A Mixed LT and BEM for Unsteady Harmonic Acoustic Problems of Anisotropic Trigonometrically Graded Materials

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Abstract—**The primary aim of this paper is to obtain numerical solutions for the unsteady Helmholtz equation governing harmonic acoustic problems in anisotropic trigonometrically graded materials. To achieve this, the paper proposes a method that combines Laplace transform (LT) and boundary element method (BEM). Several examples of problems related to anisotropic trigonometrically graded media are presented to illustrate that the proposed method is accurate and straightforward to implement.**

Index Terms—**numerical investigation, unsteady anisotropic Helmholtz, FGMs, boundary element method, Laplace transform**

I. Introduction

We will consider interior harmonic acoustic problems of anisotropic functionally graded materials governed by a Helmholtz type equation with variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[\tau_{ij}\left(\mathbf{x}\right)\frac{\partial P\left(\mathbf{x},t\right)}{\partial x_j}\right]+\omega^2\left(\mathbf{x}\right)P\left(\mathbf{x},t\right)=\psi\left(\mathbf{x}\right)\frac{\partial P\left(\mathbf{x},t\right)}{\partial t} \tag{1}$$

where $i,j=1,2,\ P,\tau_{ij},\omega^2,\psi$ represent the velocity potential, diffusivity coefficient, wave number and rate of change respectively. The coefficients $[\tau_{ij}]$ is a symmetric matrix with positive determinant, and summation convention holds for repeated indices so that explicitly equation (1) takes the form

$$\frac{\partial}{\partial x_1}\left(\tau_{11}\frac{\partial P}{\partial x_1}\right)+\frac{\partial}{\partial x_1}\left(\tau_{12}\frac{\partial P}{\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(\tau_{12}\frac{\partial P}{\partial x_1}\right)$$
$$+\frac{\partial}{\partial x_2}\left(\tau_{22}\frac{\partial P}{\partial x_2}\right)+\omega^2 P=\psi\frac{\partial c}{\partial t}$$

Equation (1) is usually used to model harmonic acoustic problems (see for examples [1]–[3]).

Over the last decade, there has been a significant increase in the study of functionally graded materials (FGMs) for various applications. FGMs are artificial materials with properties that vary according to a mathematical function in both time and position, and are designed to meet specific practical requirements. Equation (1) is therefore relevant for FGMs. Thus, solving equation (1) is important for FGMs.

Numerical solutions to the Helmholtz equation have been extensively studied in the past, and these studies are typically categorized based on the anisotropy and inhomogeneity of the medium. For example, [4]–[6] dealt with isotropic equations with constant coefficients (homogeneous media), while [7] solved an isotropic equation with variable coefficients (inhomogeneous media). Recently, Azis and his colleagues have focused on steady-state problems involving different types of anisotropic equations in inhomogeneous media, such as the Helmholtz equation (see for example [3]), DCR equation (see for example [8]), modified Helmholtz equation (see for example [9]), deformation problems (see for example [10]), and scalar elliptic equation (see for example [11]). These works have explored classes of inhomogeneities that differ from the constant-plus-variable inhomogeneity.

This paper is intended to extend the recently published works in [3] for steady anisotropic Helmholtz type equation with spatially variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[\tau_{ij}\left(\mathbf{x}\right)\frac{\partial P\left(\mathbf{x},t\right)}{\partial x_j}\right]+\omega^2\left(\mathbf{x}\right)P\left(\mathbf{x},t\right)=0$$

to unsteady anisotropic Helmholtz type equation with spatially variable coefficients of the form (1).

II. Statement of problem

Solutions $P\left(\mathbf{x},t\right)$ and derivatives for equation (1) are to be determined within a time set $t\geq 0$ and a spatial domain Σ in R^2 with a smooth closed boundary $\partial\Sigma$. On $\partial\Sigma_1$ $P\left(\mathbf{x},t\right)$ is given and on $\partial\Sigma_2$

$$F\left(\mathbf{x},t\right)=\tau_{ij}\left(\mathbf{x}\right)\frac{\partial P\left(\mathbf{x},t\right)}{\partial x_i}n_j \tag{2}$$

is specified where $\partial\Sigma=\partial\Sigma_1\cup\partial\Sigma_2$ and $\mathbf{n}=(n_1,n_2)$ denotes the outward pointing normal to $\partial\Sigma$. The initial condition is taken to be

$$P\left(\mathbf{x},0\right)=0 \tag{3}$$

III. The boundary integral equation

The coefficients τ_{ij},ω^2,ψ are assumed to be of the form

$$\tau_{ij}\left(\mathbf{x}\right)=\overline{\tau}_{ij}\gamma(\mathbf{x}) \tag{4}$$
$$\omega^2\left(\mathbf{x}\right)=\overline{\omega}^2\gamma(\mathbf{x}) \tag{5}$$
$$\psi\left(\mathbf{x}\right)=\overline{\psi}\gamma(\mathbf{x}) \tag{6}$$

where the $\overline{\tau}_{ij},\overline{\omega}^2,\overline{\psi}$ are constants and γ is a differentiable function of $\mathbf{x}$. Further we assume that

$$\gamma(\mathbf{x})=\left[A\cos\left(c_0+c_ix_i\right)+B\sin\left(c_0+c_ix_i\right)\right]^2 \tag{7}$$

where $A,B,\ c_0$ and c_i are constants. Therefore if

$$\overline{\tau}_{ij}c_ic_j+\lambda=0 \tag{8}$$

then (7) satisfies

$$\overline{\tau}_{ij}\frac{\partial^2\gamma^{1/2}}{\partial x_i\partial x_j}-\lambda\gamma^{1/2}=0 \tag{9}$$

