Arithmetic Aggregation Operators on Type-2 Picture Fuzzy Sets and Their Application in Decision Making

Yongwei Yang, and Jibo Li

Abstract—**The aggregation operators are important tools in the process of information aggregation. The aim of this paper is to investigate arithmetic aggregation operators under the type-2 picture fuzzy environment with the help of new operations on type-2 picture fuzzy numbers. The type-2 picture fuzzy set is an extended version of Cuong's picture fuzzy set, which not only considers the degree of acceptance or rejection, but also takes into account the neutral degree and invalid degree during the analysis. Under these environments, by using the new operations on type-2 picture fuzzy numbers, several series of arithmetic aggregation operators for type-2 picture fuzzy sets are proposed, namely type-2 picture fuzzy weighted average operators, type-2 picture fuzzy ordered weighted average operators, and type-2 picture fuzzy hybrid average operators, along with their desirable properties. Furthermore, a decision-making approach based on these operators is also presented to make the evaluation result more objective. Finally, an illustrative example is provided to demonstrate the practicality and reasonableness of the proposed method.**

Index Terms—**Tyep-2 picture fuzzy set, Aggregation operator, Tyep-2 picture fuzzy number, Multiple attribute decision making.**

I. Introduction

MULTIPLE attribute decision making (MADM), also known as multiple objective decision making with limited schemes, refers to the decision making problem of choosing the best alternative scheme or ranking the schemes when considering multiple attributes. Its theory and method are widely used in many fields such as engineering, technology, economy, management and military affairs. Multiple attribute decision making mainly solves two problems: evaluation and selection. In order to achieve a specific goal, decision makers make decisions on the future action after analyzing, calculating and judging the factors that affect the realization of the goal with the help of certain tools, techniques and methods. However, in the process of information processing, it is impossible to avoid contact with fuzzy information or uncertain information. For dealing with ambiguity and uncertainty in our day-to-day life, Zadeh [1] conceived the concept of fuzzy sets in 1965. Fuzzy set is a popular method to describe the uncertain information of attributes, and it has been applied in all fields of the decision-making process. Regarding the method of temperature prediction, Wang et al. used a fuzzy neural network temperature prediction method with the genetic algorithm to predict the greenhouse temperature [2]. Fuzzy sets come in many forms, such as type-1 and type-2 fuzzy sets. Gong et al. proposed an integrated decision-making framework for financial technology selection under the interval type-2 fuzzy environment [3]. A prediction model on short-term load of power system was put forward based on the interval type-2 fuzzy logic system [4]. Realizing that modelling and reasoning capabilities of fuzzy cognitive maps relying on type-1 fuzzy sets are limited, Farsi et al. extend the capability of fuzzy cognitive maps for capturing greater uncertainties in the interrelations of the modelled concepts by introducing a new reasoning algorithm that uses type-2 fuzzy sets based on z slices [5].

It is necessary to mention that in a fuzzy set, a function of memberships on a scale of $[0,1]$ is used to show the degree of membership (DM) of an element in a set with the degree of non-membership (DNM) calculated by subtracting DM from 1, i.e., DNM = 1-DM. However, in a fuzzy set, only the membership grade of an element is chosen independently, and the non-membership grade is by default taken as 1 minus the membership grade. This means that choosing a non-membership grade independently is not allowed in a fuzzy set. Realizing this, Atanassov [6] refined the concept from fuzzy sets to intuitionistic fuzzy sets by ascribing membership degrees and non-membership degrees separately, ensuring that the sum of the two degrees does not exceed 1. Intuitionistic fuzzy sets are more concrete and flexible than fuzzy sets in various circumstances. They have been successfully applied in different practical fields, such as decision-making ([7], [8]), pattern recognition [9], and medical diagnosis [10].

Although Atanassov's intuitionistic fuzzy sets model revealed Zadeh's notion of fuzzy sets, there are some scenarios where there are more than two options, including the degrees of abstinence and refusal. For example, in a voting model, voters'opinions on candidates include several types: yes, abstain, no, and refusal. However, the intuitionistic fuzzy set only focuses on those who vote for or against, neglecting the neutrals and abstainers. Recognizing these shortcomings of intuitionistic fuzzy sets, Cuong [11] introduced the notion of picture fuzzy sets as a direct generalization of Zadeh's fuzzy sets and Atanassov's intuitionistic fuzzy sets. The picture fuzzy set is characterized by three functions that express the degree of membership, the degree of neutral membership, and the degree of non-membership. The only constraint is that the sum of the three degrees must not exceed 1. The

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picture fuzzy set has the capability to effectively model ambiguous and imprecise information in the real world, and it can be applied to problems or models that fuzzy sets and intuitionistic fuzzy sets cannot adequately describe or handle.

Since its proposal, the picture fuzzy set has received a lot of attention. Numerous scholars have conducted in-depth research on the theories and applications of picture fuzzy sets, and a significant number of research results have been obtained ([12]-[15]). Among these, the measurement theory includes entropy, correlation coefficient, distance, and similarity, which are hot topics in the research of picture fuzzy sets. They serve as important tools for the quantitative analysis of picture fuzzy information. Wei [16] ranked the alternatives corresponding to the cross entropy values and selected the most desirable one(s) by using the proposed picture fuzzy weighted cross entropy between the ideal alternative and an alternative. However, in some cases, due to insufficient knowledge about the problem domain and time constraints, the knowledge about criteria weights is unknown. To solve these complex real-world situations, Arya and Kumar [17] developed a new TODIM-VIKOR method based on picture fuzzy entropy measures. In [18], Pinar and Boran proposed a novel distance measure for q-rung picture fuzzy sets, which combine q-rung orthopair fuzzy sets and picture fuzzy sets. Additionally, the integration of multi-source information through information fusion theory has been applied to management decision-making, pattern recognition, and other fields. The aggregation operator, as a simple and fast information fusion method, has become a classic method for dealing with Multiple Attribute Decision Making (MADM) problems. Wang et al. [19] proposed some picture fuzzy geometric operators from a probabilistic perspective and applied these operators to MADM problems under a picture fuzzy environment. Khan et al. introduced basic aggregation operators, namely picture fuzzy Einstein weighted and Einstein ordered weighted operators, which employed Einstein norms operations [20]. Tian et al. constructed an extended picture fuzzy multi-objective optimization using ratio analysis and the full multiplicative form method, which are based on Schweizer-Sklar prioritized aggregation operators and picture fuzzy sets [21].

The picture fuzzy set is an extended version of the intuitionistic fuzzy set, which not only considers the degree of acceptance or rejection but also takes into account the neutral degree during analysis. However, in the voting model, Cuong's picture fuzzy set (referred to as type-1 picture fuzzy sets for discrimination and subsequent discussion) attributes the "invalid" degree (grouping "invalid voting papers") into the refusal membership degree in some cases. The loss of information hinders its ability to handle decision making problems effectively in certain voting scenarios. Yang [22] analyzed the limitations of Cuong's picture fuzzy set model in dealing with the special voting model and extended the original picture fuzzy set by introducing the invalid membership degree, and gave the concept of refined picture fuzzy sets, known as type-2 picture fuzzy sets. While aggregation operators are commonly used in multiple attribute decision making problems, type-2 picture fuzzy sets as a new generation of type-1 picture fuzzy sets, still have not any aggregation operators yet. Therefore, this paper develops arithmetic aggregation operators, namely type-2 picture fuzzy weighted, ordered weighted, and hybrid weighted aggregation operators, to aggregate the different preferences of decision-makers during the decision making process. To achieve this, the remainder of this paper is organized as follows: in the next section, we introduce some basic concepts related to intuitionistic fuzzy sets and picture fuzzy sets. Simultaneously, we explain the rationale behind introducing type-2 fuzzy sets. Section III presents new operational laws of type-2 picture fuzzy numbers. In Section IV, we propose type-2 picture fuzzy arithmetic aggregation operators, namely type-2 picture fuzzy weighted, ordered weighted, and hybrid weighted aggregation operators, and investigate the properties of these new operators. Section V presents a method for multiple attribute decision-making based on the new operators in a type-2 picture fuzzy environment. In order to illustrate the application of the proposed method, a practical problem is considered in Section VI. Finally, a concrete conclusion is drawn in Section VII.

