,\alpha_2,...,\alpha_n)$ and $IIFWG_\omega(\alpha_1,\alpha_2,...,\alpha_n)$ degenerate into interval-valued intuitionistic fuzzy arithmetic average operator $IIFA_\omega(\alpha_1,\alpha_2,...,\alpha_n)$ and interval-valued intuitionistic fuzzy geometric average operator $IIFG_\omega(\alpha_1,\alpha_2,...,\alpha_n)$, respectively. That are Formulas (6) and (7)

III. Some new definition, theorems and relative proofs

A. A new theorem for interval-valued intuitionistic fuzzy number

Theorem 1 For any three interval-valued intuitionistic fuzzy numbers

$\alpha_1=([a_1,b_1],[c_1,d_1])$, $\alpha_2=([a_2,b_2],[c_2,d_2])$, and

$\alpha_3=([a_3,b_3],[c_3,d_3])$, we have the following properties.

(1) Closure property: $\alpha_1+\alpha_2$ and $\alpha_1\alpha_2$ are both interval-valued intuitionistic fuzzy numbers;

(2) Commutative law: $\alpha_1+\alpha_2=\alpha_2+\alpha_1$, $\alpha_1\cdot\alpha_2=\alpha_2\cdot\alpha_1$;

(3) Associative law:

$\alpha_1+\alpha_2+\alpha_3=\alpha_1+(\alpha_2+\alpha_3)$, $\alpha_1\alpha_2\alpha_3=\alpha_1(\alpha_2\alpha_3)$;

(4) De Morgan law:

$(\alpha_1+\alpha_2)^C=\alpha_1^C\alpha_2^C$, $\left(\alpha_1^C\alpha_2^C\right)=\alpha_1^C+\alpha_2^C$.

Our above proof process is inspired by Atanassov and Gargov (1989).

Proof:

(1) If we have $\mu_{\alpha_1+\alpha_2}(x)\subseteq[0,1]$, $\upsilon_{\alpha_1+\alpha_2}(x)\subseteq[0,1]$, and $\sup\mu_{\alpha_1+\alpha_2}(x)+\sup\upsilon_{\alpha_1+\alpha_2}(x)\le1$, then we can prove that $\alpha_1+\alpha_2$ is the interval-valued intuitionistic fuzzy number. Since

$$IIFWA_\omega(\alpha_1,\alpha_2,...,\alpha_n)=\sum_{j=1}^n\omega_j\alpha_j=\left(\left[1-\prod_{j=1}^n(1-a_j)^{\omega_j},1-\prod_{j=1}^n(1-b_j)^{\omega_j}\right],\left[\prod_{j=1}^n c_j^{\omega_j},\prod_{j=1}^n d_j^{\omega_j}\right]\right), \quad (4)$$

$$IIFWG_\omega(\alpha_1,\alpha_2,...,\alpha_n)=\sum_{j=1}^n\omega_j\alpha_j=\left(\left[\prod_{j=1}^n a_j^{\omega_j},\prod_{j=1}^n b_j^{\omega_j}\right],\left[1-\prod_{j=1}^n(1-c_j)^{\omega_j},1-\prod_{j=1}^n(1-d_j)^{\omega_j}\right]\right), \quad (5)$$

$$IIFA_\omega(\alpha_1,\alpha_2,...,\alpha_n)=\sum_{j=1}^n\omega_j\alpha_j=\left(\left[1-\prod_{j=1}^n(1-a_j)^{\frac{1}{n}},1-\prod_{j=1}^n(1-b_j)^{\frac{1}{n}}\right],\left[\prod_{j=1}^n c_j^{\frac{1}{n}},\prod_{j=1}^n d_j^{\frac{1}{n}}\right]\right), \quad (6)$$

$$IIFG_\omega(\alpha_1,\alpha_2,...,\alpha_n)=\sum_{j=1}^n\omega_j\alpha_j=\left(\left[\prod_{j=1}^n a_j^{\frac{1}{n}},\prod_{j=1}^n b_j^{\frac{1}{n}}\right],\left[1-\prod_{j=1}^n(1-c_j)^{\frac{1}{n}},1-\prod_{j=1}^n(1-d_j)^{\frac{1}{n}}\right]\right). \quad (7)$$

$$\alpha_1+\alpha_2 = \left(\left[a_1+a_2-a_1a_2, b_1+b_2-b_1b_2\right],\left[c_1c_2, d_1d_2\right]\right) = \left(\left[1-(1-a_1)(1-a_2), 1-(1-b_1)(1-b_2)\right],\left[c_1c_2, d_1d_2\right]\right), \quad (8)$$

$$0\le a_1\le b_1\le 1,\ \ 0\le a_2\le b_2\le 1, \quad (9)$$

we have

$$0\le 1-(1-a_1)(1-a_2)\le 1,\ \ 0\le 1-(1-b_1)(1-b_2)\le 1. \quad (10)$$

By (5), we have

$$1-(1-a_1)(1-a_2)\le 1-(1-b_1)(1-b_2). \quad (11)$$

Correspondingly, we obtain

$$a_1+a_2-a_1a_2\le b_1+b_2-b_1b_2. \quad (12)$$

By given conditions, it is obvious that

$$0\le c_1\le d_1\le 1,\ 0\le c_2\le d_2\le 1, \quad (13)$$

Then

$$0\le c_1c_2\le 1,\ 0\le d_1d_2\le 1. \quad (14)$$

Since

$$c_1c_2\le d_1d_2, \quad (15)$$

$$b_1+b_2-b_1b_2+d_1d_2=1-(1-b_1)(1-b_2)+d_1d_2, \quad (16)$$

$$b_1+d_1\le 1,\ b_2+d_2\le 1, \quad (17)$$

we can obtain

$$1-b_1\ge d_1,\ 1-b_2\ge d_2. \quad (18)$$

Then

$$(1-b_1)(1-b_2)\ge d_1d_2, \quad (19)$$

$$1-(1-b_1)(1-b_2)+d_1d_2\le 1. \quad (20)$$

We can obtain that $\alpha_1+\alpha_2$ is the interval-valued intuitionistic fuzzy number. Similarly, we also can prove that $\alpha_1\alpha_2$ is the interval-valued intuitionistic fuzzy number.

(2) Since

$$\alpha_1+\alpha_2=\left(\left[a_1+a_2-a_1a_2, b_1+b_2-b_1b_2\right],\left[c_1c_2, d_1d_2\right]\right), \quad (21)$$

$$\alpha_2+\alpha_1=\left(\left[a_2+a_1-a_2a_1, b_2+b_1-b_2b_1\right],\left[c_2c_1, d_2d_1\right]\right), \quad (22)$$

we have $\alpha_1+\alpha_2 = \alpha_2+\alpha_1$.

Since

$$\alpha_1\cdot\alpha_2=\left(\left[a_1a_2, b_1b_2\right],\left[c_1+c_2-c_1c_2, d_1+d_2-d_1d_2\right]\right), \quad (23)$$

$$\alpha_2\cdot\alpha_1=\left(\left[a_2a_1, b_2b_1\right],\left[c_2+c_1-c_2c_1, d_2+d_1-d_2d_1\right]\right), \quad (24)$$

we have $\alpha_1\cdot\alpha_2 = \alpha_2\cdot\alpha_1$.

(3) On the one hand, since

$$\alpha_1+\alpha_2=\left(\left[a_1+a_2-a_1a_2, b_1+b_2-b_1b_2\right],\left[c_1c_2, d_1d_2\right]\right), \quad (25)$$

$$\alpha_2+\alpha_3=\left(\left[a_2+a_3-a_2a_3, b_2+b_3-b_2b_3\right],\left[c_2c_3, d_2d_3\right]\right), \quad (26)$$

we have

$$\begin{aligned}&\alpha_1+\alpha_2+\alpha_3\\&=([a_1+a_2-a_1a_2+a_3-(a_1+a_2-a_1a_2)a_3, b_1+b_2-b_1b_2+b_3-(b_1+b_2-b_1b_2)b_3],[c_1c_2c_3, d_1d_2d_3]))\\&=([a_1+a_2+a_3-a_1a_2-a_1a_3-a_2a_3+a_1a_2a_3, b_1+b_2+b_3-b_1b_2-b_1b_3-b_2b_3+b_1b_2b_3],[c_1c_2c_3, d_1d_2d_3]).\end{aligned} \quad (27)$$

On the other hand, we obtain

$$\begin{aligned}&\alpha_1+(\alpha_2+\alpha_3)\\&=([a_1+a_2+a_3-a_2a_3-a_1(a_2+a_3-a_2a_3), b_1+b_2+b_3-b_2b_3-b_1(b_2+b_3-b_2b_3)],[c_1c_2c_3, d_1d_2d_3])\\&=([a_1+a_2+a_3-a_2a_3-a_1a_2-a_1a_3+a_1a_2a_3), b_1+b_2+b_3-b_2b_3-b_1b_2-b_1b_3+b_1b_2b_3)],[c_1c_2c_3, d_1d_2d_3])\\&=([a_1+a_2+a_3-a_2a_3-a_1a_2-a_1a_3+a_1a_2a_3), b_1+b_2+b_3-b_2b_3-b_1b_2-b_1b_3+b_1b_2b_3)],[c_1c_2c_3, d_1d_2d_3]),\end{aligned} \quad (28)$$

so, we have

$$\alpha_1+\alpha_2+\alpha_3=\alpha_1+(\alpha_2+\alpha_3). \quad (29)$$

On the one hand, since

$$\alpha_1\cdot\alpha_2=\left(\left[a_1a_2, b_1b_2\right],\left[c_1+c_2-c_1c_2, d_1+d_2-d_1d_2\right]\right), \quad (30)$$

$$\alpha_2\cdot\alpha_3=\left(\left[a_2a_3, b_2b_3\right],\left[c_2+c_3-c_2c_3, d_2+d_3-d_2d_3\right]\right), \quad (31)$$

we have

$$\begin{aligned}&\alpha_1\alpha_2\alpha_3\\&=([a_1a_2a_3, b_1b_2b_3],[c_1+c_2-c_1c_2+c_3-(c_1+c_2-c_1c_2)c_3, d_1+d_2-d_1d_2+d_3-(d_1+d_2-d_1d_2)d_3])\\&=([a_1a_2a_3, b_1b_2b_3],[c_1+c_2+c_3-c_1c_2-c_1c_3-c_2c_3+c_1c_2c_3, d_1+d_2+d_3-d_1d_2-d_1d_3-d_2d_3+d_1d_2d_3]).\end{aligned} \quad (32)$$

