Strong (Weak) Full cc-domination in a Graph

Anusha Laxmana, Sayinath Udupa Nagara Vinayaka*, Ravi Shankar Bhat and Prathviraj Nagaraja

***Abstract*—A clique is a maximal complete subgraph of a graph. The cc-degree (clique-clique degree) of a clique K ($d_{cc}(K)$) is the number of cliques adjacent to K. A clique C strongly clique-dominates a clique K if C is adjacent to K and $d_{cc}(C) \geq d_{cc}(K)$. Let $\mathcal{C}(G)$ be the set of all cliques in a graph G. A set $\mathcal{S} \subseteq \mathcal{C}(G)$ is a strong clique-clique dominating set (SCCD-set) of G if every clique in $\mathcal{C}(G)$-$\mathcal{S}$ is strongly clique-dominated by at least one clique in $\mathcal{S}$. The strong clique-clique domination number $\gamma_{scc}(G)$ is the cardinality of a smallest SCCD-set of G. Similarly, the weak clique-clique domination number $\gamma_{wcc}(G)$ is defined. In this paper, we study some properties of these strong (weak) clique-clique domination parameters and obtain Gallai-type results. We present an algorithm to find $\gamma_{scc}(G)$ ($\gamma_{wcc}(G)$) and obtain some bounds for the newly defined parameters. Further, we define and study clique-clique domination balanced graphs and clique-posets.**

***Index Terms*—clique-clique domination, cc-degree, full clique-clique domination, clique-poset.**

I. Introduction

The terminologies and notations used here are as in [6], [11]. Throughout the paper, all graphs are finite, simple, and undirected. Let $G = (V, E)$ be a graph. Two vertices in G are said to dominate each other if they are adjacent. Then a subset D of V is called a dominating set of G if every vertex $v \in V - D$ is dominated by some vertex $u \in D$. Then the cardinality of a minimum dominating set is called the domination number $\gamma(G)$. For a survey on domination, refer to [1], [7]. The strong (weak) domination was first introduced by E. Sampathkumar and L. Pushpa Latha [10]. A vertex v strongly (weakly) dominates a vertex u if v and u are adjacent and $deg(v) \geq deg(u)$ ($deg(v) \leq deg(u)$). A set $D \subseteq V$ is a strong-dominating set (sd-set) of G if every vertex in $V - D$ is strongly dominated by at least one vertex in D. Similarly, a weak-dominating set (wd-set) is defined. The strong (weak) domination number γ_s (γ_w) of G is the cardinality of a smallest sd-set (wd-set). This concept is extended to edges by R. S. Bhat et al. [5]. The strong (weak) domination is well studied in [2], [9].

A clique is a maximal complete subgraph of a graph. If a vertex v is in a clique C then we say that v is incident on C. Two cliques, C_1 and C_2 are said to be adjacent to each other if there is a vertex $v \in V$ incident on both C_1 and C_2. The cc-degree (clique-clique degree) [3] of a clique K, denoted by $d_{cc}(K)$ is the number of cliques adjacent to K. An ordered set of cliques $(C_1, C_2, \ldots, C_m)$ in a graph G is called a clique-path if C_i is adjacent with C_{i+1}, for every i, $1 \leq i \leq m-1$. S. G. Bhat [3] defined the clique-clique dominating sets and clique-clique full sets of a graph as follows: Let $\mathcal{C}(G)$ be the set of all cliques in a graph G. A set $\mathcal{S} \subseteq \mathcal{C}(G)$ is a clique-clique dominating set (CCD-set) of G if every clique in $\mathcal{C}(G)$-$\mathcal{S}$ is adjacent to at leat one clique in $\mathcal{S}$. The clique-clique domination number $\gamma_{cc}(G)$ is the number of cliques in a smallest clique-clique dominating set of G. A set $\mathcal{L} \subseteq \mathcal{C}(G)$ is called as a clique-clique full set of G if every clique in $\mathcal{L}$ adjacent to a clique in $\mathcal{C}(G)$-$\mathcal{L}$. The clique-clique full number of $f_{cc}(G)$ is the cardinality of a largest clique-clique full set of G. For a survey on cliques, refer to [3], [4].

Motivated by the study of the strong (weak) edge-edge domination number $\gamma_{see}(G)$ ($\gamma_{wee}(G)$) of a graph by R. S. Bhat et al. [5], we define the following:

II. Strong (weak) clique-clique dominating sets of a graph

Definition 1. *Let C_1 and C_2 be any two cliques of a graph G. We say that C_1 strongly clique dominates (sc-dominates) C_2 and C_2 weakly clique dominates (wc-dominates) C_1 if C_1 is adjacent to C_2 and $d_{cc}(C_1) \geq d_{cc}(C_2)$.*

Definition 2. *A set $\mathcal{S} \subseteq \mathcal{C}(G)$ is a strong clique-clique dominating set (SCCD-set) of G if every clique in $\mathcal{C}(G)$-$\mathcal{S}$ is sc-dominated by at least one clique in $\mathcal{S}$. The strong clique-clique domination number $\gamma_{scc}(G)$ is the cardinality of a smallest SCCD-set of G.*
Similarly, we define the weak clique-clique domination number $\gamma_{wcc}(G)$.

Example 1.

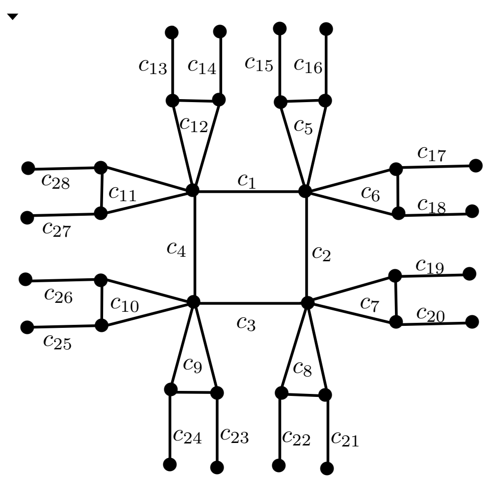

Fig. 1. Graph G with $\gamma_{scc}(G) = 10$ and $\gamma_{wcc}(G) = 18$.