II. Preliminaries

In the section, we remind some basic concepts related to fuzzy sets, intuitionistic fuzzy set, picture fuzzy sets.

A fuzzy set is defined by Zadeh, which handles uncertainty based on the view of gradualness effectively.

Definition 2.1: [1] Let X be a universe of discourse. Then a fuzzy set is defined as:

$$A = \{< x, \mu_A(x) > | x \in X\},$$

which is characterized by a membership function $\mu_A : X \to [0,1]$, where $\mu(A)(x)$ denotes the degree of membership of the element x to A.

Definition 2.2: Let X be a universe of discourse. An intuitionistic fuzzy set A in X is characterized by a membership function $\mu_A : X \to [0,1]$ and a non-membership function $\nu_A : X \to [0,1]$. For each element $x \in X$, $\mu_A(x)$ and $\nu_A(x)$ satisfy $\mu_A(x)+\nu_A(x) \leq 1$. An intuitionistic fuzzy set A can be denoted by the following symbol:

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\}.$$

By adding an extra membership function in intuitionistic fuzzy sets, namely, the degree of the neutral membership function, the type-1 picture fuzzy set is obtained.

Definition 2.3: [11] Let X be a universe of discourse. A type-1 picture fuzzy set A on the universe X is an object of the form

$$A = \{< x, \mu_A(x), \eta_A(x), \nu_A(x) > | x \in X\},$$

where $\mu_A(x) \in [0,1]$ is called the "degree of positive membership of x to the set A", $\eta_A(x) \in [0,1]$ is called the "degree of neutral membership of x to the set A", $\nu_A(x) \in [0,1]$ is called the "degree of negative membership x to the set A", and they satisfy the following condition:

$$\mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1.$$

For any $x \in X$, $\xi_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x))$ could be called the degree of refusal membership of x to the set A, which is reflection of lack of commitment or uncertainty associated with the membership or non-membership or both in A. For convenience, we call $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha)$

as a type-1 picture fuzzy number, where $\mu_\alpha \in [0,1], \eta_\alpha \in [0,1], \nu_\alpha \in [0,1]$ and $\mu_\alpha + \eta_\alpha + \nu_\alpha \leq 1$.

In the situation where $\eta_A(x) = 0$ for any $x \in X$, the type-1 picture fuzzy set reverts back to the intuitionistic fuzzy set. Similarly, when both both $\eta_A(x) = \nu_A(x) = 0$, the type-1 picture fuzzy set transforms into the fuzzy set. Consequently, the type-1 picture fuzzy set is proposed as an extension of the intuitionistic fuzzy set. It comprises four components: the degree of positive membership, the degree of neutral membership, the degree of negative membership, and the degree of refusal membership. This generalization of the intuitionistic fuzzy set offers greater versatility in addressing practical problems and real-life situations. In situations involving human opinions with various responses such as "yes", "abstain", "no", and "refusal", the type-1 picture fuzzy set model may be more appropriate. While in some cases of the election model, the type-1 picture fuzzy set model may lead to incorrect results due to omission of some information.

Example 2.4: A university's secondary college is conducting an election to select 1 dean. The college has issued 50 voting papers for the candidates α and β. The voting results are divided into five groups along with the number of papers:

α: "vote for" (35), "abstain" (6), "vote against" (4), "invalid voting papers" (3), and "refusal of voting" (2);

β: "vote for" (35), "abstain" (6), "vote against" (4), "invalid voting papers" (1), and "refusal of voting" (4);

Here, the group "abstain" indicates that the voting paper is a white paper rejecting both "agree" and "disagree" for the candidate but still counts as a vote. According to the type-1 picture fuzzy set model, the refusal degree is the ratio of the sum of the group "invalid voting papers" and the group "refusal of voting" to the total number of issued votes. Therefore, the type-1 picture fuzzy numbers for α and β are as follows, respectively,

$$\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha) = (0.70, 0.12, 0.08),$$
$$\beta = (\mu_\beta, \eta_\beta, \nu_\beta) = (0.70, 0.12, 0.08).$$

It is evident that α and β have the same type-1 picture fuzzy numbers, which means that candidates α and β have the same advantages in terms of votes. However, this does not align with our conclusion. In fact, if we denote the group "participation" as the sum of "vote for", "abstain" ,"vote against" , and "invalid voting papers", then we can get that "the number of participation" for α exceeds "the number of participation" for β. This suggests that more voters are interested in the candidate α than candidate the β and also indicates that the candidate α possesses greater social influence.

Therefore, the candidate α has more advantages than the candidate β when the values of the three indicators "vote for", "abstain", "vote against" are the same respectively.

In the context of the voting model, it is not always appropriate to include the "invalid" degree (group "invalid voting papers") in the membership degree of the "refusal" category. Yang proposed an approach where the "invalid" degree is treated as an independent membership function, resulting in an improved form of Cuong's picture fuzzy sets. This enhanced form is referred to as the type-2 picture fuzzy set.

Definition 2.5: [22] Let X be a universe of discourse. A type-2 picture fuzzy set A on the universe X is an object of the form

$$A = \{< x, \mu_A(x), \eta_A(x), \nu_A(x), \vartheta_A(x) > | x \in X\},$$

where $\mu_A(x) \in [0,1]$ is called the "degree of positive membership of x to the set A", $\eta_A(x) \in [0,1]$ is called the "degree of neutral membership of x to the set A", $\nu_A(x) \in [0,1]$ is called the "degree of negative membership of x to the set A", $\vartheta_A(x) \in [0,1]$ is called the "degree of invalid membership x to the set A", and they satisfy the following condition:

$$\mu_A(x) + \eta_A(x) + \nu_A(x) + \vartheta_A(x) \leq 1.$$

For any $x \in X$, $\rho_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x) + \vartheta_A(x))$ could be called the degree of refusal membership of x in A. If $\vartheta_A(x) = 0$ for any $x \in X$, then the type-2 picture fuzzy set is reduced to the type-1 picture fuzzy set. Thus, the type-2 picture fuzzy set is a generalized form of the type-1 picture fuzzy set.

As we mentioned above, voting can serve as a useful example to explain the type-2 picture fuzzy set model. Human voters can be categorized into five groups: "vote for", "abstain", "vote against", "invalid", and "refusal of the voting". Invalid votes refer to the votes that fail to meet the established requirements among the collected votes during an election. In the election, the confirmation of invalid votes mainly depends on whether the symbol of the voter's will on the ballot paper conforms to the provisions of the election method, and the ballot paper with irregular and unrecognizable filling is invalid. Furthermore, individuals who refuse to participate in the voting process can indicate a lack of interest in the election. From another perspective, it also shows the social popularity and influence of the candidate.

For convenience, the quadruplet $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \vartheta_\alpha)$ is called a type-2 picture fuzzy number, and the conditions are satisfying: $\mu_\alpha \in [0,1], \eta_\alpha \in [0,1], \nu_\alpha \in [0,1], \vartheta_\alpha \in [0,1]$ and $\mu_\alpha + \eta_\alpha + \nu_\alpha + \vartheta_\alpha \leq 1$.

Definition 2.6: [22] Let A, B be two type-2 picture fuzzy sets on a universe X. Some basic operations on type-2 picture fuzzy sets are defined as follows

- Intersection
$$A \sqcap B = \left\{ \left\langle \begin{matrix} x, \mu_A(x) \barwedge \mu_B(x), \eta_A(x) \barwedge \eta_B(x), \\ \nu_A(x) \veebar \nu_B(x), \vartheta_A(x) \barwedge \vartheta_B(x) \end{matrix} \right\rangle \middle| x \in X \right\};$$
- Union
$$A \sqcup B = \left\{ \left\langle \begin{matrix} x, \mu_A(x) \veebar \mu_B(x), \eta_A(x) \veebar \eta_B(x), \\ \nu_A(x) \barwedge \nu_B(x), \vartheta_A(x) \veebar \vartheta_B(x) \end{matrix} \right\rangle \middle| x \in X \right\};$$
- Complement of A:
$$A^c = \{\langle x, \nu_A(x), \eta_A(x), \mu_A(x), \vartheta_A(x) \rangle | x \in X\};$$
- Inclusion
$A \sqsubseteq B$ if and only if for any $x \in X$, $\mu_A(x) \leq \mu_B(x), \eta_A(x) \leq \eta_B(x), \nu_A(x) \geq \nu_B(x), \vartheta_A(x) \leq \vartheta_B(x)$;
- Equality
$A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

where $\veebar$ denotes maximum and $\barwedge$ denotes minimum.