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Substitution of (4)-(6) into (1) gives

$$\overline{\tau}_{ij}\frac{\partial}{\partial x_i}\left(\gamma\frac{\partial P}{\partial x_j}\right)+\overline{\omega}^2\gamma P=\overline{\psi}\gamma\frac{\partial P}{\partial t} \tag{10}$$

Let

$$P(\mathbf{x},t)=\gamma^{-1/2}(\mathbf{x})\,\sigma(\mathbf{x},t) \tag{11}$$

therefore use of (4) and (11) in (2) yields

$$F(\mathbf{x},t)=-F_\gamma(\mathbf{x})\,\sigma(\mathbf{x},t)+\gamma^{1/2}(\mathbf{x})\,F_\sigma(\mathbf{x},t) \tag{12}$$

where

$$F_\gamma(\mathbf{x})=\overline{\tau}_{ij}\frac{\partial\gamma^{1/2}}{\partial x_j}n_i \qquad F_\sigma(\mathbf{x})=\overline{\tau}_{ij}\frac{\partial\sigma}{\partial x_j}n_i$$

Also, (10) may be written in the form

$$\overline{\tau}_{ij}\frac{\partial}{\partial x_i}\left[\gamma\frac{\partial\left(\gamma^{-1/2}\sigma\right)}{\partial x_j}\right]+\overline{\omega}^2\gamma^{1/2}\sigma=\overline{\psi}\gamma\frac{\partial\left(\gamma^{-1/2}\sigma\right)}{\partial t}$$

which can be simplified

$$\overline{\tau}_{ij}\frac{\partial}{\partial x_i}\left(\gamma^{1/2}\frac{\partial\sigma}{\partial x_j}+\gamma\sigma\frac{\partial g^{-1/2}}{\partial x_j}\right)+\overline{\omega}^2\gamma^{1/2}\sigma=\overline{\psi}\gamma^{1/2}\frac{\partial\sigma}{\partial t}$$

Use of the identity

$$\frac{\partial\gamma^{-1/2}}{\partial x_i}=-\gamma^{-1}\frac{\partial g^{1/2}}{\partial x_i}$$

implies

$$\overline{\tau}_{ij}\frac{\partial}{\partial x_i}\left(\gamma^{1/2}\frac{\partial\sigma}{\partial x_j}-\sigma\frac{\partial\gamma^{1/2}}{\partial x_j}\right)+\overline{\omega}^2\gamma^{1/2}\sigma=\overline{\psi}\gamma^{1/2}\frac{\partial\sigma}{\partial t}$$

Rearranging and neglecting the zero terms yield

$$\gamma^{1/2}\overline{\tau}_{ij}\frac{\partial^2\sigma}{\partial x_i\partial x_j}-\sigma\overline{\tau}_{ij}\frac{\partial^2\gamma^{1/2}}{\partial x_i\partial x_j}+\overline{\omega}^2\gamma^{1/2}\sigma=\overline{\psi}\gamma^{1/2}\frac{\partial\sigma}{\partial t}$$

Equation (9) then implies

$$\overline{\tau}_{ij}\frac{\partial^2\sigma}{\partial x_i\partial x_j}+\left(\overline{\omega}^2-\lambda\right)\sigma=\overline{\psi}\frac{\partial\sigma}{\partial t} \tag{13}$$

Taking the Laplace transform of (11), (12), (13) and applying the initial condition (3) we obtain

$$\begin{aligned}\sigma^*(\mathbf{x},s) &= \gamma^{1/2}(\mathbf{x})\,P^*(\mathbf{x},s) &(14)\\ F_{\sigma^*}(\mathbf{x},s) &= \left[F^*(\mathbf{x},s)+F_\gamma(\mathbf{x})\,\sigma^*(\mathbf{x},s)\right]\gamma^{-1/2}(\mathbf{x}) &(15)\end{aligned}$$

$$\overline{\tau}_{ij}\frac{\partial^2\sigma^*}{\partial x_i\partial x_j}+\left(\overline{\omega}^2-\lambda-s\overline{\psi}\right)\sigma^*=0 \tag{16}$$

where s is the variable of the Laplace-transformed domain.

An integral equation for the solution of (16) can be written as

$$\alpha(\mathbf{x}_0)\,\sigma^*(\mathbf{x}_0,s)=\int_{\partial\Sigma}\left[\Theta(\mathbf{x},\mathbf{x}_0)\,\sigma^*(\mathbf{x},s)\right.$$
$$\left.-\Psi(\mathbf{x},\mathbf{x}_0)\,F_{\sigma^*}(\mathbf{x},s)\right]dS(\mathbf{x}) \tag{17}$$

where $\mathbf{x}_0=(a,b)$, $\alpha=0$ if $(a,b)\notin\Sigma\cup\partial\Sigma$, $\alpha=1$ if $(a,b)\in\Sigma$, $\alpha=\frac{1}{2}$ if $(a,b)\in\partial\Sigma$ and $\partial\Sigma$ has a continuously turning tangent at (a,b). The so called fundamental solution Ψ in (17) is any solution of the equation