Consider the graph G given in the Figure 1. Note that $\{C_5, C_6, C_7, C_8, C_9, C_{10}, C_{11}, C_{12}\}$,

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$\{C_1, C_3, C_5, C_6, C_7, C_8, C_9, C_{10}, C_{11}, C_{12}\}$ and $\{C_5, C_9, C_{13}, C_{14}, \ldots, C_{28}\}$ are some smallest CCD-set, SCCD-set and WCCD-set of G, respectively. Hence, $\gamma_{cc}(G) = 8$, $\gamma_{scc}(G) = 10$ and $\gamma_{wcc}(G) = 18$.

Remark 1. *If a graph G is triangle-free, then $\gamma_{scc}(G) = \gamma_{see}(G)$ and $\gamma_{wcc}(G) = \gamma_{wee}(G)$.*

Proposition 1. *Let $\mathcal{S}$ be a SCCD-set (WCCD-set) of G. Then $\mathcal{S}$ is a minimal SCCD-set (WCCD-set) of G if, and only if, for any $C \in \mathcal{S}$ any one of the following two conditions holds.*

(i) No clique in $\mathcal{S}$ sc-dominates (wc-dominates) C.
(ii) There exists a clique in $\mathcal{C}(G) - \mathcal{S}$ which is uniquely sc-dominated (wc-dominated) by C.

Proof: Let $\mathcal{S}$ be a minimal SCCD-set of G. Then for any $C \in \mathcal{S}$, $\mathcal{S} - \{C\}$ is not a SCCD-set of G. This implies that, there exists $K \in \mathcal{C}(G) - \mathcal{S}$ such that K is not sc-dominated by any clique in $\mathcal{S} - \{C\}$. Then either $K = C$ or $K \in \mathcal{C}(G) - \mathcal{S}$. Suppose $K = C$, clearly (i) holds. If $K \in \mathcal{C}(G) - \mathcal{S}$, then K is not sc-dominated by any clique in $\mathcal{S} - \{C\}$. Since $\mathcal{S}$ is a SCCD-set of G, C uniquely sc-dominates K. Hence (ii) holds. Conversely, Let $\mathcal{S}$ be a SCCD-set of G satisfying (i) and (ii). Suppose $\mathcal{S}$ is not minimal SCCD-set, then there exists $C \in \mathcal{S}$ such that $\mathcal{S} - \{C\}$ is a SCCD-set of G. This implies that C is sc-dominated by a clique in $\mathcal{S} - \{C\}$. That means, C does not satisfy the condition (i). Also, since $\mathcal{S} - \{C\}$ is a SCCD-set of G, every $\mathcal{C}(G) - (\mathcal{S} - \{C\})$ is sc-dominated by the cliques in $\mathcal{C}(G) - \mathcal{S}$. This means that C does not satisfy the condition (ii), which is a contradiction to our assumption. Hence, $\mathcal{S}$ is a minimal SCCD-set of G. ∎

Remark 2. *We observe that the numbers $\gamma_{wcc}(G)$ and $\gamma_{scc}(G)$ are not comparable in general. For example, consider the graph H given in the Figure 2. We observe that $\{C_2, C_3, C_4, C_7, C_8, C_{10}\}$ and $\{C_1, C_5, C_6, C_9, C_{11}\}$ are the smallest SCCD-set and WCCD-set of H, respectively. Hence, $\gamma_{wcc}(H) = 5 < 6 = \gamma_{scc}(H)$. Whereas for the graph G given in the Figure 1, we have $\gamma_{scc}(G) < \gamma_{wcc}(G)$.*

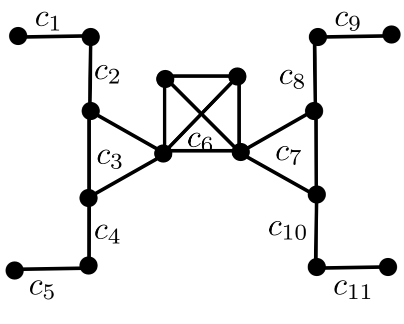

Fig. 2. Graph H with $\gamma_{wcc}(H) < \gamma_{scc}(H)$.

A. Construction of a graph G with arbitrarily large difference between $\gamma_{cc}(G)$ and $\gamma_{scc}(G)$ ($\gamma_{wcc}(G)$)

Consider a graph G_1 given in the Figure 3. For G_1, $\mathcal{D} = \{C_5, C_6, C_7, C_8\}$ is the smallest CCD-set, $\mathcal{S} = \{C_1, C_3, C_5, C_6, C_7, C_8\}$ is a smallest SCCD-set and $\mathcal{W} = \{C_1, C_3, C_9, C_{10}, C_{11}, C_{12}\}$ is a smallest WCCD-set of G_1. Hence, $\gamma_{scc}(G_1) - \gamma_{cc}(G_1) = 2$ and $\gamma_{wcc}(G_1) - \gamma_{cc}(G_1) = 2$. Let $G'_2 = G_1$ and we rename the clique

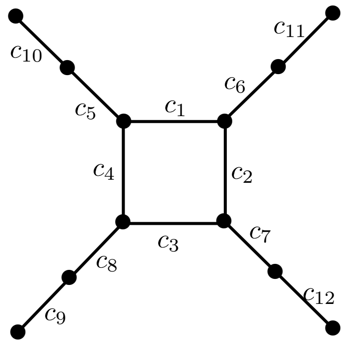

Fig. 3. Graph G_1 with $\gamma_{scc}(G_1) - \gamma_{cc}(G_1) = 2$.