III. Operational Laws of Type-2 Picture Fuzzy Numbers

To aggregate type-2 picture fuzzy preference information, some operational laws of type-2 picture fuzzy numbers are introduced in the section.

Definition 3.1: Let $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \vartheta_\alpha)$ be a type-2 picture fuzzy number. A score function S of a type-2 picture fuzzy number can be represented as follows:

$$S(\alpha) = \mu_\alpha - \nu_\alpha.$$

Definition 3.2: Let α be a type-2 picture fuzzy number. An accuracy function H of a type-2 picture fuzzy number can be defined as:

$$H(\alpha) = \mu_\alpha + \eta_\alpha + \nu_\alpha.$$

According to Definition 3.2, it is easy to see that $H(\alpha)$ is used to evaluate the degree of accuracy of the type-2 picture fuzzy number α. The larger the value of $H(\alpha)$, the more the degree of accuracy of the type-2 picture fuzzy number α.

Definition 3.3: Let α be a type-2 picture fuzzy number. A participation function P of a type-2 picture fuzzy number can be given by:

$$P(\alpha) = \mu_\alpha + \eta_\alpha + \nu_\alpha + \vartheta_\alpha.$$

We can also use voting as a fitting example to explain the above definitions. In this scenario, $S(\alpha) = \mu_\alpha - \nu_\alpha$ represents the goal difference, $H(\alpha) = \mu_\alpha + \eta_\alpha + \nu_\alpha$ can be interpreted as the effective degree of voting, and $P(\alpha) = \mu_\alpha + \eta_\alpha + \nu_\alpha + \vartheta_\alpha$ can be understood as the voting participation degree. When $S(\alpha)$ increases, it indicates that there are more individuals are voting in favor of the candidate α, resulting in a decrease in the number of people voting against α. When $H(\alpha)$ increases, we can deduce that a greater number of people have voted either for or against α, while fewer people have cast invalid votes or opted for refusal. So, $H(\alpha)$ depicts the effective degree of voting. An increase in $P(\alpha)$ reveals a heightened interest among individuals to vote for α, thereby indicating a greater social influence and higher popularity for the candidate α.

Based on score functions, accuracy functions and participation functions, a comparison method of type-2 picture fuzzy numbers is proposed.

Definition 3.4: Let $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \vartheta_\alpha)$ and $\beta = (\mu_\beta, \eta_\beta, \nu_\beta, \vartheta_\beta)$ be two type-2 picture fuzzy numbers, $S(\alpha)$ and $S(\beta)$ be the scores of α and β, respectively. Let $H(\alpha)$ and $H(\beta)$ be the accuracy degrees of α and β, respectively. And let $P(\alpha)$ and $P(\beta)$ be the participation degrees of α and β, respectively.

- If $S(\alpha) < S(\beta)$, then β is superior than α, denoted by $\alpha < \beta$.
- If $S(\alpha) = S(\beta)$, and $H(\alpha) < H(\beta)$, then β is superior than α, denoted by $\alpha < \beta$.
- If $S(\alpha) = S(\beta)$, $H(\alpha) = H(\beta)$, $P(\alpha) < P(\beta)$, then β is superior than α, denoted by $\alpha < \beta$.
- If $S(\alpha) = S(\beta)$, $H(\alpha) = H(\beta)$, $P(\alpha) = P(\beta)$, then α and β represent the same information, denoted by denoted by $\alpha = \beta$.

Motivated by the operations of the type-1 picture fuzzy numbers [19], we explain the type-2 picture fuzzy number from a view point of probability. In the voting model, for a type-2 picture fuzzy number $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \vartheta_\alpha)$, it represents that voters are divided into five groups: vote for (its ratio is denoted as μ), abstain (its ratio is denoted as η), vote against (its ratio is denoted as ν), invalid (its ratio is denoted as ϑ), refusal (its ratio is denoted as ρ).

In order to combine two type-2 picture fuzzy numbers $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \vartheta_\alpha)$ and $\beta = (\mu_\beta, \eta_\beta, \nu_\beta, \vartheta_\beta)$, we can construct the joint probability as Table I.

We come to choose those who vote for α or β, then

$$\begin{aligned}\mu_{\alpha\oplus\beta} =& \mu_\alpha\mu_\beta + \mu_\alpha\eta_\beta + \mu_\alpha\nu_\beta + \mu_\alpha\vartheta_\beta + \mu_\alpha\rho_\beta \\ &+ \eta_\alpha\mu_\beta + \nu_\alpha\mu_\beta + \vartheta_\alpha\mu_\beta + \rho_\alpha\mu_\beta \\ =& \mu_\alpha + \mu_\beta - \mu_\alpha\mu_\beta \\ =& 1 - (1-\mu_\alpha)(1-\mu_\beta).\end{aligned}$$

Those who are abstain for α and abstain for β can be viewed as abstain for α and β. That is, $\eta_{\alpha\oplus\beta} = \eta_\alpha\eta_\beta$. We can also choose those who vote against α and β as $\nu_{\alpha\oplus\beta} = \nu_\alpha\nu_\beta$. Those who are invalid for α or β as $\vartheta_{\alpha\oplus\beta} = \vartheta_\alpha\eta_\beta + \vartheta_\alpha\vartheta_\beta + \eta_\alpha\vartheta_\beta = (\eta_\alpha + \vartheta_\alpha)(\eta_\beta + \vartheta_\beta) - \eta_\alpha\eta_\beta$. The rest products in Table I are considered as $\rho_{\alpha\oplus\beta}$. We can obtain that

$$\begin{aligned}\lambda\alpha :=& \underbrace{\alpha \oplus \alpha \oplus \cdots \oplus \alpha}_{\lambda} \\ =& \left(1 - (1-\mu_\alpha)^\lambda, \eta_\alpha^\lambda, \nu_\alpha^\lambda, (\eta_\alpha + \vartheta_\alpha)^\lambda - \vartheta_\alpha^\lambda\right).\end{aligned}$$

After simplify these expressions, we can introduce the following definition.

Definition 3.5: Let $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \vartheta_\alpha)$ and $\beta = (\mu_\beta, \eta_\beta, \nu_\beta, \vartheta_\beta)$ be two type-2 picture fuzzy numbers. Then

(1) $\alpha^c = (\nu_\alpha, \eta_\alpha, \mu_\alpha, \vartheta_\alpha)$, where c denotes the complement;

(2) $\alpha \vee \beta = \begin{pmatrix} \max\{\mu_\alpha, \mu_\beta\}, \max\{\eta_\alpha, \eta_\beta\}, \\ \min\{\nu_\alpha, \nu_\beta\}, \max\{\vartheta_\alpha, \vartheta_\beta\} \end{pmatrix}$;

(3) $\alpha \wedge \beta = \begin{pmatrix} \min\{\mu_\alpha, \mu_\beta\}, \min\{\eta_\alpha, \eta_\beta\}, \\ \max\{\nu_\alpha, \nu_\beta\}, \min\{\vartheta_\alpha, \vartheta_\beta\} \end{pmatrix}$;

(4) $\alpha \leq_p \beta$ if and only if $\mu_\alpha \leq \mu_\beta, \eta_\alpha \leq \eta_\beta, \nu_\alpha \geq \nu_\beta, \vartheta_\alpha \leq \vartheta_\beta$;

(5) $\alpha \oplus \beta = \begin{pmatrix} 1 - (1-\mu_\alpha)(1-\mu_\beta), \eta_\alpha\eta_\beta, \\ \nu_\alpha\nu_\beta, (\eta_\alpha + \vartheta_\alpha)(\eta_\beta + \vartheta_\beta) - \vartheta_\alpha\vartheta_\beta \end{pmatrix}$;

(6) $\lambda\alpha = \left(1 - (1-\mu_\alpha)^\lambda, \eta_\alpha^\lambda, \nu_\alpha^\lambda, (\eta_\alpha + \vartheta_\alpha)^\lambda - \eta_\alpha^\lambda\right)$.

Based on Definition 3.5, we can derive the following properties easily.

