Adaptive Synchronization for Inertial Cohen-Grossberg Neural Networks with Distributed Delays

Yanyan Wang and Zhilian Yan

Abstract—**The paper discusses adaptive synchronization for inertial Cohen-Grossberg neural networks (ICGNNs) with distributed delays. The ICGNN is changed into a first-order differential equation model by making use of a variable substitution method. A criterion is proposed to guarantee the adaptive synchronization of the drive-response ICGNNs by means of an appropriate Lyapunov functional and the LaSalle invariant principle. A concrete example is offered to illustrate the validity of the obtained results.**

Index Terms—**distributed delay, inertial neural network, Cohen-Grossberg network, synchronization.**

I. INTRODUCTION

NEURAL networks (NNs) exhibit various dynamic behaviors, including stability, bifurcation, chaos, and synchronization. Among these behaviors, synchronization is closely related to a number of application areas, such as secure communication and image encryption. Over the past few decades, the synchronization of NNs has been extensively studied. Building upon the drive-response framework introduced in [1], numerous references on this topic have been published; see, e.g., [1–4].

Cohen-Grossberg NN (CGNN) proposed by the authors in [5] is a particularly important class of NN since some famous models can be looked upon as their particular case [5–7]. Because of the broad applications in distributed computation, message processing and secure communication, the stability and the synchronization problems of CGNNs were universally followed with interest questions; see, e.g., [8, 9].

Meanwhile, many dynamic phenomena such as bifurcation and oscillation are related to time delay [10, 11]. Delayed NNs are one of the most widely used and rapidly developing NNs at present. About CGNNs with time delay, a lot of studies have been carried out; the results can be found in [12–20]. Particularly, in [16], Song and Cao discussed robust stability for time-varying-delayed CGNNs containing the reaction-diffusion sections; and the stability was discussed respectively for Takagi-Sugeno fuzzy CGNNs having multiple delays and impulsive CGNNs in [17] and [18]. In [19], the author's attention was focused on lag synchronization and dissipativity for fuzzy CGNNs containing time delays and discontinuous activations. In addition, the spatial extent is the objective existence for NNs. Compared to systems with finite time delays which are put to use widely, systems with infinite distributed delay can more accurately describe the actual interactions between neurons. In [21–24], dynamic behaviors for systems with distributed delays (DDs) were investigated.

It's a remarkable fact that most models that are studied now are the differential equations with the first derivative of the states, and the inertial terms are ignored, which can be written in the form of the second derivative of the states. However, in [25] and [26], the phenomenons about chaotic behaviors, bifurcation were observed when the impact of the inertial terms was considered. In recent, dynamic behavior about the inertial NNs (INNs) has been discussed. The synchronization and stability were studied for inertial time-delayed NNs in [27–32]. In [31], Tang and Jian focused on complex-valued inertial delayed NNs and studied synchronization problem aided by matrix measure and impulsive control. Huang and Cao studied the synchronization phenomenon for CGNNs containing inertial terms and time-delay, taking advantage of adaptive feedback controller in [32].

This paper will mainly be committed to studying the adaptive asymptotical synchronization of ICGNNs, which are second-order differential equation models with DDs. Drawing support from the variable substitution method, the ICGNN is changed into a first-order differential equation model. Then, a criterion is presented to ensure the adaptive synchronization of the drive-response ICGNNs by means of an appropriate Lyapunov functional (LF) and the LaSalle invariant principle. A conclusion is also given for the ICGNN model containing both discrete delays and DDs. Finally, through a concrete example, we illustrate the validity and the correctness of the obtained results.

The remainder structure arrangement is as below: In Section II, the authors describe the drive ICGNN model, the response ICGNN model, some assumptions, and the necessary definition and lemma. Throughout Section III, two results are provided. A concrete example is offered to illustrate the validity the acquired results in Section IV. In the end, Section V draws the conclusions.

II. PRELIMINARIES

At first, the following ICGNNs with DDs are considered:

$$\begin{aligned}\frac{d^2x_i(s)}{ds^2} &= -l_i\frac{dx_i(s)}{ds} - a_i(x_i(s))[h_i(x_i(s))\\&\quad -\sum_{j=1}^{m} b_{ij}g_j(x_j(s))\end{aligned}$$

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$$-\sum_{j=1}^{m} p_{ij}\int_{-\infty}^{s}\chi_{ij}(s-z)g_j(x_j(z))dz+J_i],$$
$$i=1,2,\ldots,m, \tag{1}$$

here, $x_i(s)$ means the state for the ith neuron, $\frac{d^2x_i(s)}{ds^2}$ denotes inertial term, $a_i(x_i(s))$ means amplification function, $h_i(x_i(s))$ stands for an appropriate behaved function, $g_j(x_j(s))$ stands activation function [33], $\chi_{ij}(\cdot))$ is delay kernel, J_i denotes external input for the jth neuron, b_{ij} denotes connection weight, p_{ij} denotes delay connection weight, and l_i is a positive constant.