C_i in G'_2 by C_{2_i}, for all i, $1 \le i \le 12$. We construct a graph G_2 by joining a pendant vertex of G_1 with a pendant vertex of G'_2, as shown in the Figure 4. Then $\mathcal{D} \cup \{C_{2_5}, C_{2_6}, C_{2_7}, C_{2_8}\}$, $\mathcal{S} \cup \{C_{2_1}, C_{2_3}, C_{2_5}, C_{2_6}, C_{2_7}, C_{2_8}\}$ and $\mathcal{W} \cup \{C_{2_1}, C_{2_3}, C_{2_9}, C_{2_{10}}, C_{2_{11}}, C_{2_{12}}\}$ are some smallest CCD-set, smallest SCCD-set and smallest WCCD-set of G_2, respectively. Note that, $\gamma_{scc}(G_2) - \gamma_{cc}(G_2) = 4$ and $\gamma_{wcc}(G_2) - \gamma_{cc}(G_2) = 4$.

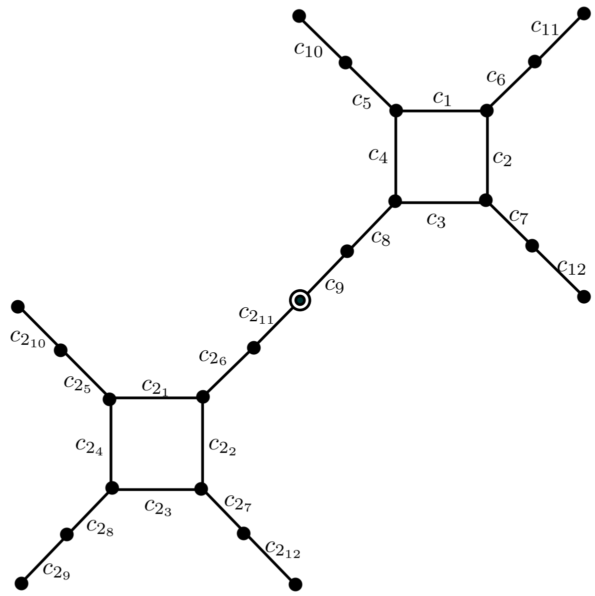

Fig. 4. Graph G_2 with $\gamma_{scc}(G_2) - \gamma_{cc}(G_2) = 4$.

For any $n \ge 3$, let Let $G'_n = G_1$ and we rename the clique C_i in G'_n by C_{n_i}, for all i, $1 \le i \le 12$. Let G_n be the graph obtained by joining a pendant vertex of G_{n-1} with a pendant vertex of G'_n. Then, $\mathcal{D} \cup \bigcup_{i=2}^{n} \{C_{i_5}, C_{i_6}, C_{i_7}, C_{i_8}\}$, $\mathcal{S} \cup \bigcup_{i=2}^{n} \{C_{i_1}, C_{i_3}, C_{i_5}, C_{i_6}, C_{i_7}, C_{i_8}\}$ and $\mathcal{W} \cup \bigcup_{i=2}^{n} \{C_{i_1}, C_{i_3}, C_{i_9}, C_{i_{10}}, C_{i_{11}}, C_{i_{12}}\}$ are some smallest CCD-set, smallest SCCD-set and smallest WCCD-set of G_n, respectively. Now, $\gamma_{scc}(G_n) - \gamma_{cc}(G_n) = 2n$ and $\gamma_{wcc}(G_n) - \gamma_{cc}(G_n) = 2n$. Hence, it is possible to find a graph G_n with arbitrarily large difference between $\gamma_{cc}(G)$ and $\gamma_{scc}(G)$ ($\gamma_{wcc}(G)$).

B. Some bounds on $\gamma_{scc}(G)$ and $\gamma_{wcc}(G)$

First we obtain some elementary bounds for $\gamma_{scc}(G)$ and $\gamma_{wcc}(G)$. Consider the set of all cliques $\mathcal{C}(G)$ of a graph G. Let $\Delta_{cc}(G)$=max$\{d_{cc}(K) : K \in \mathcal{C}(G)\}$ and $\delta_{cc}(G)$=min$\{d_{cc}(K) : K \in \mathcal{C}(G)\}$. A clique K of G with $d_{cc}(K) = 1$ is called as a cc-pendant clique of G. A clique L of G is called a cc-support clique if there is a pendant clique adjacent to it. A clique-complete graph is a graph in which all its cliques are adjacent to each other and we denote a clique-complete graph with n cliques by K_{C_n}. A graph G is called a clique-star if there exists a unique clique C in G with $d_{cc}(C) > 1$ and all other cliques of G have cc-degree 1. For a clique C, $d_{scc}(C)$ $(d_{wcc}(C))$ be the number of cliques sc-dominated (wc-dominated) by C. Let $\Delta_{scc}(G) = max\{d_{scc}(C) : C \in \mathcal{C}(G)\}$ and $\Delta_{wcc}(G) = max\{d_{wcc}(C) : C \in \mathcal{C}(G)\}$.

Proposition 2. *Let G be a graph and $\mathcal{C}(G)$ be the set of all cliques in G. Then*

(i) $\gamma_{cc}(G) \leq \gamma_{scc}(G) \leq |\mathcal{C}(G)| - \Delta_{cc}(G)$

(ii) $\gamma_{cc}(G) \leq \gamma_{wcc}(G) \leq |\mathcal{C}(G)| - \delta_{cc}(G)$

(iii) $\gamma_{cc}(G) \leq \gamma_{wcc}(G) \leq |\mathcal{C}(G)| - \Delta_{wcc}(G)$

Proof: Clearly, $\gamma_{cc}(G) \leq \gamma_{scc}(G)$ and $\gamma_{cc}(G) \leq \gamma_{wcc}(G)$. Let K be a clique of G with $d_{cc}(K) = \Delta_{cc}(G)$ and $N(K)$ be the set of all cliques adjacent to K. Then K sc-dominates all the cliques in $N(K)$. This implies that $\mathcal{C}(G) - N(K)$ is a SCCD-set of G. Hence, $\gamma_{scc}(G) \leq |\mathcal{C}(G) - N(K)| = |\mathcal{C}(G)| - \Delta_{cc}(G)$. With similar arguments we can prove the upper bound in (ii) and (iii). ■