Theorem 3.6: Let α and β be two type-2 picture fuzzy numbers, and $\lambda, \lambda_1, \lambda_2 > 0$. Then

(1) $\alpha \oplus \beta = \beta \oplus \alpha$,
(2) $\lambda(\alpha \oplus \beta) = \lambda\alpha \oplus \lambda\beta$,
(3) $\lambda_1\alpha \oplus \lambda_2\alpha = (\lambda_1\alpha + \lambda_2)\alpha$.

IV. Type-2 Picture Fuzzy Arithmetic Aggregation Operators

In this section, we shall develop some arithmetic aggregation operators with picture fuzzy information, such as type-2 picture fuzzy weighted averaging operator, type-2 picture fuzzy ordered weighted averaging operator and type-2 picture fuzzy hybrid average operator. Let Θ be a collection of all type-2 picture fuzzy numbers.

Definition 4.1: Let α_j $(j = 1, 2, \cdots, n)$ be a collection of type-2 picture fuzzy numbers. The type-2 picture fuzzy weighted averaging (T2PFWA) operator is a mapping $f_a : \Theta^n \to \Theta$ such that

$$f_a(\alpha_1, \alpha_2, \cdots, \alpha_n) = \bigoplus_{j=1}^{n} w_j\alpha_j \tag{IV.1}$$

where $w = (w_1, w_2, \cdots, w_n)$ be the weight vector of α_j $(j = 1, 2, \cdots, n)$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

TABLE I
THE JOINT PROBABILITY OF α AND β

	vote for β	abstain for β	against for β	invalid for β	refusal for β
vote for α	$\mu_\alpha\mu_\beta$	$\mu_\alpha\eta_\beta$	$\mu_\alpha\nu_\beta$	$\mu_\alpha\vartheta_\beta$	$\mu_\alpha\rho_\beta$
abstain for α	$\eta_\alpha\mu_\beta$	$\eta_\alpha\eta_\beta$	$\eta_\alpha\nu_\beta$	$\eta_\alpha\vartheta_\beta$	$\eta_\alpha\rho_\beta$
against for α	$\nu_\alpha\mu_\beta$	$\nu_\alpha\eta_\beta$	$\nu_\alpha\nu_\beta$	$\nu_\alpha\vartheta_\beta$	$\nu_\alpha\rho_\beta$
invalid for α	$\vartheta_\alpha\mu_\beta$	$\vartheta_\alpha\eta_\beta$	$\vartheta_\alpha\nu_\beta$	$\vartheta_\alpha\vartheta_\beta$	$\vartheta_\alpha\rho_\beta$
refusal for α	$\rho_\alpha\mu_\beta$	$\rho_\alpha\eta_\beta$	$\rho_\alpha\nu_\beta$	$\rho_\alpha\vartheta_\beta$	$\rho_\alpha\rho_\beta$

Theorem 4.2: Let $\alpha_j = (\mu_{\alpha_j}, \eta_{\alpha_j}, \nu_{\alpha_j}, \vartheta_{\alpha_j})$ $(j = 1, 2, \cdots, n)$ be a collection of type-2 picture fuzzy numbers. Then the aggregated value by using T2PFWA operator is still a type-2 picture fuzzy number, and

$$\begin{aligned} &f_a(\alpha_1, \alpha_2, \cdots, \alpha_n) \\ &= \Big(1 - \prod_{j=1}^{n}(1-\mu_{\alpha_j})^{w_j}, \prod_{j=1}^{n}\eta_{\alpha_j}^{w_j}, \\ &\quad \prod_{j=1}^{n}\nu_{\alpha_j}^{w_j}, \prod_{j=1}^{n}(\eta_{\alpha_j}+\vartheta_{\alpha_j})^{w_j} - \prod_{j=1}^{n}\eta_{\alpha_j}^{w_j}\Big) \end{aligned} \tag{IV.2}$$

Proof: We shall prove the result by using the principle of mathematical induction on n.

(1) For $n = 2$, we have

$$f_a(\alpha_1, \alpha_2) = w_1\alpha_1 \oplus w_2\alpha_2.$$

With the operational laws of type-2 picture fuzzy numbers, we can get

$$\begin{aligned} w_1\alpha_1 &= (1-(1-\mu_{\alpha_1})^{w_1}, \eta_{\alpha_1}^{w_1}, \nu_{\alpha_1}^{w_1}, (\eta_{\alpha_1}+\vartheta_{\alpha_1})^{w_1} - \eta_{\alpha_1}^{w_1}). \\ w_2\alpha_2 &= (1-(1-\mu_{\alpha_2})^{w_2}, \eta_{\alpha_2}^{w_2}, \nu_{\alpha_2}^{w_2}, (\eta_{\alpha_2}+\vartheta_{\alpha_2})^{w_2} - \eta_{\alpha_2}^{w_2}) \end{aligned}$$

It follows that

$$\begin{aligned} &f_a(\alpha_1, \alpha_2) \\ &= w_1\alpha_1 \oplus w_2\alpha_2 \\ &= \big(1-(1-\mu_{\alpha_1})^{w_1}(1-\mu_{\alpha_2})^{w_2}, \eta_{\alpha_1}^{w_1}\eta_{\alpha_2}^{w_2}, \\ &\quad \nu_{\alpha_1}^{w_1}\nu_{\alpha_2}^{w_2}, (\eta_{\alpha_1}+\vartheta_{\alpha_1})^{w_1}(\eta_{\alpha_2}+\vartheta_{\alpha_2})^{w_2} - \eta_{\alpha_1}^{w_1}\eta_{\alpha_2}^{w_2}\big) \end{aligned}$$

Thus, results holds for $n = 2$.

(2) If Equation IV.2 holds for $n = k$, that is,

$$\begin{aligned} &f_a(\alpha_1, \alpha_2, \cdots, \alpha_k) \\ &= \bigoplus_{j=1}^{k} w_j\alpha_j \\ &= \Big(1 - \prod_{j=1}^{k}(1-\mu_{\alpha_j})^{w_j}, \prod_{j=1}^{k}\eta_{\alpha_j}^{w_j}, \\ &\quad \prod_{j=1}^{k}\nu_{\alpha_j}^{w_j}, \prod_{j=1}^{k}(\eta_{\alpha_j}+\vartheta_{\alpha_j})^{w_j} - \prod_{j=1}^{k}\eta_{\alpha_j}^{w_j}\Big). \end{aligned}$$

and the aggregated value is a type-2 picture fuzzy number. Then for $n = k+1$, by the operational laws of type-2 picture fuzzy numbers, we have (IV.3). It can be seen that the aggregated value is also a type-2 picture fuzzy number, thus, the result is valid for $n = k+1$.

By (1) and (2), we know that Equation IV.2 holds for all positive integers n. The proof is completed. ■

Theorem 4.3: (Idempotency) Let α_j $(j = 1, 2, \cdots, n)$ be a collection of type-2 picture fuzzy numbers. If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \vartheta_\alpha)$, then

$$f_a(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha.$$

Proof: Since $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \vartheta_\alpha)$ and $\sum_{j=1}^{n} w_j = 1$, by Theorem 4.2 we get that

$$\begin{aligned} &f_a(\alpha_1, \alpha_2, \cdots, \alpha_n) \\ &= \Big(1 - \prod_{j=1}^{n}(1-\mu_{\alpha})^{w_j}, \prod_{j=1}^{n}\eta_{\alpha}^{w_j}, \\ &\quad \prod_{j=1}^{n}\nu_{\alpha}^{w_j}, \prod_{j=1}^{n}(\eta_{\alpha}+\vartheta_{\alpha})^{w_j} - \prod_{j=1}^{n}\eta_{\alpha}^{w_j}\Big) \\ &= \Big(1 - (1-\mu_\alpha)^{\sum_{j=1}^{n} w_j}, \eta_\alpha^{\sum_{j=1}^{n} w_j}, \\ &\quad \nu_\alpha^{\sum_{j=1}^{n} w_j}, (\eta_\alpha+\vartheta_\alpha)^{\sum_{j=1}^{n} w_j} - \eta_\alpha^{\sum_{j=1}^{n} w_j}\Big) \\ &= \Big(1-(1-\mu_\alpha), \eta_\alpha, \nu_\alpha, (\eta_\alpha+\vartheta_\alpha) - \eta_\alpha\Big) \\ &= \Big(\mu_\alpha, \eta_\alpha, \nu_\alpha, \vartheta_\alpha\Big) \\ &= \alpha, \end{aligned}$$

which completes the proof. ■

Theorem 4.4: (Monotonicity) Let α_j $(j = 1, 2, \cdots, n)$ and $\beta_j (j = 1, 2, \cdots, n)$ be collections of type-2 picture fuzzy numbers. If $\alpha_j \leq_p \beta_j$ for all j, then

$$f_a(\alpha_1, \alpha_2, \cdots, \alpha_n) \leq f_a(\beta_1, \beta_2, \cdots, \beta_n).$$