$$\overline{\tau}_{ij}\frac{\partial^2\Psi}{\partial x_i\partial x_j}+\left(\overline{\omega}^2-s\overline{\psi}-\lambda\right)\Psi=\delta(\mathbf{x}-\mathbf{x}_0)$$

and the Θ is given by

$$\Theta(\mathbf{x},\mathbf{x}_0)=\overline{\tau}_{ij}\frac{\partial\Psi(\mathbf{x},\mathbf{x}_0)}{\partial x_j}n_i$$

where δ is the Dirac delta function. For two-dimensional problems Ψ and Θ are given by

$$\Psi(\mathbf{x},\mathbf{x}_0)=\begin{cases}\frac{K}{2\pi}\ln R & \text{if } \overline{\omega}^2-s\overline{\psi}-\lambda=0\\ \frac{\imath K}{4}H_0^{(2)}(\eta R) & \text{if } \overline{\omega}^2-s\overline{\psi}-\lambda>0\\ \frac{-K}{2\pi}K_0(\eta R) & \text{if } \overline{\omega}^2-s\overline{\psi}-\lambda<0\end{cases}$$

$$\Theta(\mathbf{x},\mathbf{x}_0)=\begin{cases}\frac{K}{2\pi}\frac{1}{R}\overline{\tau}_{ij}\frac{\partial R}{\partial x_j}n_i\\ \frac{-\imath K\eta}{4}H_1^{(2)}(\eta R)\,\overline{\tau}_{ij}\frac{\partial R}{\partial x_j}n_i\\ \frac{K\eta}{2\pi}K_1(\eta R)\,\overline{\tau}_{ij}\frac{\partial R}{\partial x_j}n_i\end{cases}$$
$$\begin{cases}\text{if } \overline{\omega}^2-s\overline{\psi}-\lambda=0\\ \text{if } \overline{\omega}^2-s\overline{\psi}-\lambda>0\\ \text{if } \overline{\omega}^2-s\overline{\psi}-\lambda<0\end{cases} \tag{18}$$

where

$$\begin{aligned}K &= \ddot{\rho}/D\\ \eta &= \sqrt{|\overline{\omega}^2-s\overline{\psi}-\lambda|/D}\\ D &= \left[\overline{\tau}_{11}+2\overline{\tau}_{12}\dot{\rho}+\overline{\tau}_{22}\left(\dot{\rho}^2+\ddot{\rho}^2\right)\right]/2\\ R &= \sqrt{(\dot{x}_1-\dot{a})^2+(\dot{x}_2-\dot{b})^2}\\ \dot{x}_1 &= x_1+\dot{\rho}x_2\\ \dot{a} &= a+\dot{\rho}b\\ \dot{x}_2 &= \ddot{\rho}x_2\\ \dot{b} &= \ddot{\rho}b\end{aligned}$$

where $\dot{\rho}$ and $\ddot{\rho}$ are the real and the positive imaginary parts respectively of the complex root ρ of the quadratic

$$\overline{\tau}_{11}+2\overline{\tau}_{12}\rho+\overline{\tau}_{22}\rho^2=0$$

and $H_0^{(2)}$, $H_1^{(2)}$ denote the Hankel functions, K_0, K_1 are the modified Bessel functions, $\imath=\sqrt{-1}$. The derivatives $\partial R/\partial x_j$ needed for the calculation of the Θ in (18) are given by

$$\begin{aligned}\frac{\partial R}{\partial x_1} &= \frac{1}{R}(\dot{x}_1-\dot{a})\\ \frac{\partial R}{\partial x_2} &= \dot{\rho}\left[\frac{1}{R}(\dot{x}_1-\dot{a})\right]+\ddot{\rho}\left[\frac{1}{R}\left(\dot{x}_2-\dot{b}\right)\right]\end{aligned}$$

As can be seen in (18), the value of $\overline{\omega}^2-s\overline{\psi}-\lambda$ characterizes the fundamental solutions Ψ and Θ. Therefore a specific problem may be solved using one of the three types of fundamental solutions, depending on the value of $\overline{\omega}^2-s\overline{\psi}-\lambda$, namely the modified Helmholtz (when $\overline{\omega}^2-s\overline{\psi}-\lambda<0$), Laplace ($\overline{\omega}^2-s\overline{\psi}-\lambda=0$) or Helmholtz ($\overline{\omega}^2-s\overline{\psi}-\lambda>0$) fundamental solution.

Use of (14) and (15) in (17) yields

$$\alpha\gamma^{1/2}P^*=$$
$$\int_{\partial\Sigma}\left[\left(\gamma^{1/2}\Theta-F_\gamma\Psi\right)P^*-\left(\gamma^{-1/2}\Psi\right)F^*\right]dS \tag{19}$$

This boundary integral equation can be used to determine P^* and derivatives for all points inside the domain Σ.

The Stehfest formula can then be used for a numerical Laplace transform inversion to find the solutions P and their derivatives in the original time variable. The obtained solutions and their derivatives are for the original variable t,

TABLE I
VALUES OF V_m

V_m	$N=6$	$N=8$	$N=10$	$N=12$
V_1	1	$-1/3$	$1/12$	$-1/60$
V_2	-49	$145/3$	$-385/12$	$961/60$
V_3	366	-906	1279	-1247
V_4	-858	$16394/3$	$-46871/3$	$82663/3$
V_5	810	$-43130/3$	$505465/6$	$-1579685/6$
V_6	-270	18730	-236957.5	1324138.7
V_7		$-35840/3$	$1127735/3$	$-58375583/15$
V_8		$8960/3$	$-1020215/3$	$21159859/3$
V_9			164062.5	-8005336.5
V_{10}			-32812.5	5552830.5
V_{11}				-2155507.2
V_{12}				359251.2

which were previously transformed to the Laplace transform variable s.