Proposition 3. *Let G be a graph and $\mathcal{C}(G)$ be the set of all cliques in G. Then,*

(i) $\left\lceil \frac{|\mathcal{C}(G)|}{1+\Delta_{cc}(G)} \right\rceil \leq \gamma_{scc}(G) \leq |\mathcal{C}(G)| - \Delta_{cc}(G)$

(ii) $\left\lceil \frac{|\mathcal{C}(G)|}{1+\Delta_{wcc}(G)} \right\rceil \leq \gamma_{wcc}(G) \leq |\mathcal{C}(G)| - \Delta_{wcc}(G)$

Proof: Each clique of G can sc-dominate at most $\Delta_{cc}(G)$ cliques and itself. Hence, $\left\lceil \frac{|\mathcal{C}(G)|}{1+\Delta_{cc}(G)} \right\rceil \leq \gamma_{scc}(G)$. Similarly, (ii) can be proved. ■

The above bounds in Proposition 2 and 3 are sharp as both upper and lower bounds are attained for any clique-complete graph.

Proposition 4. *Let G be a graph with $n > 2$ cliques, n_p cc-pendant cliques and n_s cc-support cliques. If G has no K_{C_2} component, then*

(i) $n_s \leq \gamma_{scc}(G) \leq n - n_p$

(ii) $n_p \leq \gamma_{wcc}(G) \leq n - n_s$

Further, the above bounds are sharp.

Proof: Let $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{S}$ be the set of all cliques, set of all cc-pendant cliques and set of all cc-support cliques of G, respectively. Then any $N \in \mathcal{S}$ uniquely sc-dominates a clique $P \in \mathcal{P}$. This implies that either N or P belongs to every smallest SCCD set of G. Hence, $n_s \leq \gamma_{scc}(G)$. Also, if $K \in \mathcal{P}$ then $d_{cc}(K) = 1$. Since G has no K_{C_2} component, there exists a clique $L \in \mathcal{S} \subseteq \mathcal{C} - \mathcal{P}$ which sc-dominates K. This implies that $\mathcal{C} - \mathcal{P}$ is a SCCD-set of G. Thus, $\gamma_{scc}(G) \leq n - n_p$. The proof of (ii) is similar, and we omit it.

Note that both the upper and lower bounds in (i) and (ii) are attained for any clique-star graph. ■

III. Strong (weak) full clique-clique dominating sets of a graph

Definition 3. *A set $\mathcal{D} \subseteq \mathcal{C}(G)$ is a strong [weak] full clique-clique dominating set (SFCCD-set [WFCCD-set]) of G if every $K \in \mathcal{D}$ sc-dominates [wc-dominates] some $L \in \mathcal{C}(G) - \mathcal{D}$. The strong (weak) full clique-clique domination number $f_{scc}(G)$ ($f_{wcc}(G)$) of G is the maximum cardinality of a SFCCD-set (WFCCD-set) of G.*

Example 2. *Consider the graph H given in the Figure 2. Then $\{C_2, C_3, C_4, C_7, C_8, C_{10}\}$ is a largest SFCCD-set of H and $\{C_1, C_5, C_6, C_9, C_{11}\}$ is a largest WFCCD-set of H. Hence, $f_{scc}(H) = 6$ and $f_{wcc}(H) = 5$.*

Proposition 5. *Let G be a graph and $\mathcal{C}(G)$ be the set of all cliques of G. Then, for any set $\mathcal{D} \subseteq \mathcal{C}(G)$ the following holds.*

(i) $\mathcal{D}$ is a SCCD-set of G if, and only if, $\mathcal{C}(G) - \mathcal{D}$ is a WFCCD-set of G.

(ii) $\mathcal{D}$ is a WCCD-set of G if, and only if, $\mathcal{C}(G) - \mathcal{D}$ is a SFCCD-set of G.

Proof: Let $\mathcal{S}$ be a SCCD-set of G and $K \in \mathcal{C}(G) - \mathcal{S}$. Then K is sc-dominated by a clique say $C \in \mathcal{S}$. This implies that K and C are adjacent and $d_{cc}(K) \leq d_{cc}(C)$. That is K wc-dominates C. Hence, $\mathcal{C}(G) - \mathcal{S}$ is a WFCCD-set of G. Conversely, let $\mathcal{D}$ be a WFCCD-set of G and $\mathcal{S} = \mathcal{C}(G) - \mathcal{D}$. Then for any $K \in \mathcal{C}(G) - \mathcal{S} = \mathcal{D}$, there exists $L \in \mathcal{S}$ such that K wc-dominates L. That is, L sc-dominates K. Thus, $\mathcal{S}$ is a SCCD-set of G. Hence (i) follows. With similar arguments we can prove (ii). ■

Proposition 6. *(Gallai Type Results) Let G be a graph and $\mathcal{C}(G)$ be the set of all cliques of G. Then,*

(i) $\gamma_{scc}(G) + f_{wcc}(G) = |\mathcal{C}(G)|$

(ii) $\gamma_{wcc}(G) + f_{scc}(G) = |\mathcal{C}(G)|$

Proof: Let $\mathcal{S}$ be a smallest SCCD-set of G. Then by Proposition 5, $\mathcal{C}(G) - \mathcal{S}$ is a WFCCD-set of G. Hence, $f_{wcc}(G) \geq |\mathcal{C}(G)| - \gamma_{scc}(G)$. Now, suppose $\mathcal{D}$ is a largest WFCCD-set of G, then $\mathcal{C}(G) - \mathcal{D}$ is a SCCD-set of G. This implies that $\gamma_{scc}(G) \leq |\mathcal{C}(G)| - f_{wcc}(G)$. Thus, (i) follows. With similar arguments we can prove (ii). ■

Corollary 7. *For a graph G, there exists a SCCD-set $\mathcal{S}$ which is SFCCD-set if, and only if, there exists a WCCD-set $\mathcal{W}$ which is WFCCD-set.*