Proof: By Theorem 4.2, we get

$$\begin{aligned} &f_a(\alpha_1, \alpha_2, \cdots, \alpha_n) \\ &= \Big(1 - \prod_{j=1}^{n}(1-\mu_{\alpha_j})^{w_j}, \prod_{j=1}^{n}\eta_{\alpha_j}^{w_j}, \\ &\quad \prod_{j=1}^{n}\nu_{\alpha_j}^{w_j}, \prod_{j=1}^{n}(\eta_{\alpha_j}+\vartheta_{\alpha_j})^{w_j} - \prod_{j=1}^{n}\eta_{\alpha_j}^{w_j}\Big) \end{aligned}$$

and

$$\begin{aligned} &f_a(\beta_1, \beta_2, \cdots, \beta_n) \\ &= \Big(1 - \prod_{j=1}^{n}(1-\mu_{\beta_j})^{w_j}, \prod_{j=1}^{n}\eta_{\beta_j}^{w_j}, \\ &\quad \prod_{j=1}^{n}\nu_{\beta_j}^{w_j}, \prod_{j=1}^{n}(\eta_{\beta_j}+\vartheta_{\beta_j})^{w_j} - \prod_{j=1}^{n}\eta_{\beta_j}^{w_j}\Big). \end{aligned}$$

Since $\alpha_j \leq_p \beta_j$ for all j, that is, $\mu_{\alpha_j} \leq \mu_\beta, \eta_{\alpha_j} \leq \eta_\beta, \nu_{\alpha_j} \geq \nu_\beta, \vartheta_{\alpha_j} \leq \vartheta_\beta$, we have

$$1 - \prod_{j=1}^{n}(1-\mu_{\alpha_j})^{w_j} \leq 1 - \prod_{j=1}^{n}(1-\mu_{\beta_j})^{w_j},$$

$$\prod_{j=1}^{n}\eta_{\alpha_j}^{w_j} \leq \prod_{j=1}^{n}\eta_{\beta_j}^{w_j}, \prod_{j=1}^{n}\nu_{\alpha_j}^{w_j} \geq \prod_{j=1}^{n}\nu_{\beta_j}^{w_j},$$

$$
\begin{aligned}
f_a(\alpha_1,\alpha_2,\cdots,\alpha_{k+1}) &= \bigoplus_{j=1}^{k+1} w_j\alpha_j = \bigoplus_{j=1}^{k} w_j\alpha_j \oplus w_{k+1}\alpha_{k+1} \\
&= \Big(1-\prod_{j=1}^{k}(1-\mu_{\alpha_j})^{w_j}, \prod_{j=1}^{k}\eta_{\alpha_j}^{w_j}, \prod_{j=1}^{k}\nu_{\alpha_j}^{w_j}, \prod_{j=1}^{k}(\eta_{\alpha_j}+\vartheta_{\alpha_j})^{w_j} - \prod_{j=1}^{k}\eta_{\alpha_j}^{w_j}\Big) \\
&\quad \oplus \Big(1-(1-\mu_{\alpha_{k+1}})^{w_{k+1}}, \eta_{\alpha_{k+1}}^{w_{k+1}}, \nu_{\alpha_{k+1}}^{w_{k+1}}, (\eta_{\alpha_{k+1}}+\vartheta_{\alpha_{k+1}})^{w_{k+1}} - \eta_{\alpha_{k+1}}^{w_{k+1}}\Big) \\
&= \Big(1-\prod_{j=1}^{k+1}(1-\mu_{\alpha_j})^{w_j}, \prod_{j=1}^{k+1}\eta_{\alpha_j}^{w_j}, \prod_{j=1}^{k+1}\nu_{\alpha_j}^{w_j}, \prod_{j=1}^{k+1}(\eta_{\alpha_j}+\vartheta_{\alpha_j})^{w_j} - \prod_{j=1}^{k+1}\eta_{\alpha_j}^{w_j}\Big)
\end{aligned} \tag{IV.3}
$$

$$\prod_{j=1}^{n}(\eta_{\alpha_j}+\vartheta_{\alpha_j})^{w_j} \le \prod_{j=1}^{n}(\eta_{\beta_j}+\vartheta_{\beta_j})^{w_j}.$$

Therefore, by the score definition of type-2 picture fuzzy numbers, we have $S(f_a(\alpha_1,\alpha_2,\cdots,\alpha_n)) \le S(f_a(\beta_1,\beta_2,\cdots,\beta_n))$. And we consider the following cases:

1) If $S(f_a(\alpha_1,\alpha_2,\cdots,\alpha_n)) < S(f_a(\beta_1,\beta_2,\cdots,\beta_n))$, then $f_a(\alpha_1,\alpha_2,\cdots,\alpha_n) < f_a(\beta_1,\beta_2,\cdots,\beta_n)$ by Definition 3.4.
2) If $S(f_a(\alpha_1,\alpha_2,\cdots,\alpha_n)) = S(f_a(\beta_1,\beta_2,\cdots,\beta_n))$, then we get that

$$1-\prod_{j=1}^{n}(1-\mu_{\alpha_j})^{w_j} = 1-\prod_{j=1}^{n}(1-\mu_{\beta_j})^{w_j},$$

and

$$\prod_{j=1}^{n}\nu_{\alpha_j}^{w_j} = \prod_{j=1}^{n}\nu_{\beta_j}^{w_j}.$$

Noting that $\prod_{j=1}^{n}\eta_{\alpha_j}^{w_j} \le \prod_{j=1}^{n}\eta_{\beta_j}^{w_j}$, we get that $H(f_a(\alpha_1,\alpha_2,\cdots,\alpha_n)) \le H(f_a(\beta_1,\beta_2,\cdots,\beta_n)$ according to the accuracy definition of type-2 picture fuzzy numbers. Now we consider the following two cases:

a) If

$$H(f_a(\alpha_1,\alpha_2,\cdots,\alpha_n)) < H(f_a(\beta_1,\beta_2,\cdots,\beta_n)),$$

then it follows that

$$f_a(\alpha_1,\alpha_2,\cdots,\alpha_n) < f_a(\beta_1,\beta_2,\cdots,\beta_n).$$

b) If

$$H(f_a(\alpha_1,\alpha_2,\cdots,\alpha_n)) = H(f_a(\beta_1,\beta_2,\cdots,\beta_n)),$$

then it follows that

$$\prod_{j=1}^{n}\eta_{\alpha_j}^{w_j} = \prod_{j=1}^{n}\eta_{\beta_j}^{w_j}.$$

And so $\prod_{j=1}^{n}(\eta_{\alpha_j}+\vartheta_{\alpha_j})^{w_j} - \prod_{j=1}^{n}\eta_{\alpha_j}^{w_j} \le \prod_{j=1}^{n}(\eta_{\beta_j}+\vartheta_{\beta_j})^{w_j} - \prod_{j=1}^{n}\eta_{\beta_j}^{w_j}$. By the participation function of type-2 picture fuzzy numbers, we get

$$P(f_a(\alpha_1,\alpha_2,\cdots,\alpha_n)) \le P(f_a(\beta_1,\beta_2,\cdots,\beta_n)),$$

and so

$$f_a(\alpha_1,\alpha_2,\cdots,\alpha_n) \le f_a(\beta_1,\beta_2,\cdots,\beta_n).$$

Based on the above discussion, we can obtain that $f_a(\alpha_1,\alpha_2,\cdots,\alpha_n) \le f_a(\beta_1,\beta_2,\cdots,\beta_n)$. ∎

Theorem 4.5: (Boundedness) Let α_j $(j=1,2,\cdots,n)$ be a collection of type-2 picture fuzzy numbers, and let

$$\alpha^+ = \bigvee_{j=1}^{n}\alpha_j,\ \alpha^- = \bigwedge_{j=1}^{n}\alpha_j.$$

Then we have

$$\alpha^- \le f_a(\alpha_1,\alpha_2,\cdots,\alpha_n) \le \alpha^+.$$

Proof: According to Definition 3.5, we get that $\alpha^- \le_p \alpha_j \le_p \alpha^+$ for all j. By Theorem 4.3 and Theorem 4.4, we obtain that $\alpha^- = f_a(\alpha^-,\alpha^-,\cdots,\alpha^-) \le f_a(\alpha_1,\alpha_2,\cdots,\alpha_n) \le f_a(\alpha^+,\alpha^+,\cdots,\alpha^+) = \alpha^+$. Hence, the proof is completed. ∎

Theorem 4.6: (Shift Invariance) Let α_j $(j=1,2,\cdots,n)$ be a collection of type-2 picture fuzzy numbers, and β be a type-2 picture fuzzy number. Then we have
$f_a(\alpha_1\oplus\beta,\alpha_2\oplus\beta,\cdots,\alpha_n\oplus\beta) = f_a(\alpha_1,\alpha_2,\cdots,\alpha_n)\oplus\beta$.