The Stehfest formula is

$$\begin{aligned} P(\mathbf{x},t) &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N} V_m P^*(\mathbf{x},s_m) \\ \frac{\partial P(\mathbf{x},t)}{\partial x_1} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N} V_m \frac{\partial P^*(\mathbf{x},s_m)}{\partial x_1} \\ \frac{\partial P(\mathbf{x},t)}{\partial x_2} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N} V_m \frac{\partial P^*(\mathbf{x},s_m)}{\partial x_2} \end{aligned} \tag{20}$$

where

$$s_m = \frac{\ln 2}{t} m$$
$$V_m = (-1)^{\frac{N}{2}+m} \times$$
$$\sum_{k=\left[\frac{m+1}{2}\right]}^{\min\left(m,\frac{N}{2}\right)} \frac{k^{N/2}(2k)!}{\left(\frac{N}{2}-k\right)!k!(k-1)!(m-k)!(2k-m)!}$$

IV. NUMERICAL EXAMPLES

To validate the analysis developed in previous sections, several test cases are considered. Each test case belongs to a system governed by equation (1) and satisfies the initial and boundary conditions described in Section II. The coefficients of the system are assumed to be in the form of equations (4), (5) and (6), where $g(\mathbf{x})$ is a trigonometric function in the form of equation (7). Numerical solutions are obtained using the Boundary Element Method (BEM). A unit square with 320 equally-sized elements, and a time interval of $0 \le t \le 5$ are used for simplicity. A FORTRAN script is developed to compute solutions and measure the efficiency of the numerical procedure.

A standard BEM with constant elements and the Stehfest formula in (20) will be employed to obtain numerical solutions P, $\partial P/\partial x_1$ and $\partial P/\partial x_2$ at points $(\mathbf{x},t)$ inside the domain Σ for $t \ge 0$. A FORTRAN code is developed to compute the solutions and a short script is developed and embedded into the main FORTRAN code to calculate the values of $V_m, m = 1,2,\ldots,N$ in (20) for any even number N. The values of V_m for some values of N obtained by using the script are shown in Table I. All plots in figures are produced using the plotter software GNUPlot.

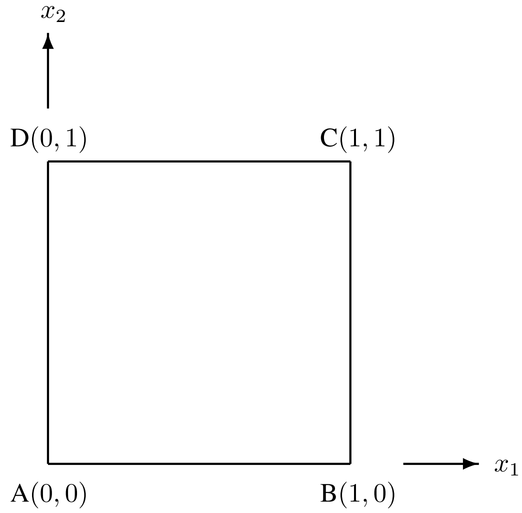

Fig. 1. The domain Σ

The Stehfest formula is used with $N = 6,8,10,12$ to investigate the convergence of error as N changes. The results show that $N = 10$ is the optimal value for which the error is stable and optimized. Increasing N from $N = 10$ to $N = 12$ leads to inaccurate solutions due to round-off errors (see for example [12]).

For all problems the inhomogeneity function is taken to be

$$\begin{aligned} \gamma^{1/2}(\mathbf{x}) &= \cos(1-0.3x_1-0.7x_2) \\ &\quad + \sin(1-0.3x_1-0.7x_2) \end{aligned}$$

and the constant anisotropy coefficient $\overline{\tau}_{ij}$

$$\overline{\tau}_{ij} = \begin{bmatrix} 1 & 0.15 \\ 0.15 & 0.75 \end{bmatrix}$$

so that 8 implies

$$\lambda = -0.5205$$

We set the constant coefficient $\overline{\omega}^2$

$$\overline{\omega}^2 = 1$$

A. Examples of exact solutions

1) Problem 1:: The exact solutions are assumed to take a separable variables form

$$P(\mathbf{x},t) = \gamma^{-1/2}(\mathbf{x}) f(\mathbf{x}) g(t)$$

where $f(\mathbf{x}), g(t)$ are continuous functions. The boundary conditions are assumed to be (see Figure 1)

F is given on side AB
F is given on side BC
P is given on side CD
F is given on side AD

Case 1:: We take

$$\begin{aligned} f(\mathbf{x}) &= 1-0.25x_1-0.75x_2 \\ g(t) &= 1-\exp(-1.75t) \end{aligned}$$