Proof: Let $\mathcal{C}(G)$ be the set of all cliques of a graph G. Then, by Proposition 5, $\mathcal{S} \subseteq \mathcal{C}(G)$ is both SCCD-set and SFCCD-set of G if, and only if, $\mathcal{W} = \mathcal{C}(G) - \mathcal{S}$ is both WCCD-set and WFCCD-set of G. ■

Corollary 8. *Let G be a graph and $\mathcal{C}(G)$ be the set of all cliques in G. If there exists a set $\mathcal{S} \subseteq \mathcal{C}(G)$ such that $\mathcal{S}$ is both SCCD-set and SFCCD-set of G, then $\gamma_{scc}(G) + \gamma_{wcc}(G) \leq |\mathcal{C}(G)|$.*

Proof: Suppose there is $\mathcal{S} \subseteq \mathcal{C}(G)$ which is both SCCD-set and SFCCD-set of G. Then by the Proposition 5, $\mathcal{C}(G) - \mathcal{S}$ is a WFCCD-set of G. This implies that, $\gamma_{wcc}(G) \leq |\mathcal{C}(G) - \mathcal{S}| = |\mathcal{C}(G)| - |\mathcal{S}|$. But, $\gamma_{scc}(G) \leq |\mathcal{S}|$. Hence, $\gamma_{wcc}(G) \leq |\mathcal{C}(G)| - \gamma_{scc}(G)$. ■

A. Algorithm to find $\gamma_{scc}(G)$ and $f_{wcc}(G)$ of a graph G with no K_{C_2} component

For any clique C of G, the open neighbourhood of C, denoted by $\mathcal{N}(C)$ is the set of all cliques in G adjacent to C and the closed neighbourhood $\mathcal{N}[C]$ of C is

$\mathcal{N}[C] = \mathcal{N}(C) \cup \{C\}$. In this section, we provide a greedy algorithm to find $\gamma_{scc}(G)$ ($\gamma_{wcc}(G)$) and $f_{scc}(G)$ ($f_{wcc}(G)$) of a graph G with no K_{C_2} component.

Input: A graph G with no K_{C_2} component and the set of all cliques $\mathcal{C}(G) = \{C_1, C_2, \ldots, C_k\}$ of G such that $d_{cc}(C_1) \geq d_{cc}(C_2) \geq \cdots \geq d_{cc}(C_k)$ and the set of all cc-support cliques $\mathcal{S}$ of G.

Output: $\gamma_{scc}(G)$ and $f_{wcc}(G)$

Algorithm:

Step 1: $\mathcal{C}(G) = \{C_1, C_2, \ldots, C_k\}$
Step 2: $\mathcal{D} = \mathcal{S}$
Step 3: if $\mathcal{D} \neq \phi$ then
Step 4: for $(i = 1; i \leq k; i = i + 1)$
Step 5: for $(j = 1; j \leq k; j = j + 1)$
Step 6: if $C_i \in \mathcal{S}$ AND $C_j \in \mathcal{N}(C_i)$ AND $d_{cc}(C_j) \leq d_{cc}(C_i)$ then
Step 7: $\mathcal{C}(G) = \mathcal{C}(G) - \{C_i, C_j\}$
Step 8: end if
Step 9: end for
Step 10: end for
Step 11: end if
Step 12: for $(i = 1; i \leq k; i = i + 1)$
Step 13: if $\mathcal{N}(C_i) \neq \phi$ then
Step 14: for $(j = i + 1; j \leq k; j = j + 1)$
Step 15: if $d_{cc}(C_i) = d_{cc}(C_j)$ then
Step 16: $\mathcal{N}_i = \phi$
Step 17: $\mathcal{N}_j = \phi$
Step 18: $temp = C_i$
Step 19: for $(l = 1; l \leq k; l = l + 1)$
Step 20: if $C_l \in \mathcal{N}(C_i)$ AND $d_{cc}(C_l) \leq d_{cc}(C_i)$ then
Step 21: $\mathcal{N}_i = \mathcal{N}_i \cup \{C_l\}$
Step 22: end if
Step 23: if $C_l \in \mathcal{N}(C_j)$ AND $d_{cc}(C_l) \leq d_{cc}(C_j)$ then
Step 24: $\mathcal{N}_j = \mathcal{N}_j \cup \{C_l\}$
Step 25: end if
Step 26: end for
Step 27: if $|\mathcal{N}_i| < |\mathcal{N}_j|$ then
Step 28: $C_i = C_j$
Step 29: $C_j = temp$
Step 30: end if
Step 31: end if
Step 32: end for
Step 33: end if
Step 34: $\mathcal{D} = \mathcal{D} \cup \{C_i\}$
Step 35: $\mathcal{C}(G) = \mathcal{C}(G) - \{C_i\}$
Step 36: for $(m = 1; m \leq k; m = m + 1)$
Step 37: if $C_m \in \mathcal{N}(C_i)$ AND $d_{cc}(C_m) \leq d_{cc}(C_i)$ then
Step 38: $\mathcal{C}(G) = \mathcal{C}(G) - \{C_m\}$
Step 39: end if
Step 40: end for
Step 41: end for
Step 42: $\gamma_{scc}(G) = |\mathcal{D}|$
Step 43: $f_{wcc}(G) = k - \gamma_{scc}(G)$

Note: Initially, the set D consists of the set of all cc-support cliques of G. The steps 3–11 of the algorithm eliminate all cliques in $\mathcal{S}$ and cliques that are sc-dominated by the elements of $\mathcal{S}$ from the set $\mathcal{C}(G)$. In steps 12–33, we look for the cliques that sc-dominates a maximum number of cliques in $\mathcal{C}(G) - S'$, where elements of the set S' are the cliques in S along with all cliques that are sc-dominated by the elements of S. Steps 35–40 eliminate all the cliques from $\mathcal{C}(G)$ that are sc-dominated by the updated set D elements. After completion of all iterations, D is a sc-dominating set of G with minimum cardinality. In a similar manner, we can construct an algorithm to find $\gamma_{wcc}(G)$ and $f_{scc}(G)$.