Proof: Let $\alpha_j = (\mu_{\alpha_j},\eta_{\alpha_j},\nu_{\alpha_j},\vartheta_{\alpha_j})$ $(j=1,2,\cdots,n)$ be a collection of type-2 picture fuzzy numbers and $\beta = (\mu_\beta,\eta_\beta,\nu_\beta,\vartheta_\beta)$ be a type-2 picture fuzzy number. Then for all j, we have

$$
\begin{aligned}
\alpha_j\oplus\beta = \Big(&1-(1-\mu_{\alpha_j})(1-\mu_\beta), \eta_{\alpha_j}\eta_\beta, \\
&(\eta_{\alpha_j}+\vartheta_{\alpha_j})(\eta_\beta+\vartheta_\beta) - \eta_{\alpha_j}\eta_\beta\Big).
\end{aligned}
$$

Since $\sum_{j=1}^{n} w_j = 1$, we have Equation IV.4. Thus $f_a(\alpha_1,\alpha_2,\cdots,\alpha_n)\oplus\beta$. ∎

Theorem 4.7: (Homogeneity) Let α_j $(j=1,2,\cdots,n)$ be a collection of type-2 picture fuzzy numbers, and $\lambda > 0$ be a real number. Then we have

$$f_a(\lambda\alpha_1,\lambda\alpha_2,\cdots,\lambda\alpha_n) = \lambda f_a(\alpha_1,\alpha_2,\cdots,\alpha_n).$$

Proof: Let $\alpha_j = (\mu_{\alpha_j},\eta_{\alpha_j},\nu_{\alpha_j},\vartheta_{\alpha_j})$ $(j=1,2,\cdots,n)$ be a collection of type-2 picture fuzzy numbers. Since

$$\lambda\alpha_j = \Big(1-(1-\mu_{\alpha_j})^\lambda, \eta_{\alpha_j}^\lambda, \nu_{\alpha_j}^\lambda, (\eta_{\alpha_j}+\vartheta_{\alpha_j})^\lambda - \eta_{\alpha_j}^\lambda\Big),$$

and

$$
\begin{aligned}
&f_a(\alpha_1,\alpha_2,\cdots,\alpha_n) \\
&\quad = \Big(1-\prod_{j=1}^{n}(1-\mu_{\alpha_j})^{w_j}, \prod_{j=1}^{n}\eta_{\alpha_j}^{w_j}, \\
&\qquad \prod_{j=1}^{n}\nu_{\alpha_j}^{w_j}, \prod_{j=1}^{n}(\eta_{\alpha_j}+\vartheta_{\alpha_j})^{w_j} - \prod_{j=1}^{n}\eta_{\alpha_j}^{w_j}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&f_a(\alpha_1 \oplus \beta, \alpha_2 \oplus \beta, \cdots, \alpha_n \oplus \beta)\\
&= \Big(1 - \prod_{j=1}^{n}(1-\mu_{\alpha_j})^{w_j}(1-\mu_\beta)^{w_j}, \prod_{j=1}^{n}\eta_{\alpha_j}^{w_j}\eta_\beta^{w_j}, \prod_{j=1}^{n}\nu_{\alpha_j}^{w_j}\nu_\beta^{w_j}, \prod_{j=1}^{n}(\eta_{\alpha_j}+\vartheta_{\alpha_j})^{w_j}(\eta_\beta+\vartheta_\beta)^{w_j} - \prod_{j=1}^{n}\eta_{\alpha_j}^{w_j}\eta_\beta^{w_j}\Big)\\
&= \Big(1 - (1-\mu_\beta)^{\sum_{j=1}^{n} w_j}\prod_{j=1}^{n}(1-\mu_{\alpha_j})^{w_j}, \eta_\beta^{\sum_{j=1}^{n} w_j}\prod_{j=1}^{n}\eta_{\alpha_j}^{w_j},\\
&\quad \nu_\beta^{\sum_{j=1}^{n} w_j}\prod_{j=1}^{n}\nu_{\alpha_j}^{w_j}, (\eta_\beta+\vartheta_\beta)^{\sum_{j=1}^{n} w_j}\prod_{j=1}^{n}(\eta_{\alpha_j}+\vartheta_{\alpha_j})^{w_j} - \eta_\beta^{\sum_{j=1}^{n} w_j}\prod_{j=1}^{n}\eta_{\alpha_j}^{w_j}\Big) \qquad \text{(IV.4)}\\
&= \Big(1 - (1-\mu_\beta)\prod_{j=1}^{n}(1-\mu_{\alpha_j})^{w_j}, \eta_\beta\prod_{j=1}^{n}\eta_{\alpha_j}^{w_j}, \nu_\beta\prod_{j=1}^{n}\nu_{\alpha_j}^{w_j}, (\eta_\beta+\vartheta_\beta)\prod_{j=1}^{n}(\eta_{\alpha_j}+\vartheta_{\alpha_j})^{w_j} - \eta_\beta\prod_{j=1}^{n}\eta_{\alpha_j}^{w_j}\Big)\\
&= \Big(1 - \prod_{j=1}^{n}(1-\mu_{\alpha_j})^{w_j}, \prod_{j=1}^{n}\eta_{\alpha_j}^{w_j}, \prod_{j=1}^{n}\nu_{\alpha_j}^{w_j}, \prod_{j=1}^{n}(\eta_{\alpha_j}+\vartheta_{\alpha_j})^{w_j} - \prod_{j=1}^{n}\eta_{\alpha_j}^{w_j}\Big) \oplus (\mu_\beta, \eta_\beta, \nu_\beta, \vartheta_\beta)\\
&= f_a(\alpha_1, \alpha_2, \cdots, \alpha_n) \oplus \beta
\end{aligned}
$$

then we get that

$$
\begin{aligned}
&f_a(\lambda\alpha_1, \lambda\alpha_2, \cdots, \lambda\alpha_n)\\
&= \Big(1 - \prod_{j=1}^{n}(1-\mu_{\alpha_j})^{\lambda w_j}, \prod_{j=1}^{n}\eta_{\alpha_j}^{\lambda w_j},\\
&\quad \prod_{j=1}^{n}\nu_{\alpha_j}^{\lambda w_j}, \prod_{j=1}^{n}(\eta_{\alpha_j}+\vartheta_{\alpha_j})^{\lambda w_j} - \prod_{j=1}^{n}\eta_{\alpha_j}^{\lambda w_j}\Big)\\
&= \lambda\Big(1 - \prod_{j=1}^{n}(1-\mu_{\alpha_j})^{w_j}, \prod_{j=1}^{n}\eta_{\alpha_j}^{w_j},\\
&\quad \prod_{j=1}^{n}\nu_{\alpha_j}^{w_j}, \prod_{j=1}^{n}(\eta_{\alpha_j}+\vartheta_{\alpha_j})^{w_j} - \prod_{j=1}^{n}\eta_{\alpha_j}^{w_j}\Big)\\
&= \lambda f_a(\alpha_1, \alpha_2, \cdots, \alpha_n).
\end{aligned}
$$

The proof is completed. ■

If there arises a need to assign weights to the ordered positions of the picture fuzzy arguments instead of weighting the arguments themselves, we propose the following type-2 picture fuzzy ordered weighted average operator.