Thus for $f(\mathbf{x})$ to satisfy (16)

$$\overline{\psi} = 1.5205/s$$

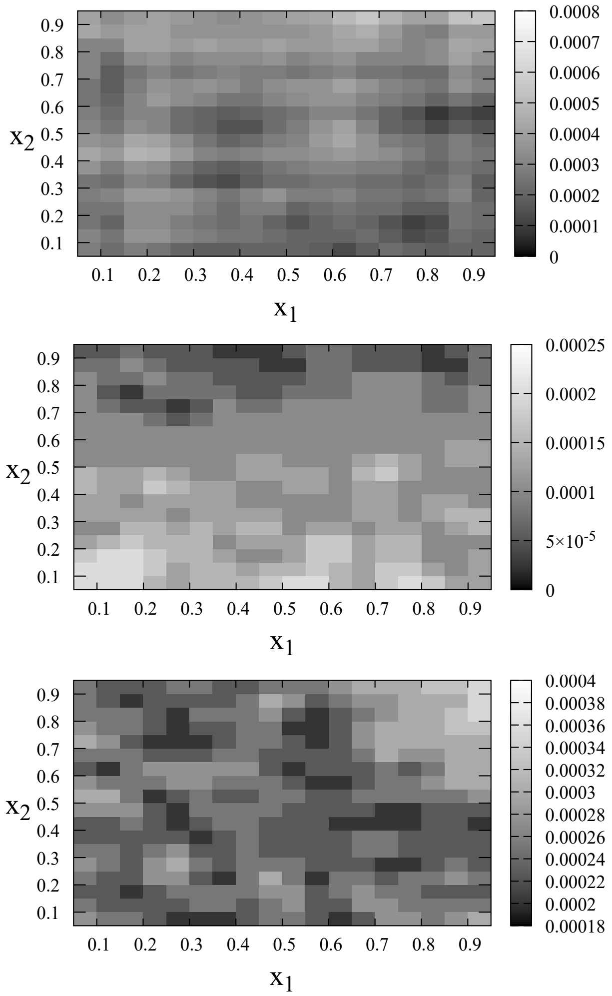

Fig. 2. The errors of interior solution P at $t = 2.5$ for the Case 1 (top), Case 2 (center), Case 3 (bottom) of Problem 1

Case 2:: For the analytical solution we take

$$\begin{aligned} f(\mathbf{x}) &= \cos\left(-1 + 0.25x_1 + 0.75x_2\right) \\ g(t) &= t/5 \end{aligned}$$

So that in order for $f(\mathbf{x})$ to satisfy (16)

$$\overline{\psi} = 0.979875/s$$

Case 3:: We take

$$\begin{aligned} f(\mathbf{x}) &= \exp\left(-1 + 0.25x_1 + 0.75x_2\right) \\ g(t) &= 0.16t\left(5 - t\right) \end{aligned}$$

Therefore (16) gives

$$\overline{\psi} = 2.061125/s$$

As depicted in Figures 2, for the numerical solutions P, the errors mainly occur in the fourth decimal place for all Cases 1, 2, amd 3. Figures 3, 4 and 5 indicate the consistency between the scattering and the flow solutions. Figures 6, 7 and 8 show that the variation of the P solution follows the way the associated function $g(t)$ changes. Specifically for the Case 1 of associated function $g(t) = 1 - \exp\left(-1.75t\right)$, the P solution will converge to 1.

For the computation of the numerical solutions the CPU elapses 5204.21875 seconds for the Case 1, 8082.859375 seconds for the Case 2, and 3119.90625 seconds for the Case 3. The longer computation time for the Cases 1 and 2 is produced by the iterative calculation of the polynomial approximation of the Hankel and Bessel functions in the fundamental solutions (18).

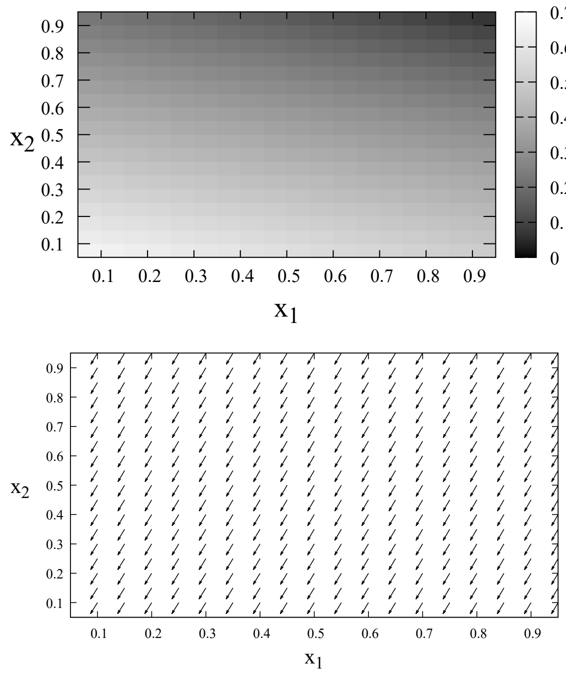

Fig. 3. The scattering P and the flow vector $(\partial P/\partial x_1, \partial P/\partial x_2)$ solutions at $t = 2.5$ for the Case 1 of Problem 1