On the other hand, we obtain

$$\begin{aligned}&\alpha_1(\alpha_2\alpha_3)\\&=([a_1a_2a_3, b_1b_2b_3],[c_1+c_2+c_3-c_2c_3-c_1(c_2+c_3-c_2c_3), d_1+d_2+d_3-d_2d_3-d_1(d_2+d_3-d_2d_3)])\\&=([a_1a_2a_3, b_1b_2b_3],[c_1+c_2+c_3-c_2c_3-c_1c_2-c_1c_3+c_1c_2c_3, d_1+d_2+d_3-d_2d_3-d_1d_2-d_1d_3+d_1d_2d_3]),\end{aligned} \quad (33)$$

we have

$$\alpha_1\alpha_2\alpha_3=\alpha_1(\alpha_2\alpha_3). \quad (34)$$

(4) Since

$$(\alpha_1+\alpha_2)^C=\left(\left[a_1+a_2-a_1a_2, b_1+b_2-b_1b_2\right],\left[c_1c_2, d_1d_2\right]\right)^C=\left(\left[c_1c_2, d_1d_2\right],\left[a_1+a_2-a_1a_2, b_1+b_2-b_1b_2\right]\right), \quad (35)$$

$$\alpha_1^{\ C}=\left(\left[c_1, d_1\right],\left[a_1, b_1\right]\right), \quad (36)$$

$$\alpha_2^{\ C}=\left(\left[c_2, d_2\right],\left[a_2, b_2\right]\right), \quad (37)$$

we can obtain

$$\left(\alpha_1+\alpha_2\right)^C=\alpha_1^C\alpha_2^C. \tag{38}$$

Similarly, we also can prove

$$\left(\alpha_1\alpha_2\right)^C=\alpha_1^C+\alpha_2^C. \tag{39}$$

Remark 1 It should be noted that the following equation does not hold:

$$\alpha_1\left(\alpha_2+\alpha_3\right)=\alpha_1\alpha_2+\alpha_1\alpha_3. \tag{40}$$

Let

$$\begin{aligned}\alpha_1&=([0.1,0.2],[0.5,0.7]),\\ \alpha_2&=([0.2,0.3],[0.4,0.6]),\\ \alpha_3&=([0.1,0.3],[0.1,0.2]),\end{aligned} \tag{41}$$

we have

$$\begin{aligned}&\alpha_1\left(\alpha_2+\alpha_3\right)=([0.028,0.102],[0.04,0.12]),\\ &\alpha_1\alpha_2+\alpha_1\alpha_3=([0.098,0.1164],[0.385,0.6688]).\end{aligned} \tag{42}$$

Obviously, we can see that

$$\alpha_1\left(\alpha_2+\alpha_3\right)\neq\alpha_1\alpha_2+\alpha_1\alpha_3. \tag{43}$$

B. A new division definition and related proof

Definition 5 For any two interval-valued intuitionistic fuzzy numbers $\alpha_1=\left(\left[a_1,b_1\right],\left[c_1,d_1\right]\right)$ and $\alpha_2=\left(\left[a_2,b_2\right],\left[c_2,d_2\right]\right)$, we can define the subtraction operator and division operator:

$$\begin{aligned}\alpha_1-\alpha_2&=\alpha_1\alpha_2^C\\ &=\left(\left[a_1c_2,b_1d_2\right],\left[c_1+a_2-c_1a_2,d_1+b_2-d_1b_2\right]\right),\end{aligned} \tag{44}$$

$$\begin{aligned}\alpha_1\div\alpha_2&=\alpha_1+\alpha_2^C\\ &=\left(\left[a_1+c_2-a_1c_2,b_1+d_2-b_1d_2\right],\left[c_1a_2,d_1b_2\right]\right).\end{aligned} \tag{45}$$

Since subtraction comes from multiplication and division from addition, $\alpha_1-\alpha_2$ and $\alpha_1\div\alpha_2$ are both interval-valued intuitionistic fuzzy numbers.

Remark 2 Drawing inspiration from the division definition of intuitionistic fuzzy numbers, we have defined division for interval intuitionistic fuzzy numbers. Subsequently, we substantiate its validity, which in turn streamlines the division operation for interval-valued intuitionistic fuzzy numbers.

Theorem 2 For the interval-valued intuitionistic fuzzy number $\alpha_1=\left(\left[a_1,b_1\right],\left[c_1,d_1\right]\right)$, $\alpha_2=\left(\left[a_2,b_2\right],\left[c_2,d_2\right]\right)$, and $\alpha_3=\left(\left[a_3,b_3\right],\left[c_3,d_3\right]\right)$, we have the following properties.

(1) $\left(\alpha_1-\alpha_2\right)-\alpha_3=\alpha_1-\left(\alpha_2+\alpha_3\right)$;

(2) $\left(\alpha_1\div\alpha_2\right)\div\alpha_3=\alpha_1\div\left(\alpha_2\alpha_3\right)$.

Proof:

(1) According to Definition 5 and Theorem 1, we can get

$$\begin{aligned}(\alpha_1-\alpha_2)-\alpha_3&=(\alpha_1\alpha_2^C)\alpha_3^C\\ &=\alpha_1\left(\alpha_2^C\alpha_3^C\right)\\ &=\alpha_1\left(\alpha_2+\alpha_3\right)^C\\ &=\alpha_1-\left(\alpha_2+\alpha_3\right).\end{aligned} \tag{46}$$

(2) According to Definition 5 and Theorem 1, we can also get

$$\begin{aligned}(\alpha_1\div\alpha_2)\div\alpha_3&=(\alpha_1+\alpha_2^C)+\alpha_3^C\\ &=\alpha_1+\left(\alpha_2^C+\alpha_3^C\right)\\ &=\alpha_1+\left(\alpha_2\alpha_3\right)^C\\ &=\alpha_1\div\left(\alpha_2\alpha_3\right).\end{aligned} \tag{47}$$

IV. The Ideas and Steps of Our New Interval-Valued Intuitionistic Fuzzy AHP

With the help of the idea of intuitionistic fuzzy AHP of Xu [35], we can construct interval-valued intuitionistic AHP, the basic ideas and specific steps are as follows.

A. The basic ideas

Our inspiration is drawn from Xu [35]'s research on Intuitionistic Fuzzy Sets (IFS) in the context of the Analytic Hierarchy Process (AHP) [35]. He extended both classical AHP and fuzzy AHP to the framework of IFS, where pairwise comparisons of decision alternatives are expressed using intuitionistic fuzzy numbers. He established a Perfect Multiplicative Consistent Interval-valued Intuitionistic Fuzzy Judgment Matrix to examine the consistency of preference relations. The weight vector of intuitionistic preference relations can be obtained using the normalized rank-sum method. Based on the weight vector and the scoring function, scores for various criteria are determined, eventually leading to a normalized weight vector. Subsequently, we endeavor to extend AHP to the realm of interval-valued intuitionistic fuzzy sets, which offers significant capabilities in describing ambiguity and uncertainty.

B. The specific steps

Step 1: Establish an interval-valued intuitionistic fuzzy judgment matrix.

First, we establish an interval-valued intuitionistic fuzzy judgment matrix $R=(r_{ij})_{n\times m}$, where $r_{ij}=([a_{ij},b_{ij}],[c_{ij},d_{ij}])$, $[a_{ij},b_{ij}]=[c_{ji},d_{ji}]$, $[c_{ij},d_{ij}]=[a_{ji},b_{ji}]$ and $[a_{ii},b_{ii}]=[c_{ii},d_{ii}]$ $=[0.5,0.5]$. $[a_{ij},b_{ij}]$ indicates the extent to which decision makers prefer the i-th index to the j-th index, $[c_{ij},d_{ij}]$ indicates the extent to which decision makers prefer the j-th index to the i-th index. Preference information can be compared between two indexes using the scaling method [36].

TABLE I
SCALING MEANING

Criterion	μ	ν
absolutely low	[0.10,0.25]	[0.65,0.75]
very low	[0.15,0.30]	[0.60,0.70]
low	[0.20,0.35]	[0.55,0.65]
on the low side	[0.25,0.40]	[0.50,0.60]
approximately equal	[0.45,0.55]	[0.30,0.45]
absolutely equal	[0.50,0.50]	[0.50,0.50]
on the high side	[0.50,0.60]	[0.25,0.40]
high	[0.55,0.65]	[0.20,0.35]
very high	[0.60,0.70]	[0.15,0.30]
absolute height	[0.65,0.75]	[0.10,0.25]

Step 2: Check the consistency by establishing consistent matrix $\overline{R}$.