Remark 3. *The time complexity of the algorithm 3.1. in worst case scenario is $O(n(1 + n + n^2))$.*

IV. Clique-clique domination balanced graphs

Definition 4. *A graph G is clique-clique domination balanced (ccd-balanced) if there exist a SCCD-set $\mathcal{S}$ and a WCCD-set $\mathcal{W}$ of G such that $\mathcal{S} \cap \mathcal{W} = \phi$.*

Definition 5. *A graph G is fully clique-clique domination balanced (fccd-balanced) if there exist a smallest SCCD-set $\mathcal{S}$ and a smallest WCCD-set $\mathcal{W}$ of G such that $\mathcal{S} \cap \mathcal{W} = \phi$ and $\mathcal{S} \cup \mathcal{W} = \mathcal{C}(G)$, where $\mathcal{C}(G)$ is the set of all cliques of G.*

Example 3. *Consider a graph G_1 given in the Figure 3. Note that $\mathcal{S} = \{C_1, C_3, C_5, C_6, C_7, C_8\}$ is a smallest SCCD-set and $\mathcal{W} = \{C_2, C_4, C_9, C_{10}, C_{11}, C_{12}\}$ is a smallest WCCD-set of G_1 such that $\mathcal{S} \cap \mathcal{W} = \phi$ and $\mathcal{S} \cup \mathcal{W} = \mathcal{C}(G_1)$. Therefore G_1 is both ccd-balanced and fccd-balanced.*

Remark 4. *Every fccd-balanced graph is ccd-balanced. But the converse need not be true. For example, the cycle C_5 is a ccd-balanced graph but not fccd-balanced.*

Proposition 9. *A graph G is ccd-balanced if, and only if, there exists a SCCD-set $\mathcal{S}$ which is a SFCCD-set of G.*

Proof: Let $\mathcal{C}(G)$ be the set of all cliques of G. Assume that G is ccd-balanced. Then there exist a SCCD-set $\mathcal{S}$ and a WCCD-set $\mathcal{W}$ of G such that $\mathcal{S} \cap \mathcal{W} = \phi$. We shall show that $\mathcal{S}$ is SFCCD-set of G. Let $C \in \mathcal{S}$. Since, $\mathcal{W}$ is a WCCD-set of G and $C \in \mathcal{C}(G) - \mathcal{W}$, there exists $K \in \mathcal{W}$ which wc-dominates C. Then K sc-dominates $C \in \mathcal{C}(G) - \mathcal{S}$. Hence, $\mathcal{S}$ is SFCCD-set of G. Conversely, suppose there exists a SCCD-set $\mathcal{S}$ which is a SFCCD-set of G. Then, by Proposition 5, $\mathcal{C}(G) - \mathcal{S}$ is WCCD-set of G. Thus, G is ccd-balanced. ∎

Proposition 10. *Let G be a graph and $\mathcal{C}(G)$ be the set of all cliques of G.*

(i) If G is ccd-balanced, then $\gamma_{scc}(G) + \gamma_{wcc}(G) \leq |\mathcal{C}(G)|$.
(ii) If G is fccd-balanced, then $\gamma_{scc}(G) + \gamma_{wcc}(G) = |\mathcal{C}(G)|$.

Proposition 11. *A graph G is fccd-balanced if, and only if, the following two conditions are satisfied.*

(i) $f_{scc}(G) + f_{wcc}(G) = |\mathcal{C}(G)|$
(ii) There exists a smallest SCCD-set $\mathcal{S}$ which is SFCCD-set of G.

Proof: Assume that G is fccd-balanced. Then, there exist a smallest SCCD-set $\mathcal{S}$ and a smallest WCCD-set $\mathcal{W}$ of G such that $\mathcal{S} \cap \mathcal{W} = \phi$ and $\mathcal{S} \cup \mathcal{W} = \mathcal{C}(G)$. Clearly, (i) holds, by Proposition 5. Since, $\mathcal{W}$ is a WCCD-set of G, $\mathcal{S} = \mathcal{C}(G) - \mathcal{W}$ is a SFCCD-set of G. Thus, (ii) holds.

Conversely, assume that the conditions (i) and (ii) are true in G. Let $\mathcal{S}$ be a smallest SCCD-set and SFCCD-set of G. Then, by Proposition 5., $\mathcal{C}(G)-\mathcal{S}$ is a WCCD-set of G. Now, by (i) and Proposition 6, $|\mathcal{C}(G)-\mathcal{S}| = |\mathcal{C}(G)| - \gamma_{scc}(G) = f_{wcc}(G) = |\mathcal{C}(G)| - f_{scc}(G) = \gamma_{wcc}(G)$. Thus, G is fccd-balnced. ■

V. Partial ordering relation on the clique set of a graph using cc-degree

A relation on a set P which is reflexive, antisymmetric and transitive is called as a partial order on P. A set P with a partial order $\leq$ defined on P is called as a partially ordered set or briefly a poset and denoted as $(P, \leq)$ [8]. An element m is called a maximal (minimal) element of a poset $(P, \leq)$ if there is no element $x \in P$ such that $x > m$ $(x < m)$. If $x, y \in P$ with $x \leq y$ or $y \leq x$, then we say that x and y are comparable. A subset S of a poset $(P, \leq)$ is called a subchain of P if which every pair of elements of S is comparable under $\leq$. The length of a poset $(P, \leq)$, denoted by $l(P)$ is defined as $l(P) = \max\{|C|-1 : \text{C is a maximal subchain of } P\}$. A lattice is a poset in which every pair of elements has a greatest lower bound and a least upper bound.