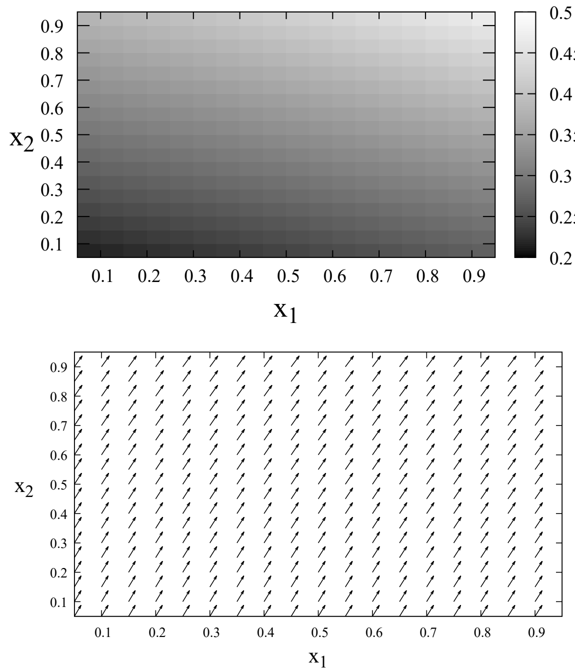

Fig. 4. The scattering P and the flow vector $(\partial P/\partial x_1, \partial P/\partial x_2)$ solutions at $t = 2.5$ for the Case 2 of Problem 1

B. Examples of no exact solutions

1) Problem 2:: The material is either inhomogeneous or homogeneous and either anisotropic or isotropic. If the material is homogeneous then

$$\gamma(\mathbf{x}) = 1$$

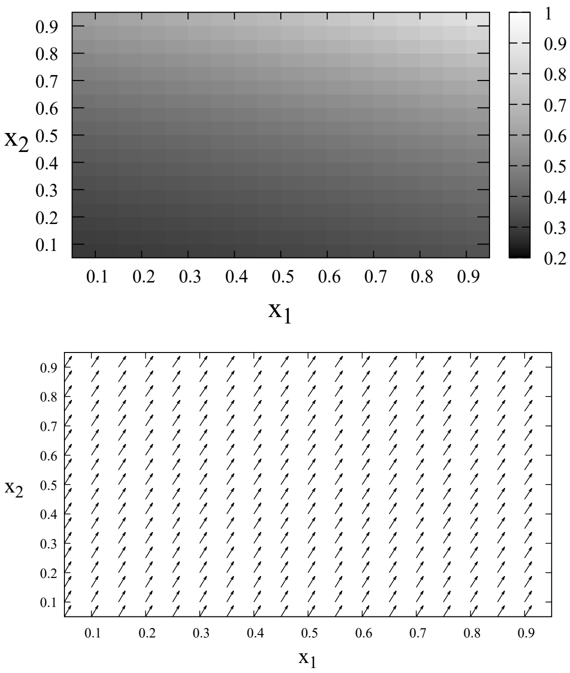

Fig. 5. The scattering P and the flow vector $(\partial P/\partial x_1, \partial P/\partial x_2)$ solutions at $t = 2.5$ for the Case 3 of Problem 1

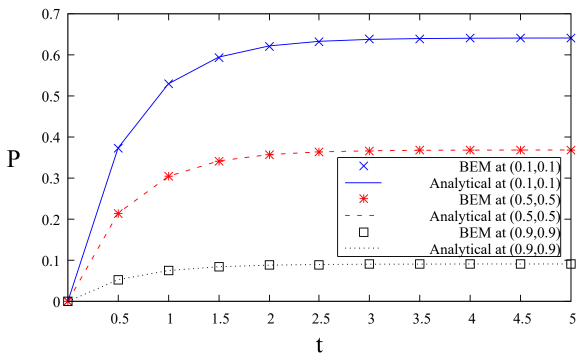

Fig. 6. Solutions P at some interior points (x_1, x_2) for the Case 1 of Problem 1

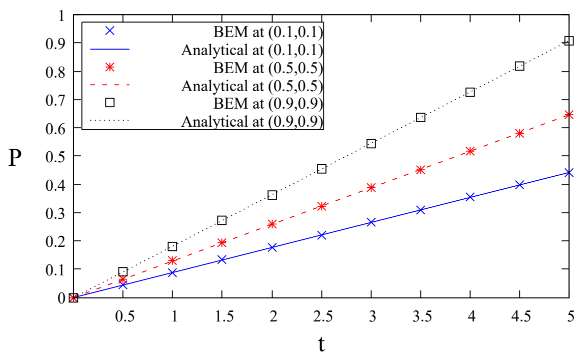

Fig. 7. Solutions P at some interior points (x_1, x_2) for the Case 2 of Problem 1

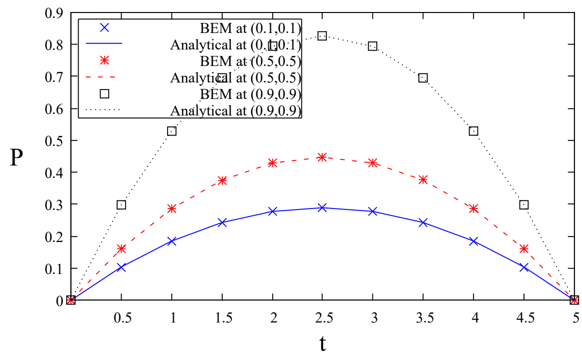

Fig. 8. Solutions P at some interior points (x_1, x_2) for the Case 3 of Problem 1

and if it is isotropic then

$$\overline{\tau}_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So that there are four cases of materials that will be considered, namely anisotropic inhomogeneous, anisotropic homogeneous, isotropic inhomogeneous and isotropic homogeneous material. The corresponding value of λ for each case is obtained from equation (8). We set $\overline{\psi} = 1$ and the boundary conditions are (see Figure 1)

$$\begin{aligned} F &= g(t) \text{ on side AB} \\ F &= 0 \text{ on side BC} \\ P &= 0 \text{ on side CD} \\ F &= 0 \text{ on side AD} \end{aligned}$$