We can check the consistency of $R=(r_{ij})_{n\times m}$ by establishing a perfect multiplicative consistent interval-valued intuitionistic fuzzy judgment matrix $\overline{R}=(\overline{r_{ij}})_{n\times m}$. There are three possible cases:

Case 1: When $j>i+1$, let $\overline{r_{ij}}=([\overline{a_{ij}},\overline{b_{ij}}],[\overline{c_{ij}},\overline{d_{ij}}])$;

Case 2: When $j=i+1$ or $j=i$, let $\overline{r_{ij}}=r_{ij}$;

Case 3: When $j<i$, let $\overline{r_{ij}}=([\overline{c_{ij}},\overline{d_{ij}}],[\overline{a_{ij}},\overline{b_{ij}}])$.

Where

$$\begin{cases}\overline{a_{ij}}=\dfrac{\sqrt[j-i-1]{\prod_{t=i+1}^{j-1}a_{it}a_{tj}}}{\sqrt[j-i-1]{\prod_{t=i+1}^{j-1}a_{it}a_{tj}}+\sqrt[j-i-1]{\prod_{t=i+1}^{j-1}(1-a_{it})(1-a_{tj})}},\\ \overline{b_{ij}}=\dfrac{\sqrt[j-i-1]{\prod_{t=i+1}^{j-1}b_{it}b_{tj}}}{\sqrt[j-i-1]{\prod_{t=i+1}^{j-1}b_{it}b_{tj}}+\sqrt[j-i-1]{\prod_{t=i+1}^{j-1}(1-b_{it})(1-b_{tj})}},\\ \overline{c_{ij}}=\dfrac{\sqrt[j-i-1]{\prod_{t=i+1}^{j-1}c_{it}c_{tj}}}{\sqrt[j-i-1]{\prod_{t=i+1}^{j-1}c_{it}c_{tj}}+\sqrt[j-i-1]{\prod_{t=i+1}^{j-1}(1-c_{it})(1-c_{tj})}},\\ \overline{d_{ij}}=\dfrac{\sqrt[j-i-1]{\prod_{t=i+1}^{j-1}d_{it}d_{tj}}}{\sqrt[j-i-1]{\prod_{t=i+1}^{j-1}d_{it}d_{tj}}+\sqrt[j-i-1]{\prod_{t=i+1}^{j-1}(1-d_{it})(1-d_{tj})}}.\end{cases} \quad (48)$$

The consistency of R is acceptable if R and $\overline{R}$ meet the following conditions: $d(R,\overline{R})<\tau$, where τ is a consistency threshold. Generally, its value is 0.1. $d(R,\overline{R})$ is the distance measure for R to $\overline{R}$, where

$$d(R,\overline{R})=\frac{1}{4(n-1)(n-2)}\sum_{i=1}^{n}\sum_{j=1}^{n}\begin{pmatrix}|\overline{a_{ij}}-a_{ij}|+|\overline{b_{ij}}-b_{ij}|\\+|\overline{c_{ij}}-c_{ij}|+|\overline{d_{ij}}-d_{ij}|\\+|\overline{1-a_{ij}-c_{ij}}-(1-a_{ij}-c_{ij})|\\+|\overline{1-b_{ij}-d_{ij}}-(1-b_{ij}-d_{ijij})|\end{pmatrix}. \quad (49)$$

Step 3: Calculate the weights of the indexes by introducing parameters.

For the interval-valued intuitionistic fuzzy number $r_{ij}=([a_{ij},b_{ij}],[c_{ij},d_{ij}])$, let $r_{ij}'=(m_{ij},n_{ij})$, preference parameters

$$m_{ij}(\lambda_{ij})=a_{ij}+\lambda_{ij}(b_{ij}-a_{ij}), \quad (50)$$

$$n_{ij}(\lambda_{ij})=c_{ij}+(1-\lambda_{ij})(d_{ij}-c_{ij}), \quad (51)$$

where m_{ij} represents the parameter membership of the interval-valued intuitionistic fuzzy number r_{ij}', n_{ij} represents the parameter non-membership of the interval-valued intuitionistic fuzzy number r_{ij}'. λ_{ij} represents the satisfaction coefficient of decision makers to the i-th index relative to the j-th index and $\lambda_{ij}\in[0,1]$. The larger the value of λ_{ij} is, the more satisfied the decision maker is with the i-th index. Then $r_{ij}=([a_{ij},b_{ij}],[c_{ij},d_{ij}])$ can be converted to

$$r_{ij}'=(m_{ij},n_{ij})=(a_{ij}+\lambda_{ij}(b_{ij}-a_{ij}),c_{ij}+(1-\lambda_{ij})(d_{ij}-c_{ij})). \quad (52)$$

Since

$m_{ij}+n_{ij}=a_{ij}+\lambda_{ij}(b_{ij}-a_{ij})+c_{ij}+(1-\lambda_{ij})(d_{ij}-c_{ij})\le b_{ij}+d_{ij}\le 1$,

thus $m_{ij}\le 1-n_{ij}$, we can convert (m_{ij},n_{ij}) to $[m_{ij},1-n_{ij}]$, then we can convert $R'=(r_{ij}')_{n\times m}=(m_{ij},n_{ij})_{n\times m}$ to $R''=(r_{ij}'')_{n\times m}=([m_{ij},1-n_{ij}])_{n\times m}$. According to the arithmetic rules of the interval, we use normalizing rank summation method [35] to obtain the weights as follows:

$$\begin{aligned}\omega'&=\frac{\sum_{j=1}^{n}r_{ij}''}{\sum_{i=1}^{n}\sum_{j=1}^{n}r_{ij}''}\\&=\frac{\sum_{j=1}^{n}[m_{ij},1-n_{ij}]}{\sum_{i=1}^{n}\sum_{j=1}^{n}[m_{ij},1-n_{ij}]}\\&=\frac{\left[\sum_{j=1}^{n}m_{ij},\sum_{j=1}^{n}(1-n_{ij})\right]}{\left[\sum_{i=1}^{n}\sum_{j=1}^{n}m_{ij},\sum_{i=1}^{n}\sum_{j=1}^{n}(1-n_{ij})\right]}\\&=\left[\frac{\sum_{j=1}^{n}m_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{n}(1-n_{ij})},\frac{\sum_{j=1}^{n}(1-n_{ij})}{\sum_{i=1}^{n}\sum_{j=1}^{n}m_{ij}}\right].\end{aligned} \quad (53)$$

Then we convert the interval $\left[\frac{\sum_{j=1}^{n}m_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{n}(1-n_{ij})},\frac{\sum_{j=1}^{n}(1-n_{ij})}{\sum_{i=1}^{n}\sum_{j=1}^{n}m_{ij}}\right]$ to its corresponding intuitionistic fuzzy number. Finally, we have

$$\omega_i'=\left(\frac{\sum_{j=1}^{n}m_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{n}(1-n_{ij})},1-\frac{\sum_{j=1}^{n}(1-n_{ij})}{\sum_{i=1}^{n}\sum_{j=1}^{n}m_{ij}}\right). \quad (54)$$

Next, we can show that

$$0\le\frac{\sum_{j=1}^{n}m_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{n}(1-n_{ij})}\le 1 \text{ and } 0\le 1-\frac{\sum_{j=1}^{n}(1-n_{ij})}{\sum_{i=1}^{n}\sum_{j=1}^{n}m_{ij}}\le 1.$$

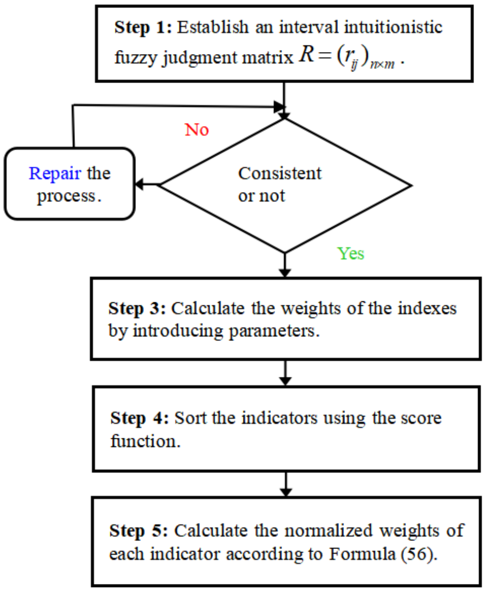

Fig. 1. Schematic diagram of the IVIFAHP.

In fact, since $\sum_{j=1}^{n} a_{ij} \le \sum_{j=1}^{n} m_{ij} \le \sum_{j=1}^{n} b_{ij}$, $\sum_{j=1}^{n} c_{ij} \le \sum_{j=1}^{n} n_{ij} \le \sum_{j=1}^{n} d_{ij}$ and $b_{ij} \le 1-d_{ij}$, then we have

$$\sum_{j=1}^{n} a_{ij} \le \sum_{j=1}^{n} m_{ij} \le \sum_{j=1}^{n} b_{ij} \le \sum_{j=1}^{n}(1-d_{ij}) \le \sum_{j=1}^{n}(1-n_{ij}) \le \sum_{j=1}^{n}(1-c_{ij}).$$

So $0 \le \frac{\sum_{j=1}^{n} m_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{n}(1-n_{ij})} \le 1$. Similarly, we can show that

$$0 \le 1-\frac{\sum_{j=1}^{n}(1-n_{ij})}{\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}} \le 1.$$

Step 4: Sort the indexes.