Definition 6. *We define a partial order $\leq$ on the clique set $\mathcal{C}(G)$ of a graph G as follows: for any two cliques K and L, $K \leq L$ if either $K = L$ or there exists a clique-path between K and L, say $(K = K_1, K_2, \ldots, K_m = L)$ such that $d_{cc}(K_1) < d_{cc}(K_2) < \cdots < d_{cc}(K_m)$. We call the poset $(\mathcal{C}(G), \leq)$ as the clique-poset of G.*

Example 4. *Consider the graph G_3 given in the Figure 5. Then $d_{cc}(c_1) = 6$, $d_{cc}(c_2) = d_{cc}(c_3) = d_{cc}(c_6) = d_{cc}(c_7) = d_{cc}(c_9) = d_{cc}(c_{10}) = 3$, $d_{cc}(c_4) = d_{cc}(c_5) = d_{cc}(c_8) = d_{cc}(c_{11}) = d_{cc}(c_{12}) = d_{cc}(c_{13}) = 2$.*

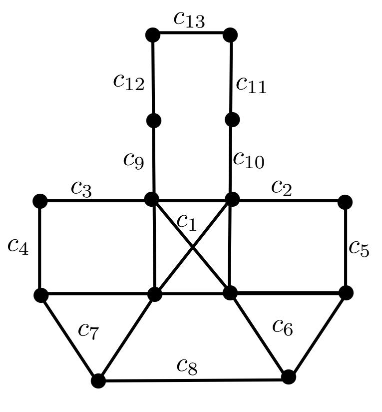

Fig. 5. Graph G_3 with $\mathcal{C}(G_3) = \{c_1, c_2, \ldots, c_{13}\}$.

Then $c_{11} < c_{10} < c_1$, $c_4 < c_3 < c_1$, $c_4 < c_7 < c_1$, $c_8 < c_7 < c_1$, $c_8 < c_6 < c_1$, $c_5 < c_6 < c_1$, $c_5 < c_2 < c_1$ and $c_{12} < c_9 < c_1$. The Hasse diagram of the clique-poset $(\mathcal{C}(G_3), \leq)$ is given in the Figure 6.

Definition 7. *(i) A clique $K \in \mathcal{C}(G)$ is called a cc-strong (cc-weak) clique if K sc-dominates (wc-dominates) all cliques adjacent to it.*

(ii) A clique $K \in \mathcal{C}(G)$ is called a cc-regular clique if it sc-dominates and wc-dominates all cliques adjacent to it.

(iii) A clique $K \in \mathcal{C}(G)$ is called a cc-balanced clique if there exist cliques L and M which are adjacent to K such that L is not sc-dominated by K and M is not wc-dominated by K.

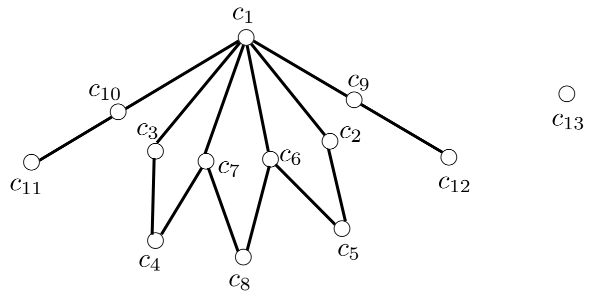

Fig. 6. clique-poset $(\mathcal{C}(G_3), \leq)$

Definition 8. *The cc-strong number of a graph G, denoted by $s_{cc}(G)$ is the number of cc-strong cliques in G. Similarly, cc-weak number $(w_{cc}(G))$, cc-regular number $(r_{cc}(G))$ and cc-balanced number $(b_{cc}(G))$ of G are defined.*

Proposition 12. *Let G be a graph. Then $s_{cc}(G) + w_{cc}(G) - r_{cc}(G) + b_{cc}(G) = |\mathcal{C}(G)|$.*

Proposition 13. *Let $(\mathcal{C}(G), \leq)$ be the clique-poset of a graph G. Then*

(i) K is a cc-strong clique of G if and only if K is a maximal element of $(\mathcal{C}(G), \leq)$.

(ii) K is a cc-weak clique of G if and only if K is a minimal element of $(\mathcal{C}(G), \leq)$.

(iii) K is a cc-regular clique of G if and only if K is not related to any element $L \in \mathcal{C}(G)$ such that $K \neq L$ with respect to $\leq$.

(iv) K is a cc-balanced clique of G if and only if K is neither a minimal nor a maximal element of $(\mathcal{C}(G), \leq)$.

Proof: (i) Assume that K is a cc-strong clique of G. Suppose K is not a maximal element of $(\mathcal{C}(G), \leq)$. Then there exists $L \in \mathcal{C}(G)$ such that $K < L$. This implies that there is a K-L clique-path in G say $(K = K_1, K_2, \ldots, K_n = L)$ such that $d_{cc}(K) < d_{cc}(K_2) < \cdots < d_{cc}(L)$. We observe that K_2 is adjacent to K and $d_{cc}(K) < d_{cc}(K_2)$, which is a contradiction to our assumption. Conversely, assume that K is a maximal element of $\mathcal{C}(G)$. Suppose there exists a clique L adjacent to K in G such that $d_{cc}(L) > d_{cc}(K)$. Then, K, L is a K-L path in G with the property $d_{cc}(K) < d_{cc}(L)$. This implies that $K < L$ in $\mathcal{C}(G)$, which is a contradiction to our assumption. Hence, K is a cc-strong clique of G. By similar argument *(ii)* can be proved.

(iii) A clique K of G is cc-regular if and only if K is both cc-strong and cc-weak clique of G if and only if K is both maximal and minimal element of $(\mathcal{C}(G), \leq)$ if and only if there is no $L, M \in \mathcal{C}(G)$ such that $L > K$ and $M < K$ if and only if K is not related to any element of $\mathcal{C}(G) - \{K\}$ with respect to $\leq$.