The amplication functions are required to meet the following condition:

$(H1)$: For $i=1,2,\ldots,m$, the constants $\bar{a}_i>0$ and $m_i>0$ satisfy

$$0<a_i(s)<\bar{a}_i, \mid a_i(\bar{s})-a_i(s)\mid\leq m_i\mid \bar{s}-s\mid.$$

The activation functions are require to meet the following two conditions:

$(H2)$: For $i=1,2,\ldots,m$, there exist constants $L_i>0$ and $G_i>0$ such that [34, 35]: $\mid g_i(s)\mid\leq G_i,\mid g_i(\bar{s})-g_i(s)\mid\leq L_i\mid \bar{s}-s\mid$.

$(H3)$: There exist constants $\gamma_i>0$ such that

$$\mid g_i(\bar{s})a_i(\bar{s})-g_i(s)a_i(s)\mid\leq\gamma_i\mid\bar{s}-s\mid, i=1,2,\ldots,m.$$

The following hypothesis about χ_{ij} is required:

$(H4)$: The functions χ_{ij} $i,j=1,2,\cdots,m$, are continuous and bounded on $[0,+\infty)$, and satisfy $\int_0^{+\infty}\chi_{ij}(s)ds=1$.

ICGNN (1) is regarded as the drive system, and the matched response system is offered as below:

$$\begin{aligned}\frac{d^2\bar{x}_i(s)}{ds^2} &= -l_i\frac{d\bar{x}_i(s)}{ds}-a_i(\bar{x}_i(s))[h_i(\bar{x}_i(s))\\ &\quad -\sum_{j=1}^{m} b_{ij}g_j(\bar{x}_j(s))\\ &\quad -\sum_{j=1}^{m} p_{ij}\int_{-\infty}^{s}\chi_{ij}(s-z)g_j(\bar{x}_j(z))dz\\ &\quad +J_i]+v_i(s),\\ &\quad i=1,2,\ldots,m,\end{aligned} \tag{2}$$

where $v_i(s)$ denotes the appropriate control input that will be given in Section III.

Considering the following variable substitution:

$$u_i(s)=\frac{dx_i(s)}{ds}+\sigma_i x_i(s),$$

$$\bar{u}_i(s)=\frac{d\bar{x}_i(s)}{ds}+\sigma_i\bar{x}_i(s),$$

where σ_i denotes a positive constant, its value will be given in the following derivation.

For ICGNNs (1) and (2), the equivalent equations are offered as below:

$$\begin{cases}\frac{dx_i(s)}{ds}=u_i(s)-\sigma_i x_i(s),\\ \frac{du_i(s)}{ds}=-\sigma_i(\sigma_i-l_i)x_i(s)+(\sigma_i-l_i)u_i(s)\\ -a_i(x_i(s))[h_i(x_i(s))-\sum\limits_{j=1}^{m} b_{ij}g_j(x_j(s))\\ -\sum\limits_{j=1}^{m} p_{ij}\int_{-\infty}^{s}\chi_{ij}(s-z)g_j(x_j(z))dz+J_i].\end{cases} \tag{3}$$

$$\begin{cases}\frac{d\bar{x}_i(s)}{ds}=\bar{u}_i(s)-\sigma_i\bar{x}_i(s),\\ \frac{d\bar{u}_i(s)}{ds}=-\sigma_i(\sigma_i-l_i)\bar{x}_i(s)+(\sigma_i-l_i)\bar{u}_i(s)\\ -a_i(\bar{x}_i(s))[h_i(\bar{x}_i(s))-\sum\limits_{j=1}^{m} b_{ij}g_j(\bar{x}_j(s))\\ -\sum\limits_{j=1}^{m} p_{ij}\int_{-\infty}^{s}\chi_{ij}(s-z)g_j(\bar{x}_j(z))dz+J_i]\\ +v_i(s).\end{cases} \tag{4}$$