We sort the indicators using the score function. Set the score function as follows.

$$f(\alpha)=\frac{1-v_{\alpha}}{1+\pi_{\alpha}}. \tag{55}$$

Step 5: Normalize the weights.

We calculate the normalized weights of each indicator according to Formula (56).

$$\omega_i=\frac{f(\omega_i')}{\sum_{i=1}^{n} f(\omega_i')}. \tag{56}$$

Remark 3 Intuitionistic fuzzy AHP proposed by Xu [35] applied AHP to intuitionistic fuzzy sets. We extend it to interval-valued intuitionistic fuzzy sets and obtain the interval intuitionistic AHP. When establishing the interval-valued intuitionistic fuzzy judgment matrix, we change the intuitionistic fuzzy assessments $r_{ij}=(\mu_{ij},v_{ij})$ into interval-valued intuitionistic fuzzy assessments $r_{ij}=([a_{ij},b_{ij}],[c_{ij},d_{ij}])$. When doing the consistency check, we also propose new $\overline{R}=(\overline{r_{ij}})_{n\times m}$ and $d(R,\overline{R})$.

Remark 4 When we calculate the weight of the evaluation index, we introduce a parameter λ_{ij} to express the satisfaction coefficient of decision makers to the i-th index compared with the j-th index, and then we can change an interval-valued intuitionistic fuzzy number into an intuitionistic fuzzy number.

Schematic diagram of the IVIFAHP is shown in Fig. 1.

V. THE LIMITATION AND IMPROVEMENT ANALYSIS OF SCORE FUNCTIONS

In the previous section, we have elucidated the construction of interval-valued intuitionistic fuzzy AHP. In this section, we will introduce the existing score functions and accurate functions, and propose a novel score function.

A. Score functions, accurate functions and their limitation analysis

The existing score functions and accurate functions have specific definitions, yet they exhibit limitations when applied to certain data scenarios. Through meticulous review and thorough exploration of existing references, we have identified seven distinct score functions and accurate functions. After an exhaustive review and analysis of the existing literature, we posit that the sorting function serves as a pivotal approach for assessing the comparative merits between two intuitionistic fuzzy numbers, thus necessitating its consideration. This paper proceeds to analyze the limitations associated with these seven score functions and accurate functions as follows.

a. The 1st kind of score function, accurate function and their limitation analysis

Xu [20] defined the score function and accurate function and gave the ordering rules in 2007.

Definition 6 [20] Suppose $\alpha=([a,b],[c,d])$ is an interval-valued intuitionistic fuzzy numbers, its score function $S(\alpha)$ and accurate function $h(\alpha)$ can be defined as follows

$$S(\alpha)=\frac{(a+b)-(c+d)}{2}, h(\alpha)=\frac{(a+b)+(c+d)}{2}. \tag{57}$$

The ordering rule: when $S(\alpha_1)>S(\alpha_2)$, the interval-valued intuitionistic fuzzy number α_1 is better; when $S(\alpha_1)=S(\alpha_2)$, we need to compare the accurate function $h(\alpha)$ (if $h(\alpha_1)>h(\alpha_2)$, the interval-valued intuitionistic fuzzy number α_1 is better).

Limitation analysis: For some special interval-valued intuitionistic fuzzy numbers satisfying $a_1+b_1=a_2+b_2$ and $c_1+d_1=c_2+d_2$, this method will fail.

For example, we let $\alpha_1=([0.35,0.45],[0.20,0.30])$, $\alpha_2=([0.30,0.50],[0.15,0.35])$, we can calculate out $S(\alpha_1)=S(\alpha_2)=0.15$, $h(\alpha_1)=h(\alpha_2)=0.65$. The above results only show α_1 is equivalent α_2. We can't judge whether α_1 or α_2 is better.

b. The 2nd kind of accurate function and its limitation analysis

After pointing out the limitations of Xu [20], Ye [21] proposed the following accurate function from the perspective of hesitation.

Definition 7 [21] Suppose $\alpha=([a,b],[c,d])$ is an interval-valued intuitionistic fuzzy number, its accurate function $h(\alpha)$ can be defined as follows:

$$h(\alpha)=\frac{a-(1-a-c)+b-(1-b-d)}{2} = a+b-1+\frac{c+d}{2}. \tag{58}$$

Limitation analysis:

(1) $\alpha_1=([0.1,0.2],[0.6,0.8]),\alpha_2=([0.4,0.5],[0.1,0.2])$, we can obviously get $\alpha_1\subset\alpha_2$ and α_2 is significantly superior to the α_1. By using Formula (58), we have $h(\alpha_1)=0,h(\alpha_2)=-0.0850,\ h(\alpha_1)>h(\alpha_2)$. This is obviously unrealistic.

(2) This function does not make full use of the change information of the upper and lower bounds of membership and non-membership. When the midpoint of the membership degree and the non-membership degree of two interval-valued intuitionistic fuzzy numbers are equal, their exact functions will be equal, and this sort method will fail.

c. The 3rd kind of accurate function and its limitation analysis

Nayagam and Sivaraman et al. [22][23] proposed two accurate functions based on Xu [20] and Ye [21].

Definition 8 [22][23] Suppose $\alpha=([a,b],[c,d])$ is an interval-valued intuitionistic fuzzy number, its accurate function $h(\alpha)$ can be defined as follows, where δ is the weight determined by the decision maker:

$$h(\alpha)=\frac{a+b-d(1-b)-c(1-a)}{2}, \tag{59}$$

$$h(\alpha)=\frac{a+b+\delta(2-a-b-c-d)}{2}. \tag{60}$$

Limitation analysis:

(1) For the first accurate function, let $\alpha_1=([0.1,0.1],[0.5,0.7]),\alpha_2=([0.2,0.2],[0.8,0.8])$, then the hesitancy of the latter is ([0.00,0.00], [0.00,0.00]). Since accurate function places more emphasis on hesitancy and the hesitancy of α_2 is smaller, we can judge $\alpha_1<\alpha_2$. However, according to this definition, we have $h(\alpha_1)=h(\alpha_2)=-0.44$. This is not consistent with common sense.

(2) For the second accurate function, the presence of δ strengthens the subjective evaluation of the decision maker, and the determination of its value is also a problem.

(3) For two accurate functions, when a,b,d are fixed, we seek the partial derivatives for c, then we can find the partial derivative is less than 0, indicating that the accurate function value increases as the lower bound c of the non-membership interval-value decreases. It is obviously questionable.

d. The 4th kind of accurate function and its limitation analysis

In 2014, Gao et al. [24] proposed the following accurate functions.

Definition 9 [24] Suppose $\alpha=([a,b],[c,d])$ is an interval-valued intuitionistic fuzzy number, its accurate function $h(\alpha)$ can be defined as follows:

$$h(\alpha)=\frac{a+b-c-d}{(1-b-d)+(1-a-c)+2} =\frac{a+b-c-d}{4-(a+b+c+d)}. \tag{61}$$

Limitation analysis: The accurate function does not make full use of the change information of the upper and lower bounds of the non-membership, when the midpoint of membership is the same as that of the non-membership, the accurate function will be the same, and the sort method of the interval-valued intuitionistic fuzzy number will fail. For example, we let $\alpha_1=([0.35,0.45],[0.20,0.30])$ and $\alpha_2=([0.30,0.50],[0.15,0.35])$. We have $h(\alpha_1)=h(\alpha_2)=0$. The α_1 and α_2 cannot be judged in this case.

e. The 5th kind of accurate function and its limitation analysis

In 2015, Kang et al. [25] modified the accurate function and gave the following three definitions.

Definition 10 [25] Suppose $\alpha=([a,b],[c,d])$ is an interval-valued intuitionistic fuzzy number, its accurate function $h(\alpha)$ can be defined as follows, where δ_1 and δ_2 are weights determined by the intention of the decision maker.

$$h(\alpha)=\frac{a-c+b-d-c(1-b-d)-d(1-a-c)}{2}, \tag{62}$$

$$h(\alpha)=\frac{a+b(1-a-c)+b+a(1-b-d)}{2}, \tag{63}$$

$$h(\alpha)=\frac{a+\delta_1(1-a-c)+b+\delta_2(1-b-d)}{2}. \tag{64}$$

Limitation analysis:

(1) When a,b,d are fixed, we seek the partial derivatives for c, then we can find the partial derivative is less than 0, indicating that the accurate function value increases as the lower bound c of the non-membership interval-value decreases. which is obviously questionable.

(2) For the three accurate functions, the δ strengthens the subjective evaluation of the decision maker, and the determination of its value is also a problem.

(3) For Formula (62), let $\alpha_1=([0.1,0.5],[0.00,0.00])$, $\alpha_2=([0.20,0.40],[0.00,0.00])$ and $h(\alpha_1)=h(\alpha_2)=0.30$. Then α_1 and α_2 cannot be judged in this case.

(4) For Formula (63), let $\alpha_1=([0.00,0.00],[1.00,1.00])$, $\alpha_2=([0.00,0.00],[0.00,0.00])$ and $h(\alpha_1)=h(\alpha_2)=0$. Then α_1 and α_2 cannot be judged in this case.

f. The 6th kind of sore function and its limitation analysis

New score functions were presented by Wang and Chen [26] in 2017.