Four cases of $g(t)$ will be considered, namely

Case 1: $g(t) = 1$
Case 2: $g(t) = 1 - \exp(-1.75t)$
Case 3: $g(t) = t/5$
Case 4: $g(t) = 0.16t\,(5 - t)$

In fact, for the case of isotropic and homogeneous material the system is geometrically symmetric about the axis $x_1 = 0.5$. And this is verified by the results in Figures 9 and 10. In addition, Figure 9 also shows the effect of anisotropy and inhomogeneity on the asymmetry of the solution P. And Figure 10 indicates that the solution P tends to follow the variation of the function $g(t)$ associated for the boundary condition on the side AB.

Figure 11 shows again the anisotropy as well as the inhomogeneity give effects on the solution P and the tendency of the solution P to agree the variation of the corresponding function $g(t)$. In particular, for bigger t the boundary conditions on the side AB with $g(t) = g_1(t) = 1$ and $g(t) = g_2(t) = 1 - \exp(-1.75t)$ are identical. This is verified by the results in Figure 11, the two plots for the cases when $g(t) = g_1(t) = 1$ and $g(t) = g_2(t) = 1 - \exp(-1.75t)$ will coincide as t goes to infinity.

V. Conclusion

Several two-dimensional transient problems for anisotropic Functionally Graded Materials (FGMs) governed by a modified Helmholtz-type equation with time-space dependent coefficients of the form (1) have been studied. The coefficients $\tau_{ij}(\mathbf{x}), \omega^2(\mathbf{x}), \psi(\mathbf{x})$ are assumed to take the forms (4), (5) and (6), respectively. By considering $\gamma(\mathbf{x})$ as a trigonometric

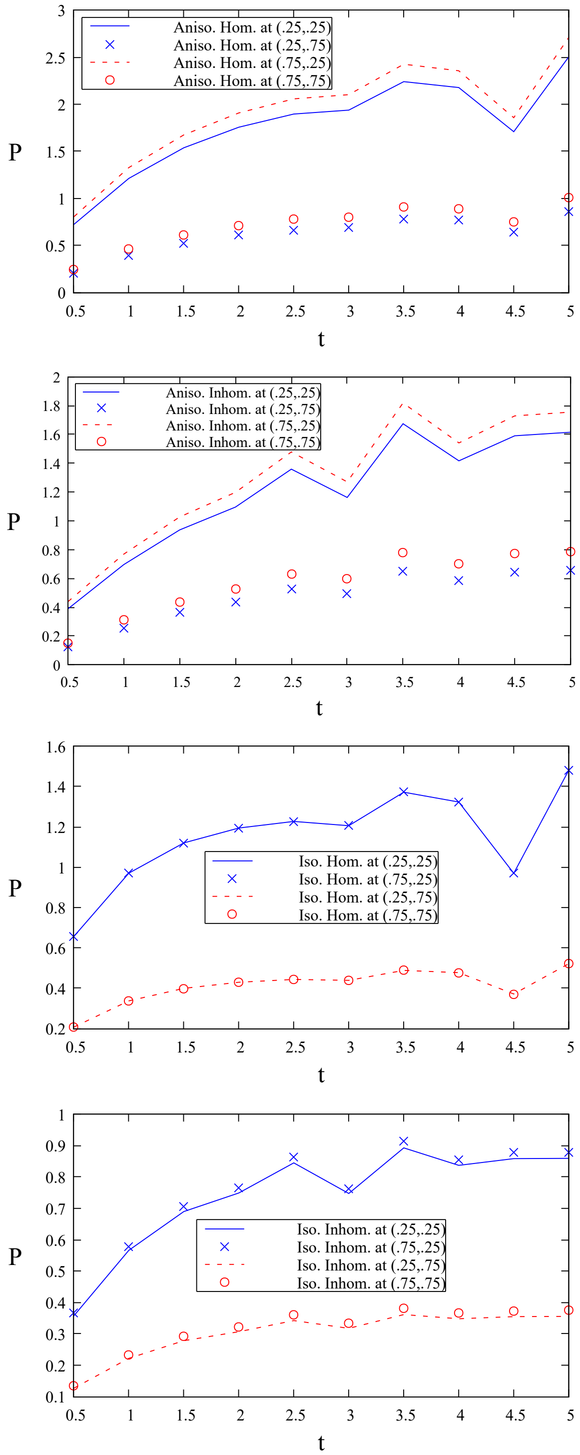

Fig. 9. Symmetry of solution P when $f(t) = 1$ for Problem 2

function of the form (7) and using the transformation (11), the space-dependent coefficients equation (1) is reduced to an equation with time-dependent coefficients (13). Taking a Laplace transform of (13) results in a constant coefficients equation (16), which can be written in the form of a boundary-only integral equation (17). This equation is then solved using a standard Boundary Element Method (BEM) to obtain the solutions P^*. These BEM solutions are then numerically inverse transformed using the Stehfest formula (20) to get the solutions P.