(iv) Let K be a cc-balanced clique of G. Then there exist cliques L and M adjacent to K such that $d_{cc}(L) < d_{cc}(K) < d_{cc}(M)$. This implies that $L < K < M$ in $\mathcal{C}(G)$. Hence, K is neither a minimal nor a maximal element of $(\mathcal{C}, \leq)$. Conversely, suppose there is $K \in \mathcal{C}(G)$ such that L is neither a minimal nor a maximal element. Then there exists $L, M \in \mathcal{C}(G)$ such that $L < K < M$. Then there is a L-K clique-path in G say $(L = K_1, K_2, \ldots, K_n = K)$ such

that $d_{cc}(L) < d_{cc}(K_2) < \cdots < d_{cc}(K_{n-1}) < d_{cc}(K)$ and a K-M clique-path in G say $(K = L_1, L_2, \ldots, L_k = M)$ such that $d_{cc}(K) < d_{cc}(L_2) < \cdots < d_{cc}(M)$. We observe that K_{n-1} and L_2 are adjacent to K such that $d_{cc}(K_{n-1}) < d_{cc}(K) < d_{cc}(L_2)$. Hence, K is a cc-balanced clique of G. ■

Corollary 14. *If the clique-poset of a graph G is a lattice, then $s_{cc}(G) = 1$, $w_{cc}(G) = 1$, $r_{cc}(G) = 0$ and $b_{cc}(G) = |\mathcal{C}(G)| - 2$.*

Remark 5. *(i) Let $\Delta_{cc}(G)$ and $\delta_{cc}(G)$ denote the maximum cc-degree and minimum cc-degree of a graph G. Then any clique K of G with $d_{cc}(K) = \Delta_{cc}$ ($d_{cc}(K) = \delta_{cc}$) is a maximal (minimal) element of the clique-poset of G. But, the converse need not be true.*

(ii) Let G be a graph and $l(\mathcal{C}(G))$ be the length of the clique-poset $(\mathcal{C}(G), \leq)$ of G. Then, $l(\mathcal{C}(G)) \leq \Delta_{cc}(G) - \delta_{cc}(G)$.

(iii) If two graphs G and H are isomorphic then their clique-posets are order isomorphic. But the converse need not be true. For example, consider the graphs G and H given in the Figure 7.

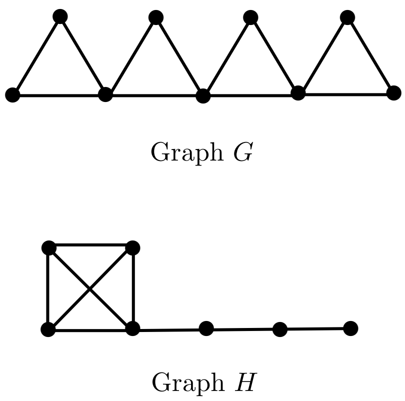

Fig. 7. Graphs G and H for the counter-example

Note that the graphs G and H are not isomorphic, but their clique-posets are order isomorphic. Then, the Hasse diagrams of the clique-posets $(\mathcal{C}(G), \leq)$ and $(\mathcal{C}(H), \leq)$ are same and it is given in the Figure 8.

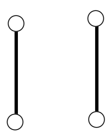

Fig. 8. Hasse diagram of the clique-posets $(\mathcal{C}(G), \leq)$ and $(\mathcal{C}(H), \leq)$

Proposition 15. *For a connected graph G, the Hasse diagram of the clique-poset $(\mathcal{C}(G), \leq)$ is connected if and only if for any $L, M \in \mathcal{C}(G)$, there exists a clique-path $(L = K_1, K_2, \ldots, K_n = M)$ in G such that $d_{cc}(K_i) \neq d_{cc}(K_{i+1})$, for all i, $1 \leq i \leq n-1$.*

Proof: Assume that the Hasse diagram of the clique-poset $(\mathcal{C}(G), \leq)$ is connected. Suppose there are two cliques L and M such that every L-M clique-path in G contains some adjacent cliques of same cc-degree. Then L and M are not comparable in the poset $(\mathcal{C}(G), \leq)$. We shall show that lowerbound and upperbound of L and M do not exist in $(\mathcal{C}(G), \leq)$. In fact, suppose that a lowerbound of L and M exists in $\mathcal{C}(G)$. We choose a lowerbound B of L and M such that B is a maximal element of the set $\{K \in \mathcal{C}(G) : K$ is a lowerbound of L and $M\}$. Then, $B < L$ and $B < M$. This implies that there exist clique-paths $(B = L_1, L_2, \ldots, L_k = L)$ such that $d_{cc}(L_1) < d_{cc}(L_2) < \cdots < d_{cc}(L_k)$ and $(B = K_1, K_2, \ldots, K_n = M)$ such that $d_{cc}(K_1) < d_{cc}(K_2) < \cdots < d_{cc}(K_n)$. Then $(L = L_k, L_{k-1}, \ldots, L_2, L_1 = B = K_1, K_2, \ldots, K_n = M)$ is a L-M clique-path in G such that adjacent cliques have the distinct cc-degrees, which is not possible. Similarly, we can prove that no upperbound L and M exists in $\mathcal{C}(G)$. This implies that L and M do not lie in the same component of the Hasse diagram of $(\mathcal{C}(G), \leq)$, which is a contradiction to our assumption. Thus, there exists a clique-path $(L = K_1, K_2, \ldots, K_n = M)$ in G such that $d_{cc}(K_i) \neq d_{cc}(K_{i+1})$, for all i, $1 \leq i \leq n-1$. Conversely, assume that for any $L, M \in \mathcal{C}(G)$, there exists a clique-path $(L = K_1, K_2, \ldots, K_n = M)$ in G such that $d_{cc}(K_i) \neq d_{cc}(K_{i+1})$, for all i, $1 \leq i \leq n-1$. Consider K_i and K_{i+1}. Then either $d_{cc}(K_i) > d_{cc}(K_{i+1})$ or $d_{cc}(K_i) < d_{cc}(K_{i+1})$. This implies that either $K_i > K_{i+1}$ or $K_i < K_{i+1}$ in $(\mathcal{C}(G), \leq)$. Then K_i and K_{i+1} lie in the same component of the Hasse diagram of $(\mathcal{C}(G), \leq)$, for all i, $1 \leq i \leq n-1$. Therefore, L and M lie in the same component of the Hasse diagram of $(\mathcal{C}(G), \leq)$. Hence, the Hasse diagram of the clique-poset $(\mathcal{C}(G), \leq)$ is connected. ■

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