Definition 11 [26] Suppose $\alpha=([a,b],[c,d])$ is an interval-valued intuitionistic fuzzy number, its score function $S(\alpha)$ can be defined as follows:

$$S(\alpha)=\frac{a+b+\sqrt{bd}\,(1-a-c)+\sqrt{ac}\,(1-b-d)}{2}. \quad (65)$$

Limitation analysis: For some special interval-valued intuitionistic fuzzy numbers, this method will fail. For example, two interval-valued intuitionistic fuzzy numbers $\alpha_1=([0.00,0.00],[0.20,0.30]),\alpha_2=([0.00,0.00],[0.10,0.20])$, we have $S(\alpha_1)=S(\alpha_2)=0$, then α_1 and α_2 cannot be judged in this case.

g. The 7th kind of sore function, accurate function and its limitation analysis

In 2018, Wang and Chen [27]proposed new score and accurate functions.

Definition 12 [27] Suppose $\alpha=([a,b],[c,d])$ is an interval-valued intuitionistic fuzzy number, its score function $S(\alpha)$ and accurate function $h(\alpha)$ can be defined as follows:

$$S(\alpha)=\frac{(a+b)(a+c)-(c+d)(b+d)}{2},$$
$$h(\alpha)=\frac{(1-a+b)(1-a-c)+(1-c+d)(1-b-d)}{2}. \quad (66)$$

Limitation analysis: For some special interval-valued intuitionistic fuzzy numbers, this method will fail. For example, let $\alpha_1=([0,b_1],[0,0]),\alpha_2=([0,b_2],[0,0])$, we have $S(\alpha_1)=S(\alpha_2)=0,\ h(\alpha_1)=h(\alpha_2)=1$ for any $b_1,b_2\in[0,1]$.

B. Our new score function, theorems and proofs

In the above cases that the sorting function has the possibility of failure, we give the new score function drawn from the score function of intuitionistic fuzzy sets of Li [37]. Then the proof of relevant theorems is given.

Definition 16 Suppose $\alpha=([a,b],[c,d])$ is an interval-valued intuitionistic fuzzy number, its score function $S(\alpha)$ can be defined as follows:

$$S(\alpha)=(a+b-1)+\frac{2-a-b-c-d}{2+c+d}. \quad (67)$$

The formula considers membership, non-membership and hesitancy of the interval-valued intuitionistic fuzzy set. Let 88

$$\psi_A=\frac{a+b}{2},\ \Delta_A=\frac{c+d}{2}, \quad (68)$$

then this formula satisfies the following theorems.

Theorem 3 Score function $S(\alpha)$ is monotonically increasing about ψ_A and is monotonically decreasing about Δ_A.

Proof:
Since

$$S(\alpha)=(a+b-1)+\frac{2-a-b-c-d}{2+c+d}=2\psi_A-1+\frac{1-\psi_A-\Delta_A}{1+\Delta_A}, \quad (69)$$

$$\frac{\partial S}{\partial\psi_A}=2-\frac{1}{1+\Delta_A}>0, \quad (70)$$

$$\frac{\partial S}{\partial\Delta_A}=\frac{-2+\psi_A}{(1+\Delta_A)^2}<0, \quad (71)$$

we can get score function $S(\alpha)$ is monotonically increasing about ψ_A and is monotonically decreasing about Δ_A.

Theorem 4 Score function $S(\alpha)$ is bounded and $S(\alpha)\in[-1,1]$.

Proof:
According to Theorem 3, $S(\alpha)$ is monotonically increasing about ψ_A and is monotonically decreasing about Δ_A. So if and only if $\psi_A=1,\ \Delta_A=0$, $S(\alpha)$ has the maximum value of 1; if and only if $\psi_A=0,\ \Delta_A=1$, $S(\alpha)$ has the minimum value of -1. So $S(\alpha)\in[-1,1]$, i.e., $S(\alpha)$ is bounded.

Theorem 5 Suppose $\alpha_1=([a_1,b_1],[c_1,d_1])$ and $\alpha_2=([a_2,b_2],[c_2,d_2])$ are both interval-valued intuitionistic fuzzy numbers, when $\alpha_1\supseteq\alpha_2$, $S(\alpha_1)\ge S(\alpha_2)$, when $\alpha_1\subseteq\alpha_2$, $S(\alpha_1)\le S(\alpha_2)$.

Proof:
When

$$\alpha_1\supseteq\alpha_2,\ a_1\ge a_2,b_1\ge b_2,\ c_1\le c_2,d_1\le d_2, \quad (72)$$

we have

$$\psi_{\alpha_1}=\frac{a_1+b_1}{2}\ge\psi_{\alpha_2}=\frac{a_2+b_2}{2},$$
$$\Delta_{\alpha_1}=\frac{c_1+d_1}{2}\le\Delta_{\alpha_2}=\frac{c_2+d_2}{2}, \quad (73)$$

According to Theorem 3, we have

$$S(\alpha_1)\ge S(\alpha_2). \quad (74)$$

Similarly, we also can prove $S(\alpha_1)\le S(\alpha_2)$ when $\alpha_1\subseteq\alpha_2$.

Remark 5 The score function proposed by Li [37] is based on the intuitionistic fuzzy set. Here we extend it to the interval-valued intuitionistic fuzzy set. The original score function is

$$S(\alpha)=(2\mu_\alpha-1)+\frac{\pi_\alpha}{1+v_\alpha}. \quad (75)$$

Where $\alpha=(\mu_\alpha,v_\alpha)$, μ_α represents the membership of α, v_α is the non-membership of α and π_α is the hesitating degree of α. When $\alpha=([a,b],[c,d])$, we let

$$\mu_\alpha=\frac{a+b}{2},$$
$$v_\alpha=\frac{c+d}{2}, \quad (76)$$
$$\pi_\alpha=1-(\frac{a+b}{2}+\frac{c+d}{2})=\frac{2-a-b-c-d}{2}.$$

Finally, we obtain the score function defined above

$$S(\alpha)=(a+b-1)+\frac{2-a-b-c-d}{2+c+d}. \quad (77)$$

C. Comparison of our new score function with existing sort functions

By contrasting the existing score and accurate functions with the novel score function, we generate the ensuing table (TABLE II). The outcomes indicate that certain interval-valued intuitionistic fuzzy numbers cannot be adequately appraised using the established sort functions, whereas the novel sort function attains sensible sorting outcomes.

(1) For two interval-valued intuitionistic fuzzy numbers $\alpha_1 = ([0.35, 0.45],[0.20, 0.30]), \alpha_2 = ([0.30, 0.50],[0.15, 0.35])$, we have $\alpha_1 = \alpha_2$, according to the Formula (57) (Xu, 2007), then we cannot judge which is better. In fact, if two interval-valued intuitionistic fuzzy numbers satisfying $a_1 + b_1 = a_2 + b_2$ and $c_1 + d_1 = c_2 + d_2$, this function will fail. But we can obtain $\alpha_1 < \alpha_2$ by our new function.

(2) For two interval-valued intuitionistic fuzzy numbers $\alpha_1 = ([0.10, 0.20],[0.60, 0.80]), \alpha_2 = ([0.40, 0.50],[0.01, 0.02])$, we can obviously get $\alpha_1 \subset \alpha_2$ and α_2 is significantly superior to the α_1. But according to Formula (58), we have $\alpha_1 = \alpha_2$ which is not practical. Then we obtain the reasonable result $\alpha_1 < \alpha_2$ by our new function. Furthermore, if two interval-valued intuitionistic fuzzy numbers satisfying $a_1 + b_1 = a_2 + b_2$ and $c_1 + d_1 = c_2 + d_2$, the sort function (Ye, 2009) cannot judge which is better either. But our new function can handle this situation effectively.

(3) For two interval-valued intuitionistic fuzzy numbers $\alpha_1 = ([0.10, 0.10],[0.50, 0.70]), \alpha_2 = ([0.20, 0.20],[0.80, 0.80])$, the hesitancy of the latter is ([0.00], [0.00]). Intuitively, we can judge $\alpha_1 < \alpha_2$. However, according to Formula (59), we have $\alpha_1 = \alpha_2$ which is not consistent with common sense. The result of our new function is $\alpha_1 < \alpha_2$, which is reasonable.

(4) For two interval-valued intuitionistic fuzzy numbers $\alpha_1 = ([0.35, 0.45],[0.20, 0.30]), \alpha_2 = ([0.30, 0.50],[0.15, 0.35])$. We have $\alpha_1 = \alpha_2$ according to Formula (61). Then α_1 and α_2 cannot be judged in this case. In fact, the function does not make full use of the information of the upper and lower bounds of the non-membership which means if two interval-valued intuitionistic fuzzy numbers satisfy $c_1 + d_1 = c_2 + d_2$, the two function values will be equal and the function will fail. We can obtain $\alpha_1 < \alpha_2$ by our new function, which means that our new function can make full use of the information of the upper and lower bounds of the non-membership.

(5) For two interval-valued intuitionistic fuzzy numbers $\alpha_1 = ([0.10, 0.50],[0.00, 0.00]), \alpha_2 = ([0.20, 0.40],[0.00, 0.00])$, we cannot judge which is better according to Formula (62). In fact, if two interval-valued intuitionistic fuzzy numbers satisfy $a_1 + b_1 = a_2 + b_2$ and $c_1 + d_1 = c_2 + d_2$, this function will fail. But we have $\alpha_1 < \alpha_2$ according to our new function.

(6) For two interval-valued intuitionistic fuzzy numbers $\alpha_1 = ([0.00, 0.00],[0.20, 0.30]), \alpha_2 = ([0.00, 0.00],[0.10, 0.20])$, we have $S(\alpha_1) = S(\alpha_2) = 0$, according to Formula (65), we cannot judge which is better. In fact, if $a = b = 0$, then we will have $S(\alpha) = 0$. But our new function does not be influenced by this case and we can obtain $\alpha_1 < \alpha_2$.