Define error signals $e_i(s)=\bar{x}_i(s))-x_i(s)$ and $\bar{e}_i(s)=\bar{u}_i(s))-u_i(s)$ for the drive-response ICGNNs, from the model (3) and the model (4), we get

$$\begin{cases}\frac{de_i(s)}{ds}=\bar{e}_i(s)-\sigma_i e_i(s),\\ \frac{d\bar{e}_i(s)}{ds}=-\sigma_i(\sigma_i-l_i)e_i(s)+(\sigma_i-l_i)\bar{e}_i(s)\\ -(a_i(\bar{x}_i(s))h_i(\bar{x}_i(s))-a_i(x_i(s))h_i(x_i(s)))\\ +a_i(\bar{x}_i(s)\sum\limits_{j=1}^{m} b_{ij}(g_j(\bar{x}_j(s))-g_j(x_j(s)))\\ +a_i(\bar{x}_i(s)\sum\limits_{j=1}^{m} p_{ij}\int_{-\infty}^{s}\chi_{ij}(s-z)(g_j(\bar{x}_j(z))\\ -g_j(x_j(z)))dz\\ +(a_i(\bar{x}_i(s))-a_i(x_i(s)))\sum\limits_{j=1}^{m} b_{ij}g_j(x_j(s))\\ +(a_i(\bar{x}_i(s))-a_i(x_i(s)))\sum\limits_{j=1}^{m} p_{ij}\int_{-\infty}^{t}\chi_{ij}(s-z)\\ g_j(x_j(z))dz\\ +(a_i(\bar{x}_i(s))-a_i(x_i(s)))J_i+v_i(s).\end{cases} \tag{5}$$

Definition 1. *Drive-response ICGNNs (1) and (2) are asymptotically synchronized if the trivial solution of the error model (5) is asymptotically stable, i.e., for $i=1,2,\ldots,m$,*

$$\lim_{s\to+\infty}e_i(s)=\lim_{s\to+\infty}(\bar{x}_i(s))-x_i(s))=0.$$

Lemma 1. *[36] (Young's Inequality) Assume that $a>0$, $b>0$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, the inequality*

$$ab\leq\frac{1}{p}a^p+\frac{1}{q}b^q$$

holds.

III. Main Results

In this section, the external control input is designed as below:

For $i=1,2,\ldots,m$,

$$v_i(s)=\omega_i(s)\mid e_i(s)\mid+\nu_i(s)\bar{e}_i(s), \tag{6}$$

where $\omega_i(s)$ and $\nu_i(s)$ meet with $\omega_i(s)\geq 0$, $\frac{d\omega_i(s)}{ds}=-r_i\mid e_i(s)\bar{e}_i(s)\mid$, $\frac{d\nu_i(s)}{ds}=-s_i\bar{e}_i^2(s)$, s_i and r_i are any positive constants.

The conditions guaranteeing the asymptotic synchronization are given as below:

Theorem 1. *When the suppositions $(H_1)-(H_4)$ are true, drive-response ICGNNs (1) and (2) are asymptotically synchronized, when the external control terms are designed as (6), and σ_i are offered as follows:*

$$\begin{aligned}\sigma_i &\geq 1+\frac{1}{2}(\gamma_i+\bar{a}_i\sum_{j=1}^{m}\mid b_{ji}\mid L_i\\ &\quad +m_i\sum_{j=1}^{m}(\mid b_{ji}\mid+\mid p_{ji}\mid)G_i+m_i\mid J_i\mid).\end{aligned} \tag{7}$$

Proof: To prove the conclusion that we want to obtain, the following LF is given:

$$V(s) = \sum_{i=1}^{m}[e_i^2(s)+\bar{e}_i^2(s) + \bar{a}_i\sum_{j=1}^{m}\mid p_{ij}\mid\int_0^{+\infty}\chi_{ij}(z)\int_{s-z}^{s}\mid g_j(\bar{x}_j(t)) - g_j(x_j(t))\mid^2 dtdz + \frac{(\omega_i(s)+\kappa_{1i})^2}{r_i} + \frac{(\nu_i(s)+\kappa_{2i})^2}{s_i}], \tag{8}$$

where κ_{1i} and κ_{2i} are constans which satisfy the follow conditions:

$$\kappa_{2i} \geq \frac{1}{2}(1+\gamma_i+\bar{a}_i\sum_{j=1}^{m}L_i\mid b_{ji}\mid + 2\bar{a}_i\sum_{j=1}^{m}\mid p_{ji}\mid + m_i\sum_{j=1}^{m}(\mid b_{ji}\mid+\mid p_{ji}\mid)G_i + m_i\mid J_i\mid) + \sigma_i - l_i. \tag{9}$$