Definition 4.8: Let α_j $(j = 1, 2, \cdots, n)$ be a collection of type-2 picture fuzzy numbers. The type-2 picture fuzzy ordered weighted averaging (T2PFOWA) operator of dimension n is a mapping $f_o : \Theta^n \to \Theta$, that has an associated weight vector $w = (w_1, w_2, \cdots, w_n)$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Furthermore,

$$f_o(\alpha_1, \alpha_2, \cdots, \alpha_n) = \bigoplus_{j=1}^{n} w_j\alpha_{\sigma(j)}$$

where $(\sigma_1, \sigma_2, \cdots, \sigma_n)$ is a permutation of $(1, 2, ..., n)$ such that $\alpha_{\sigma(j-1)} \leq \alpha_{\sigma(j)})$ for all $j = 1, 2, ..., n$.

According to the operational laws of type-2 picture fuzzy numbers, we can get the following theorems. As their proofs are similar to the ones listed above, we omit them here.

Theorem 4.9: Let $\alpha_j = (\mu_{\alpha_j}, \eta_{\alpha_j}, \nu_{\alpha_j}, \vartheta_{\alpha_j})$ $(j = 1, 2, \cdots, n)$ be a collection of type-2 picture fuzzy numbers. Then the aggregated value by using a T2PFOWA operator is still a type-2 picture fuzzy number, and

$$
\begin{aligned}
&f_o(\alpha_1, \alpha_2, \cdots, \alpha_n)\\
&= \Big(1 - \prod_{j=1}^{n}(1-\mu_{\alpha_{\sigma(j)}})^{w_j}, \prod_{j=1}^{n}\eta_{\alpha_{\sigma(j)}}^{w_j},\\
&\quad \prod_{j=1}^{n}\nu_{\alpha_{\sigma(j)}}^{w_j}, \prod_{j=1}^{n}(\eta_{\alpha_{\sigma(j)}}+\vartheta_{\alpha_{\sigma(j)}})^{w_j} - \prod_{j=1}^{n}\eta_{\alpha_{\sigma(j)}}^{w_j}\Big)
\end{aligned}
$$

Theorem 4.10: Let α_j $(j = 1, 2, \cdots, n)$ be a collection of type-2 picture fuzzy numbers.

(1) (Idempotency) If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha, \vartheta_\alpha)$, then
$$f_o(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha.$$
(2) (Boundedness) Let $\alpha^+ = \bigvee_{j=1}^{n} \alpha_j$, $\alpha^- = \bigwedge_{j=1}^{n} \alpha_j$. Then we have
$$\alpha^- \leq f_o(\alpha_1, \alpha_2, \cdots, \alpha_n) \leq \alpha^+.$$
(3) (Monotonicity) Let $\beta_j (j = 1, 2, \cdots, n)$ be a collection of type-2 picture fuzzy numbers. If $\alpha_j \leq_p \beta_j$ for all j, then
$$f_o(\alpha_1, \alpha_2, \cdots, \alpha_n) \leq f_o(\beta_1, \beta_2, \cdots, \beta_n).$$
(4) (Shift Invariance) Let β be an another type-2 picture fuzzy number. Then we have
$$f_o(\alpha_1 \oplus \beta, \alpha_2 \oplus \beta, \cdots, \alpha_n \oplus \beta) = f_o(\alpha_1, \alpha_2, \cdots, \alpha_n) \oplus \beta.$$
(5) (Homogeneity) Let $\lambda > 0$ be a real number. Then we have
$$f_o(\lambda\alpha_1, \lambda\alpha_2, \cdots, \lambda\alpha_n) = \lambda f_o(\alpha_1, \alpha_2, \cdots, \alpha_n).$$

According to Definitions 4.1 and 4.8, it is obvious that the T2PFWA operator assigns weights to the type-2 picture fuzzy number itself. On the other hand, the T2PFOWA operator assigns weights to the ordered positions of type-2 picture fuzzy numbers, rather than weighting the arguments themselves. Building upon this concept, we aim to incorporate the concept of T2POWA into the T2PFWA operator, and introduce the type-2 picture fuzzy hybrid average (T2PFHA) operator.

Definition 4.11: Let α_j $(j = 1, 2, \cdots, n)$ be a collection of type-2 picture fuzzy numbers. A type-2 picture fuzzy hybrid average (T2PFHA) operator is a mapping $f_h : \Theta^n \to \Theta$, such that

$$f_h(\alpha_1, \alpha_2, \cdots, \alpha_n) = \bigoplus_{j=1}^{n} w_j \tilde{\alpha}_{\sigma(j)}$$

where $w = (w_1, w_2, \cdots, w_n)$ is an associated weight vector with $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$, and $\tilde{\alpha}_{\sigma(j)}$ is the j-th largest element of the type-2 picture fuzzy arguments $\tilde{\alpha}_j$ $(\tilde{\alpha}_j = n\omega_j\alpha_j, j = 1,2,...,n)$, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ is the weighting vector of type-2 picture fuzzy arguments α_j $(j = 1,2,...,n)$ with $\omega_j \in [0,1]$, $\sum_{j=1}^{n} \omega_j = 1$, and n is the balancing coefficient.

Let $\alpha_j = (\mu_{\alpha_j}, \eta_{\alpha_j}, \nu_{\alpha_j}, \vartheta_{\alpha_j})$ $(j = 1,2,\cdots,n)$ be a collection of type-2 picture fuzzy numbers. Similar to Theorem 4.2, we have

$$\begin{aligned} & f_a(\alpha_1, \alpha_2, \cdots, \alpha_n) \\ = & \Big(1 - \prod_{j=1}^{n}(1-\mu_{\tilde{\alpha}_j})^{w_j}, \prod_{j=1}^{n} \eta_{\tilde{\alpha}_j}^{w_j}, \\ & \prod_{j=1}^{n} \nu_{\tilde{\alpha}_j}^{w_j}, \prod_{j=1}^{n}(\eta_{\tilde{\alpha}_j} + \vartheta_{\tilde{\alpha}_j})^{w_j} - \prod_{j=1}^{n} \eta_{\alpha_j}^{w_j}\Big) \end{aligned}$$

V. Models for MADM with the Proposed Arithmetic Aggregation Operators on Type-2 Picture Fuzzy Sets

In modern decision science, multiple attribute decision making (MADM) plays a crucial role and finds extensive applications in various fields, including economics, management, and medicine. Based on the proposed arithmetic aggregation operators for type-2 picture fuzzy numbers, we aim to develop operators specifically tailored for MADM problems within the type-2 picture fuzzy framework. Since their procedures exhibit similarities, we will focus on discussing the T2PFWG operator in this context.

Consider a typical MADM problem in which $A = \{A_i, A_2, \cdots, A_m\}$ is a series of alternatives and $G = \{G_1, G_2, \cdots, G_n\}$ is a set of attributes, with the weight vector being $w = \{w_1, w_2, \cdots, w_n\}$ satisfying $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$. For the attribute G_j $(j = 1,2,\cdots,n)$ of the alternative A_i $(i = 1,2,\cdots,m)$, decision makers are required to use a type-2 picture fuzzy number $\tilde{r}_{ij} = (\mu_{ij}, \eta_{ij}, \nu_{ij}, \vartheta_{ij})$ express their preference information with anonymity. Therefore, a type-2 picture fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij})_{m\times n}$ can be obtained.

In the following section, we shall utilize the T2PFWG operator to solve this problem.

Step 1. Standardize the original decision matrix. Generally, attributes can be divided into two types, benefit attributes (the bigger the attribute values the better) and cost attributes (the smaller the attribute values the better) in MADM. Thus, by transforming the attribute values of cost type into the attribute values, the original decision matrix $\tilde{R}$ can be standardized to a normalized type-2 picture fuzzy decision matrix $R = (r_{ij})_{m\times n} = (\mu_{ij}, \eta_{ij}, \nu_{ij}, \vartheta_{ij})_{m\times n}$ as follows

$$r_{ij} = (\mu_{ij}, \eta_{ij}, \nu_{ij}, \vartheta_{ij}) = \begin{cases} \tilde{r}_{ij}, & G_j \in I_1, \\ \tilde{r}_{ij}^c, & G_j \in I_2, \end{cases} \tag{V.1}$$

where I_1 and I_2 represent the benefit type attribute and the cost type attribute, respectively.