(7) For two interval-valued intuitionistic fuzzy numbers $\alpha_1 = ([0.00, 0.00],[0.00, 0.00]), \alpha_2 = ([0.00, 1.00],[0.00, 0.00])$, we have $\alpha_1 = \alpha_2$ according to Formula (66). In this case, we cannot judge which is better. In fact, if $\alpha_1 = ([0.00, b_1],[0.00, 0.00]), \alpha_2 = ([0.00, b_2],[0.00, 0.00])$, we have $S(\alpha_1) = S(\alpha_2) = 0$, $h(\alpha_1) = h(\alpha_2) = 1$ for any $b_1, b_2 \in [0,1]$. But we can judge which is better according to our new function if $b_1 \neq b_2$.

VI. A NOVEL FUZZY PORTFOLIO MODEL WITH IMPROVED INTERVAL-VALUED INTUITIONISTIC AHP AND SCORE FUNCTION

In this section, building upon the framework of interval-valued intuitionistic fuzzy AHP, we have formulated a novel fuzzy portfolio model using the improved interval-valued intuitionistic AHP method and the new score function. We have substantiated the feasibility and effectiveness of this model and methodology through numerical examples.

A. Model building

Suppose $A = \{A_1, A_2, \ldots, A_n\}$ is a portfolio collection of interval-valued intuitionistic fuzzy portfolio model problems, $C = \{C_1, C_2, \ldots, C_m\}$ is an attribute set, $\omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ is the weight vector of the attribute, ω_j represents the weight of C_j satisfying $\sum_{j=1}^{m} \omega_j = 1$, where the weight is unknown. The decision maker measures portfolio $A_i\ (i = 1, 2, \ldots, n)$ about the attribute $C_j\ (i = 1, 2, \ldots, m)$ and gives interval-valued intuitionistic fuzzy number $r_{ij} = ([a_{ij}, b_{ij}],[c_{ij}, d_{ij}])$, $[a_{ij}, b_{ij}]$ representing the degree to which the decision maker is satisfied with the portfolio A_i with respect to attribute C_j, $[c_{ij}, d_{ij}]$ representing the degree to which the decision maker is not satisfied with the portfolio A_i with respect to attribute C_j. Then combine $r_{ij} = ([a_{ij}, b_{ij}],[c_{ij}, d_{ij}])$ and form the decision matrix $R = (r_{ij})_{m \times n}, i = 1, 2, .., n,\ j = 1, 2, \ldots, m$.

To address these problems, we introduce a multi-attribute decision-making approach grounded in the improved interval-valued intuitionistic fuzzy AHP and the novel score function. The method comprises the following specific steps.

Step 1: According to the interval-valued intuitionistic fuzzy judgment matrix $R' = (r_{ij}')_{n \times m}$, we can calculate the weight vector $\omega' = \{\omega_1', \omega_2', \ldots, \omega_m'\}$ of attribute index by Formulae (50)-(56).

$$e_j=\frac{1}{n}\sum_{i=1}^{n}\frac{2-\left|\mu_{ij}^L(x)-\nu_{ij}^L(x)\right|-\left|\mu_{ij}^U(x)-\nu_{ij}^U(x)\right|+\pi_{ij}^U(x)+\pi_{ij}^U(x)}{2+\left|\mu_{ij}^L(x)-\nu_{ij}^L(x)\right|+\left|\mu_{ij}^U(x)-\nu_{ij}^U(x)\right|+\pi_{ij}^U(x)+\pi_{ij}^U(x)}. \tag{78}$$

$$r_i=\left([a_i,b_i],[c_i,d_i]\right)=\sum_{j=1}^{m}\omega_{ij}\alpha_{ij}=\left(\left[1-\prod_{j=1}^{m}\left(1-a_{ij}\right)^{\omega_{ij}},1-\prod_{j=1}^{m}\left(1-b_{ij}\right)^{\omega_{ij}}\right],\left[\prod_{j=1}^{m}c_{ij}^{\omega_{ij}},\prod_{j=1}^{m}d_{ij}^{\omega_{ij}}\right]\right). \tag{81}$$

Step 2: According to the decision matrix R, we can calculate the weight vector $\omega^*=\left\{\omega_1^*,\omega_2^*,\dots,\omega_m^*\right\}$ of the attribute by Formulae (78)-(79).

$$\omega_j^*=\frac{1-e_j}{m-\sum_{j=1}^{m}e_j}. \tag{79}$$

Step 3: From $\omega'=\{\omega_1',\omega_2',\dots,\omega_m'\}$ and $\omega^*=\left\{\omega_1^*,\omega_2^*,\dots,\omega_m^*\right\}$, we can obtain the combination weight vector $\omega=\left\{\omega_1,\omega_2,...,\omega_m\right\}$ of the attribute index.

$$\omega_i=k\omega_i'+(1-k)\omega_i^*. \tag{80}$$

Step 4: Calculate the combination attribute values of A_i by Formulae (81).

Calculate the score function value $S(A_i)$ of A_i by using Formula (67).

Step 5: We sort the alternative portfolios according to the score function value $S(A_i)$.

The schematic diagram of the entropy method-interval-valued intuitionistic fuzzy AHP are as follows (Fig. 2).

B Numerical example

To better choose the portfolio, we usually evaluate the portfolio with income, risk, and Sharp ratio, and this paper uses income, risk, and Sharp ratio as attributes, then we consider an investor who wants to select a portfolio. There are two kinds of portfolios $A=\{A_1,A_2\}$ and three attributes which are rate of return C_1, risk C_2 and Sharpe ratio C_3. After the data processing, we can obtain the interval-valued intuitionistic fuzzy judgment matrix $R'=(r_{ij}')_{n\times m}$ as shown in TABLE III.

After data processing, we can obtain the interval-valued intuitionistic fuzzy decision matrix as shown in TABLE IV.

Step 1: According to the judgment matrix $R'=(r_{ij}')_{n\times m}$, we can calculate the weight vector of the attribute index by using Formulae (50)-(56). ($\lambda_{ij}=0.5$). It is $\omega'=\left\{0.4121,0.3435,0.2444\right\}$.

Step 2: We use Formulae (78) and (79) to calculate the weight vector as shown below: $\omega^*=\left\{0.3222,0.2482,0.4296\right\}$.

Step 3: From $\omega'=\left\{0.4121,0.3435,0.2444\right\}$ and $\omega^*=\left\{0.3222,0.2482,0.4296\right\}$, we use Formula (80) ($k=0.5$) to obtain the combination weighting vector as $\omega=\left\{0.36715,0.29585,0.337\right\}$.

Step 4: We use Formula (81) to calculate the combination attribute values of portfolio $A_i\left(i=1,2\right)$ as:

$r_1=([0.5907,0.6956],[0.1583,0.2849])$,

$r_2=([0.5658,0.6666],[0.1228,0.2617])$.

Step 5: We can calculate the score function value of combination attribute value $S(A_i)$ of A_i by using Formula (67):

$S(A_1)=0.3970$, $S(A_2)=0.3931$.

Step 6: We sort the alternative portfolios according to the score function value $S(A_i)$. We can get that $A_1 \succ A_2$, so the portfolio A_1 is selected: $S(A_1)>S(A_2)$.

After performing calculations, we derive the sorting function values for each portfolio. These computations culminate in the presentation of the sorting outcomes, which are presented in their entirety within TABLE V.

Referring to TABLE V, a notable observation emerges: among the six formulas, Portfolio A_1 emerges as the most optimal portfolio. Intriguingly, this finding aligns the new score function with the outcomes of these six established formulas. To provide a clearer and more intuitive representation of the ranking outcomes yielded by these existing sorting functions, we present the original portfolio score charts in Fig. 3 below. This graphical depiction offers a visual understanding of the portfolio assessments conducted using the diverse scoring methodologies.

Through meticulous data comparison, a noteworthy pattern emerges: solely in the case of Formula (57), the original score of Portfolio A_2 surpasses that of Portfolio A_1 $(0.4240>0.4216)$. Conversely, under the remaining six formulae, the original scores of Portfolio A_1 outweigh those of Portfolio A_2. Among these, Formula (65) yields the highest original scores (Portfolio A_1 with a score of 1.404 and Portfolio A_2 with a score of 1.3810, favoring Portfolio A_1), while Formula (66) produces the lowest scores (Portfolio A_1 with a score of 0.2644 and Portfolio A_2 with a score of 0.2458, still favoring Portfolio A_1). Impressively, our new score function echoes these outcomes, indicating that Portfolio A_1 maintains a higher score than Portfolio A_2 (Portfolio A_1 scoring 0.3970 and Portfolio A_2 scoring 0.3931).

To mitigate the influence of dimensionality on function value results, we've employed normalization for the function values obtained under each formula. Fig. 4 offers normalized score charts for various portfolios.

In stark contrast to the original scores, the normalized scores under different sorting standards exhibit divergent trends in the ups and downs of the two investment portfolios. The bar chart visually presents the normalized scores, highlighting the sole instance where the normalized score of Portfolio A_1, under Formula (57), outperforms that of

Portfolio A_2 $(0.5014 > 0.4986)$. In all other cases, the normalized scores of Portfolios A_1 outshine those of Portfolio A_2. Notably, Formula (58) yields the highest normalized score for Portfolio A_1 (Portfolio A_1 scoring 0.5447 and Portfolio A_2 scoring 0.4553) while Formula (57) yields the lowest normalized score for Portfolio A_1 (Portfolio A_1 scoring 0.4986 and Portfolio A_2 scoring 0.5014, with Portfolio A_2 having the higher score). Remarkably, our new score function maintains consistency with these six formulas (Portfolio A_1 scoring 0.5025 and Portfolio A_2 scoring 0.4975, with Portfolio A_1's score being higher).