Several problems with trigonometric gradation functions $\gamma(\mathbf{x})$ have been solved. Based on the results obtained, it can

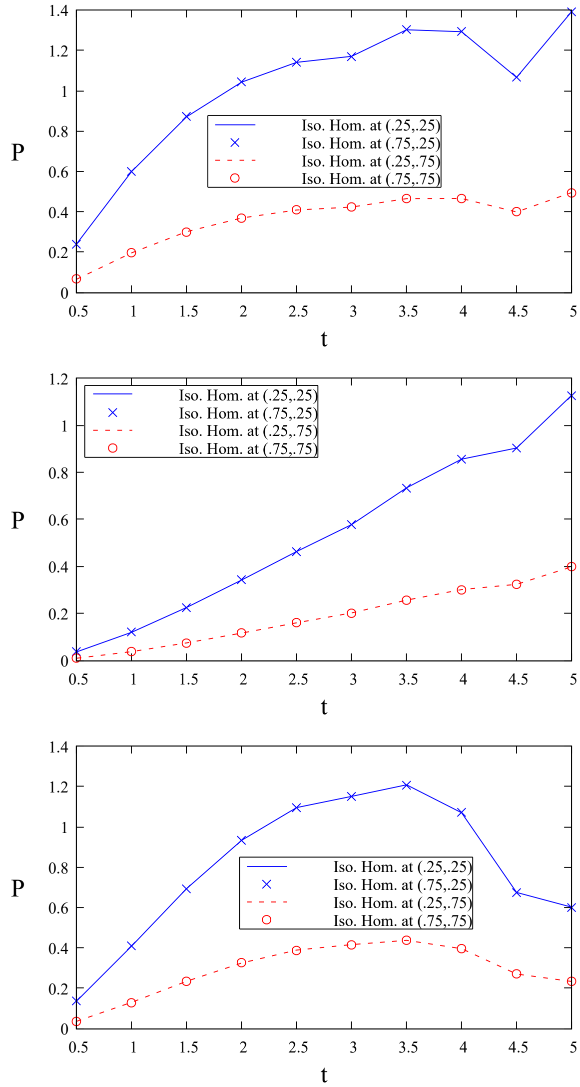

Fig. 10. Symmetry of solution P when $f(t) = 1 - \exp(-1.75t)$ (top), $f(t) = t/5$ (center) and $f(t) = 0.16t\,(5-t)$ (bottom) for Problem 2

be concluded that the combined BEM and Stehfest formula provide quite accurate solutions.

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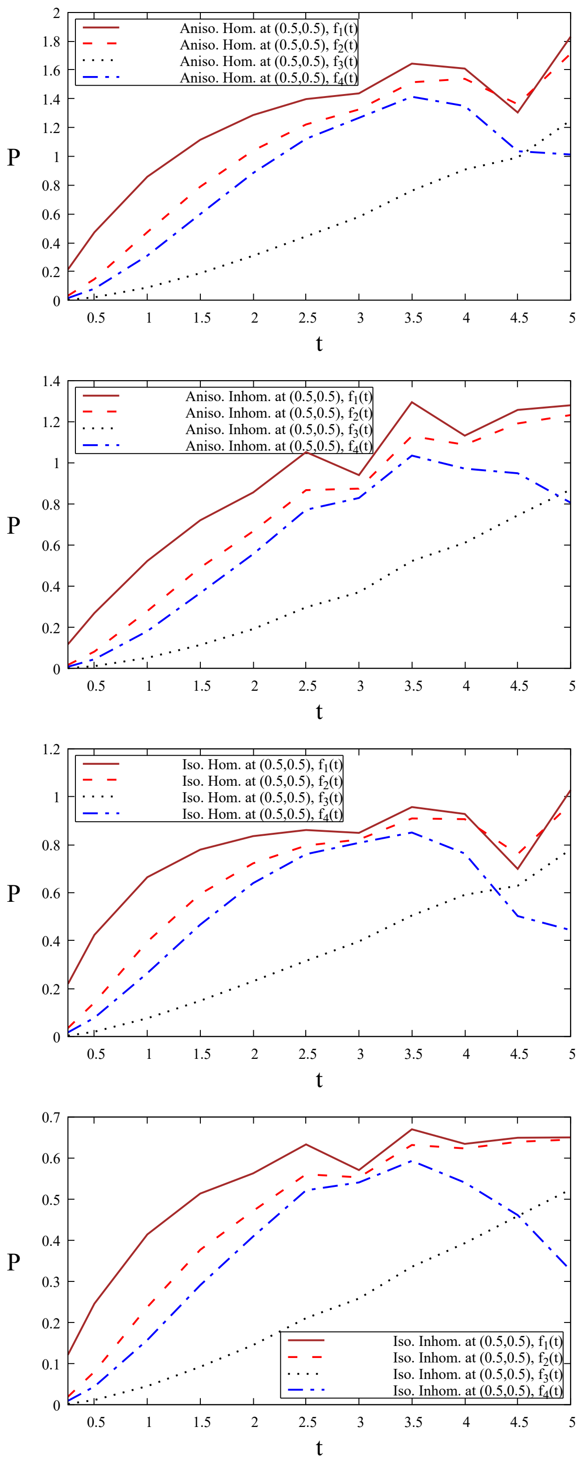

Fig. 11. Solutions P at $(x_1, x_2) = (0.5, 0.5)$ for Problem 2

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