Step 2. For the alternatives A_i $(i = 1,2,\cdots,m)$, we utilize the decision information given in the normalized type-2 picture fuzzy decision matrix R, and the T2PFWA operator

$$\begin{aligned} \alpha_i = & f_a(r_{i1}, r_{i2}, \cdots, r_{in}) \\ = & \Big(1 - \prod_{j=1}^{n}(1-\mu_{ij})^{w_j}, \prod_{j=1}^{n} \eta_{ij}^{w_j}, \\ & \prod_{j=1}^{n} \nu_{ij}^{w_j}, \prod_{j=1}^{n}(\eta_{ij} + \vartheta_{ij})^{w_j} - \prod_{j=1}^{n} \eta_{ij}^{w_j}\Big) \end{aligned} \tag{V.2}$$

to derive the overall preference values α_i $(i = 1,2,\cdots,m)$.

Step 3. According to Definition 3.1, we calculate the scores $S(\alpha_i)$ $(i = 1,2,\cdots,m)$ of the overall type-2 picture fuzzy numbers α_i $(i = 1,2,\cdots,m)$ to rank all the alternatives A_i $(i = 1,2,\cdots,m)$. If there are same values of $S(\alpha_i)$ and $S(\alpha_j)$, then we need to calculate the accuracy degrees $H(\alpha_i)$ and $H(\alpha_j)$, respectively. Furthermore, if the values of $H(\alpha_i)$ and $H(\alpha_j)$ are same, then the participation degrees $P(\alpha_i)$ and $P(\alpha_j)$ are needed to calculate, respectively.

Step 4. Rank all the alternatives A_i $(i = 1,2,\cdots,m)$ in accordance with the values of $S(\alpha_i)$ $(i = 1,2,\cdots,m)$ and select the best one(s).

Step 5. End.

VI. Numerical Examples for Using the Proposed Method

In this section, we will provide a numerical example for potential evaluation of archivists with picture fuzzy information in order to illustrate the method proposed in this paper. Effective archives management is crucial for facilitating the use of archives. Through sorting and organization, archives can be categorized and assessed for quality, thereby facilitating tasks such as identification, storage, statistical analysis, and retrieval of archives. This ensures smooth operations at every stage of the archival process. Archivists, as maintenance and coordination staff, play a vital role in the development of archival management by leveraging their skills and abilities. In order to improve the management of archival affairs, clarify the tasks and responsibilities of archivists, and enhance their motivation and creativity, the personnel department of an organization will evaluate the service quality of archivists.

Let's consider an example where there are five archivists denoted as A_i $(i = 1,2,\cdots,5)$ in a university who need to undergo assessment. A group of experts has selected six attributes for evaluating the service quality of these archivists: (1) G_1 is the knowledge and skills, (2) G_2 represents cognitive ability, (3) G_3 represents moral characteristics, (4) G_4 represents service ability, (5) G_5 represents management ability; (6) G_6 represents service effectiveness. To ensure independent evaluations, decision makers are are required to evaluate the five archivists A_i $(i = 1,2,\cdots,5)$ based on the aforementioned six attributes. The evaluations are recorded in the form of a type-2 picture fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij})_{5\times 6}$ as presented in Table II, where r_{ij} $(i = 1,2,\cdots,5, j = 1,2,\cdots,6)$ are in the form of type-2 picture fuzzy numbers. The weight vector of the six attributes $G_i (i = 1,2,\cdots,6)$ is $w = (0.17, 0.12, 0.15, 0.2, 0.14, 0.22)$.

We apply the developed approach to evaluate the best one(s), which can be described as following.

TABLE II
THE INTUITIONISTIC FUZZY DECISION MATRIX

	G_1	G_2	G_3	G_4	G_5	G_6
A_1	(0.79,0.08,0.07,0.04)	(0.83,0.04,0.08,0.02)	(0.77,0.14,0.05,0.03)	(0.84,0.05,0.07,0.02)	(0.87,0.02,0.07,0.02)	(0.69,0.14,0.12,0.03)
A_2	(0.81,0.05,0.12,0.01)	(0.79,0.12,0.01,0.07)	(0.83,0.01,0.09,0.02)	(0.87,0.03,0.09,0.01)	(0.79,0.11,0.04,0.03)	(0.72,0.12,0.13,0.02)
A_3	(0.85,0.02,0.11,0.02)	(0.86,0.04,0.05,0.04)	(0.78,0.13,0.04,0.04)	(0.82,0.09,0.06,0.03)	(0.81,0.14,0.01,0.04)	(0.74,0.08,0.09,0.05)
A_4	(0.87,0.01,0.09,0.03)	(0.78,0.13,0.04,0.05)	(0.81,0.09,0.03,0.05)	(0.89,0.02,0.08,0.01)	(0.84,0.06,0.06,0.03)	(0.82,0.04,0.12,0.02)
A_5	(0.82,0.02,0.14,0.02)	(0.79,0.07,0.11,0.01)	(0.85,0.01,0.12,0.01)	(0.83,0.07,0.03,0.06)	(0.83,0.07,0.06,0.01)	(0.86,0.07,0.05,0.01)

Step 1. Since the six attributes $G_i (i = 1, 2, \cdots, 6)$ are all benefit attributes, so we get the normalized type-2 picture fuzzy decision matrix $R = (r_{ij})_{5\times 6} = \tilde{R}$ by using Equation V.1.

Step 2. Based on the Table II and Equation V.2, we can get the the overall preference value α_i of each archivist A_i $(i = 1, 2, \cdots, 5)$, as

$$\begin{aligned}
\alpha_1 &= f_a(r_{11}, r_{12}, \cdots, r_{16}) \\
&= (0.79975, 0.06789, 0.07614, 0.02881), \\
\alpha_2 &= f_a(r_{21}, r_{22}, \cdots, r_{26}) \\
&= (0.80640, 0.05333, 0.07028, 0.02435), \\
\alpha_3 &= f_a(r_{31}, r_{32}, \cdots, r_{36}) \\
&= (0.80936, 0.06926, 0.05209, 0.03837), \\
\alpha_4 &= f_a(r_{41}, r_{42}, \cdots, r_{46}) \\
&= (0.84322, 0.03788, 0.06808, 0.02897), \\
\alpha_5 &= f_a(r_{51}, r_{52}, \cdots, r_{56}) \\
&= (0.83443, 0.04225, 0.06916, 0.02140).
\end{aligned}$$

Step 3. The scores $S(\alpha_i)$ $(i = 1, 2, \cdots, 5)$ of the overall type-2 picture fuzzy numbers α_i $(i = 1, 2, \cdots, 5)$ are calculated by Definition 3.1 as:

$$\begin{aligned}
S(\alpha_1) &= 0.72361, \\
S(\alpha_2) &= 0.73613, \\
S(\alpha_3) &= 0.75727, \\
S(\alpha_4) &= 0.77514, \\
S(\alpha_5) &= 0.76528.
\end{aligned}$$

Step 4. The ordering of the score values $S(\alpha_i)$ $(i = 1, 2, \cdots, 5)$ is

$$S(\alpha_4) > S(\alpha_5) > S(\alpha_3) > S(\alpha_2) > S(\alpha_1).$$

Thus, the ranking of alternatives A_i $(i = 1, 2, \cdots, 5)$ is

$$A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1,$$

where $\succ$ means preferred to. Thus, the best archivist is A_4.

Step 5. End.

VII. CONCLUSION

In this study, based on the new operational laws of type-2 picture fuzzy numbers, some arithmetic aggregation operators under the type-2 picture fuzzy environment, including type-2 picture fuzzy weighted, ordered weighted and hybrid weighted aggregation operators, are developed, and their related properties are discussed. Then, an extended picture fuzzy MADM method based on the new aggregation operators is constructed to make the evaluation result more objectively. Finally, an illustrative example is provided to demonstrate the validity and applicability of the constructed method. In future research, other type-2 picture fuzzy aggregation operators and the corresponding picture fuzzy MADM methods, as well as their applications in the decision making [23], risk analysis [24] and many other fields under uncertain environments ([25]) will be explored.

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