Fig. 3 and Fig. 4 collectively display the sorting outcomes obtained through the application of the seven existing sorting functions. Among these, Portfolio A_1 stands out as the preferable choice. Furthermore, our new score function aligns with the result that A_1 ranks higher than A_2. When coupled with the analysis presented in Section V, it becomes unquestionable that the new score function yields more favorable results in practical application.

TABLE II
COMPARISON OF OUR NEW SCORE FUNCTION WITH EXISTING SORT FUNCTIONS

Formula	Reference	Examples	Results of the sort	Remark	Our new function (67)
(57)	(Xu, 2007)	$\alpha_1 = ([0.35, 0.45], [0.20, 0.30])$ $\alpha_2 = ([0.30, 0.50], [0.15, 0.35])$	$\alpha_1 = \alpha_2$	Unable to judge	$\alpha_1 < \alpha_2$
(58)	(Ye, 2009)	$\alpha_1 = ([0.10, 0.20], [0.60, 0.80])$ $\alpha_2 = ([0.40, 0.50], [0.01, 0.02])$	$\alpha_1 > \alpha_2$	In fact, $\alpha_1 < \alpha_2$	$\alpha_1 < \alpha_2$
(59)	(Nayagam et al., 2011)	$\alpha_1 = ([0.10, 0.10], [0.50, 0.70])$ $\alpha_2 = ([0.20, 0.20], [0.80, 0.80])$	$\alpha_1 = \alpha_2$	Unable to judge	$\alpha_1 < \alpha_2$
(61)	(Gao et al., 2014)	$\alpha_1 = ([0.35, 0.45], [0.20, 0.30])$ $\alpha_2 = ([0.30, 0.50], [0.15, 0.35])$	$\alpha_1 = \alpha_2$	Unable to judge	$\alpha_1 < \alpha_2$
(62)	(Kang, 2015)	$\alpha_1 = ([0.10, 0.50], [0.00, 0.00])$ $\alpha_2 = ([0.20, 0.40], [0.00, 0.00])$	$\alpha_1 = \alpha_2$	Unable to judge	$\alpha_1 > \alpha_2$
(65)	(Wang and Chen, 2017)	$\alpha_1 = ([0.00, 0.50], [0.00, 0.00])$ $\alpha_2 = ([0.00, 0.60], [0.00, 0.00])$	$\alpha_1 = \alpha_2$	Unable to judge	$\alpha_1 < \alpha_2$
(66)	(Wang and Chen, 2018)	$\alpha_1 = ([0.00, 0.00], [0.00, 0.00])$ $\alpha_2 = ([0.00, 1.00], [0.00, 0.00])$	$\alpha_1 = \alpha_2$	In fact, $\alpha_1 < \alpha_2$	$\alpha_1 < \alpha_2$

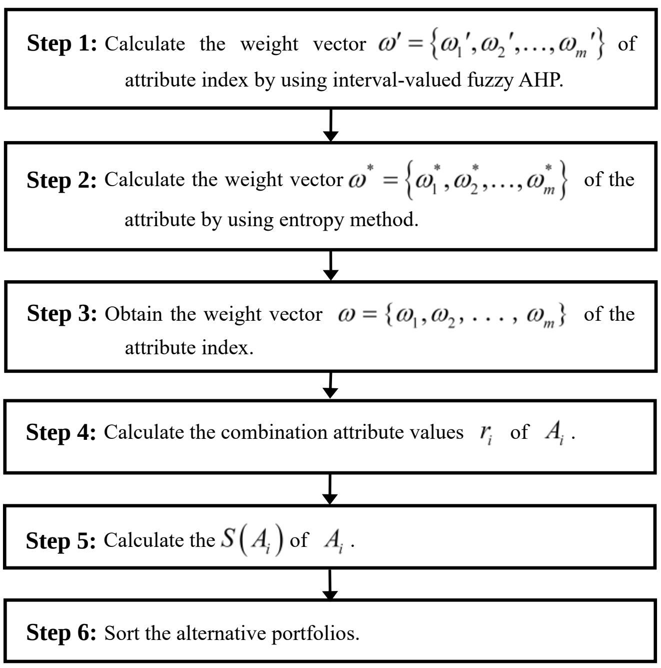

Fig. 2. Schematic diagram of the entropy method-interval-valued intuitionistic fuzzy AHP.

TABLE III
JUDGMENT MATRIX $R' = (r_{ij}')_{n\times m}$

Attribute	C_1	C_2	C_3
C_1	([0.50,0.50], [0.50,0.50])	([0.60,0.70], [0.15,0.30])	([0.55,0.65], [0.20,0.35])
C_2	([0.15,0.30], [0.60,0.70])	([0.50,0.50], [0.50,0.50])	([0.65,0.75], [0.10,0.25])
C_3	([0.20,0.35], [0.55,0.65])	([0.10,0.25], [0.65,0.75])	([0.50,0.50], [0.50,0.50])

TABLE IV
DECISION MATRIX

Portfolio	C_1	C_2	C_3
A_1	([0.60,0.70], [0.20,0.30])	([0.40,0.50], [0.20,0.40])	([0.70,0.80], [0.10,0.20])
A_2	([0.50,0.60], [0.10,0.30])	([0.60,0.70], [0.20,0.30])	([0.60,0.70], [0.10,0.20])

TABLE V
SORTING RESULTS OF DIFFERENT PORTFOLIOS

Formulae	Original score		Normalized score		Sorting results
	A_1	A_2	A_1	A_2	
(57)	0.4216	0.4240	0.4986	0.5014	A_1 p A_2
(58)	0.5079	0.4246	0.5447	0.4553	A_1 f A_2
(59)	1.1350	1.0920	0.5097	0.4903	A_1 f A_2
(61)	0.3713	0.3558	0.5107	0.4893	A_1 f A_2
(62)	0.3843	0.3788	0.5036	0.4964	A_1 f A_2
(65)	1.4040	1.3810	0.5041	0.4959	A_1 f A_2
(66)	0.2644	0.2458	0.5182	0.4818	A_1 f A_2
Our new function (67)	**0.3970**	**0.3931**	**0.5025**	**0.4975**	A_1 f A_2

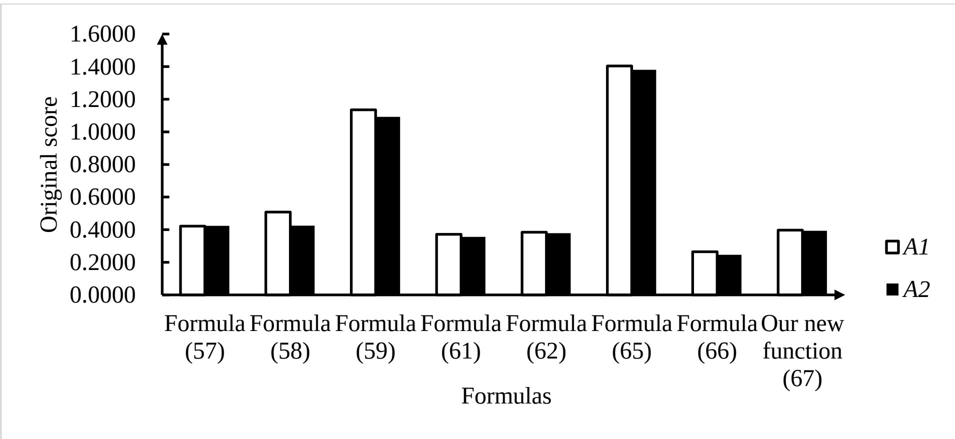

Fig. 3. Original score of different portfolios

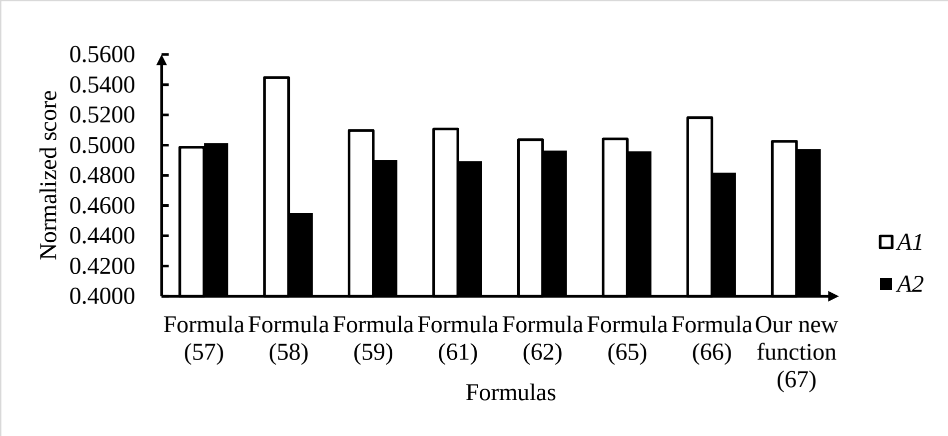

Fig. 4. Normalized score of different portfolios

VII. CONCLUSION

With the help of Xu (2007)'s intuitionistic fuzzy AHP, we apply AHP to the context of interval-valued intuitionistic fuzzy sets. Because the interval-valued intuitionistic fuzzy set is powerful in describing fuzziness and uncertainty, the interval-valued intuitionistic fuzzy AHP can describe the decision-making process more accurately, which makes the interval-valued intuitionistic fuzzy AHP have more advantages than AHP and FAHP. It can solve the problem that the classical AHP ignores the fuzziness of expert judgment. In addition, we combine interval-valued intuitionistic fuzzy AHP and entropy weight method to get the combined weight method, which overcomes the limitations of the unilateral authorization method. By the score function obtained by adjusting the parameters, we solve the problem that the previous sort functions are invalid for some interval intuitionistic fuzzy numbers. Finally, we propose a novel portfolio with the improved intuitionistic AHP and the new score function. The feasibility and validity of the model are proved by applying it to the portfolio decision problem through an example. The result of our proposed new score function is consistent with those of most existing sort functions. Therefore, our proposed score function is not only effective in practice, but also able to overcome some shortcomings of other score functions in theory. The research results of this paper will provide more theoretical and practical references for investment decision-makers. In future research, we are interested in the further theoretical discussion of the entropy weight method and the application of a new analytic hierarchy process combined with dual hesitation fuzzy sets.

REFERENCES

[1] H. M. Markowitz, "Portfolio selection," The Journal of Finance, vol. 7, no. 1, pp. 77-91, 1952.

[2] H. M. Markowitz, "Portfolio Selection: Efficient Diversification of Investments," John Wiley, pp. 12-35, 1959.

[3] H. Konno, S. R. Pliska, K. Suzuki, "Optimal portfolios with asymptotic criteria," Annals of Operations Research, vol. 45, no. 1, pp. 187-204, 1993.

[4] H. Konno, H. Yamazaki, "Mean-absolute deviation portfolio optimization model and its applications to Tokyo stock market," Management science, vol. 37, no. 5, pp. 519-531, 1991.

[5] G. M. De Athayde, R. D. Flôres Jr, "Finding a maximum skewness portfolio-a general solution to three-moments portfolio choice," Journal of Economic Dynamics and Control, vol. 28, no. 7, pp. 1335-1352, 2004.

[6] E. Jondeau, M. Rockinger, "Testing for differences in the tails of stock-market returns," Journal of Empirical Finance, vol. 10, no. 5, pp. 559-581, 2003.

[7] P. Li, Y. Han, Y. Xia, "Portfolio optimization using asymmetry robust mean absolute deviation model," Finance Research Letters, vol. 18, pp. 353-362, 2016.

[8] X. Deng, Y. K. Yuan, "A novel fuzzy dominant goal programming for portfolio selection with systematic risk and non-systematic risk," Soft Computing, vol. 25, pp. 14809-14828, 2021.

[9] L. Y. Min, Y. M. Han, Y. Xiang, "A Two-Stage Robust Omega Portfolio Optimization with Cardinality Constraints," IAENG International Journal of Applied Mathematics, vol. 53, no. 1, pp. 86-93, 2023.

[10] L. A. Zadeh, "Fuzzy sets Information and Control," vol. 8, no. 3, pp. 338-356, 1965.

[11] P. Burillo, H. Bustince, "Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets," Fuzzy Sets and Systems, vol. 118 no. 3, 305-316, 2001.

[12] E. Szmidt, J. Kacprzyk, "Entropy for intuitionistic fuzzy sets," Fuzzy sets and systems, vol. 118, no. 3, pp. 467-477, 2001.

[13] X. Deng, Y. Y. Liu, "Analysis on the Development of Digital Economy in Guangdong Province Based on Improved Entropy Method and Multivariate Statistical Analysis," Entropy, vol. 22, no. 12, pp. pp. 1-21, 2021.

[14] X. Deng, J. Y. Zhao, "Novel Hesitant Triangular Fuzzy Portfolio with Parametric Entropy Based on Credibility Theory," IAENG International Journal of Applied Mathematics, vol. 52, no. 4, 1080-1087, 2022.

[15] T. L. Saaty, L. G. Vargas, "Estimating technological coefficients by the analytic hierarchy process," Socio-Economic Planning Sciences, vol. 13, no. 6, pp. 333-336, 1979.

[16] K. T. Atanassov, "Intuitionistic fuzzy sets," Fuzzy Sets and Systems, vol. 20, no.1, pp. 87-96, 1986.

[17] K. T. Atanassov, G. Gargov, "Interval-valued valued intuitionistic fuzzy sets," Fuzzy Sets and Systems, vol. 31, no. 3, pp. 343-349, 1989.

[18] R. Sadiq, S. Tesfamariam, "Environmental decision-making under uncertainty using intuitionistic fuzzy analytic hierarchy process (IF-AHP)," Stochastic Environmental Research and Risk Assessment, vol. 23, no. 1, pp. 75-91, 2009.

[19] X. Deng, J. X. Chen, "Comparison and Analysis of Novel Score-Variance Portfolio Models based on Methods for Ranking Fuzzy Numbers," IAENG International Journal of Applied Mathematics, vol. 51, no. 3, pp. 669-679, 2021.

[20] Z. S. Xu, "Integration method of interval-valued intuitive fuzzy information and its application in decision-making," Control and decision-making, vol. 22, no. 2, pp. 215-219, 2007. (Chinese)

[21] J. Ye, "Multi-criteria fuzzy decision-making method based on a novel accuracy function under interval-valued intuitionistic fuzzy environment," Expert systems with Applications, vol. 36, no. 3, pp. 6899-6902, 2009.

[22] V. L. G. Nayagam, S. Muralikrishnan, G. Sivaraman, "Multi-criteria decision-making method based on interval-valued intuitionistic fuzzy sets," Expert Systems with Applications, vol. 38, no. 3, pp. 1464-1467, 2011.

[23] V. L. G. Nayagam, G. Sivaraman, "Ranking of interval-valued intuitionistic fuzzy sets," Applied Soft Computing, vol. 11, no. 4, pp. 3368-3372, 2011.

[24] J. W. Gao, H. H. Liu, Y. D. Gu, "interval-valued Intuition Fuzzy Multi-Criterion Decision Method Based on Foreground Theory," Theory and practice of system engineering, vol. 34, pp. 3175-3181, 2014.
[25] J. Kang, R. Lan, S.S. Wang, "The accurate function of the interval-valued intuitive ambiguity and its application to decision making," Xi'an University of Posts and Telecommunications Journal, vol. 20, no. 3, pp. 86-91. 2015; (Chinese)
[26] C. Y. Wang, S. M. Chen, "An improved multi-attribute decision making method based on new score function of interval-valued intuitionistic fuzzy values and linear programming methodology," Information Sciences, vol. 411, pp. 176-184, 2017.
[27] C. Y. Wang, S. M. Chen, "A new multiple attribute decision making method based on linear programming methodology and novel score function and novel accuracy function of interval-valued intuitionistic fuzzy values," Information Sciences, vol. 438, pp. 145-155, 2018.
[28] Q. H. Zheng, X. Y. Tian, M. Q. Yang, H. K. Su, "The email author identification system based on Support Vector Machine (SVM) and Analytic Hierarchy Process (AHP)," IAENG International Journal of Computer Science, vol. 46, no. 2, pp. 178-191, 2019.
[29] J. J. Wang, C. F. Zhang, Y. Y. Jing, G. Z. Zheng, "Using the fuzzy multi-criteria model to select the optimal cool storage system for air conditioning," Energy and Buildings, vol. 40, no.11, pp. 2059-2066, 2008.
[30] L. Yi, Y. Guo, N. Liu, J. Liu, J. Zhao, G. Jiang, "Health Status Sensing of Catenary Based on Combination Weighting and Normal Cloud Model," Arabian Journal for Science and Engineering, vol. 47, no. 3, pp. 2835-2849, 2022.
[31] Y. S. Hu, C. L. Zhu, "Credit evaluation model of road transportation enterprises based on the combination weighting method," Mathematical Problems in Engineering, pp. 1-10, 2021.
[32] F. Tan, J. Wang, Y. Y. Jiao, B. Ma, L. He, "Suitability evaluation of underground space based on finite interval cloud model and genetic algorithm combination weighting," Tunnelling and Underground Space Technology, vol. 108, pp. 1-15, 2021.
[33] Z. Wu, S. Huang, W. Hu, "Research on the selection of green building projects based on combination weighting-improved TOPSIS method," World Scientific Research Journal, vol. 6, no. 10, pp. 93-105, 20220.
[34] J. D. Genett, Jimenez-Delgado, A. F. Daniel, G. T. Diana, O. E. Pedro, G.D. Melissa, "Evaluating the Performance in the Environmental Management and Reverse Logistics in Companies of Plastic Sector: An Integration of Fuzzy AHP, DEMATEL and TOPSIS Methods," Lecture Notes in Computer Science, vol. 13097, pp. 525-545, 2021.
[35] Z. S. Xu, H. Liao, "Intuitionistic fuzzy analytic hierarchy process," IEEE transactions on fuzzy systems, vol. 22, no. 4, pp. 749-761, 2013.
[36] R. Verma, S. Chandra, "Interval-valued intuitionistic fuzzy-analytic hierarchy process for evaluating the impact of security attributes in fog-based Internet of Things paradigm," Computer Communications vol. 175, pp. 35-46, 2021.
[37] M. Li, C. Wu, L. Zhang, "Intuitionistic Fuzzy Decision Method Based on Prospect Theory and Its Application in Distribution Center Location," Journal of Xihua University (Natural Science Edition), vol. 34, no. 6, pp. 1-5+11, 2015. (Chinese)