$$\kappa_{1i} = \sigma_i(\sigma_i+l_i), \tag{10}$$

Computing the derivative of $V(s)$, the authors get:

$$\begin{aligned}\frac{dV(s)}{ds} &= \sum_{i=1}^{m}[2e_i(s)\frac{de_i(s)}{ds}+2\bar{e}_i(s)\frac{d\bar{e}_i}{ds} + \bar{a}_i\sum_{j=1}^{m}\mid p_{ij}\mid\int_0^{+\infty}\chi_{ij}(z)\mid g_j(\bar{x}_j(z)) - g_j(x_j(z))\mid^2 dz - \bar{a}_i\sum_{j=1}^{m}\mid p_{ij}\mid\int_0^{+\infty}\chi_{ij}(z) \\ &\quad \mid g_j(\bar{x}_j(s-z))-g_j(x_j(s-z))\mid^2 dz + \frac{2}{r_i}(\omega_i(s)+\kappa_{1i})(r_i\mid e_i(s)\bar{e}_i(s)\mid) + \frac{2}{s_i}(\nu_i(s)+\kappa_{2i})(-s_i\bar{e}_i^2(s))] \\ &= \sum_{i=1}^{m}[2e_i(s)(-\sigma_ie_i(s)+e_i(s)) + 2\bar{e}_i(s)(-\sigma_i(\sigma_i - l_i)e_i(s) + (\sigma_i-l_i)\bar{e}_i(s) \\ &\quad -(a_i(\bar{x}_i(s))h_i(\bar{x}_i(s)) - a_i(x_i(s))h_i(x_i(s))) + a_i(\bar{x}_i(s))\sum_{j=1}^{m}b_{ij}(g_j(\bar{x}_j(s)) - g_j(x_j(s))) \\ &\quad + a_i(\bar{x}_i(s))\sum_{j=1}^{m}p_{ij}\int_{-\infty}^{s}\chi_{ij}(s-z)(g_j(\bar{x}_j(z)) - g_j(x_j(z)))dz + (a_i(\bar{x}_i(s)) - a_i(x_i(s)))\sum_{j=1}^{m}b_{ij}g_j(x_j(s)) \\ &\quad + (a_i(\bar{x}_i(s)) - a_i(x_i(s)))\sum_{j=1}^{m}p_{ij} \times\int_{-\infty}^{s}\chi_{ij}(s-z)g_j(x_j(z))dz + (a_i(\bar{x}_i(s)) - a_i(x_i(s)))J_i \\ &\quad + \omega_i(s)\mid e_i(s)\mid + \nu_i(s)\bar{e}_i(s)) + \bar{a}_i\sum_{j=1}^{m}\mid p_{ij}\mid\int_0^{+\infty}\chi_{ij}(z)\mid g_j(\bar{x}_j(z)) - g_j(x_j(z))\mid^2 dz - \bar{a}_i\sum_{j=1}^{m}\mid p_{ij}\mid\int_0^{+\infty}\chi_{ij}(z) \\ &\quad \mid g_j(\bar{x}_j(s-z)) - g_j(x_j(s-z))\mid^2 dz - 2\omega_i(s)\mid e_i(s)\bar{e}_i(s)\mid - 2\kappa_{1i}\mid e_i(s)\bar{e}_i(s)\mid - 2\nu_i(s)\bar{e}_i^2(s) - 2\kappa_{2i}\bar{e}_i^2(s)].\end{aligned} \tag{11}$$

By the assumptions (H_1)-(H_4) and Lemma 1, one can obtain

$$\begin{aligned}\frac{dV(s)}{ds} &\leq \sum_{i=1}^{m}[(1+\gamma_i+\bar{a}_i\sum_{j=1}^{m}\mid b_{ji}\mid L_i + m_i \times\sum_{j=1}^{m}(\mid b_{ji}\mid+\mid p_{ji}\mid)G_i + m_i\mid J_i\mid - 2\sigma_i)e_i^2(s) \\ &\quad + (1+\gamma_i+\bar{a}_i\sum_{j=1}^{m}\mid b_{ji}\mid L_i + 2\bar{a}_i\sum_{j=1}^{m}\mid p_{ji}\mid + m_i\sum_{j=1}^{m}(\mid b_{ji}\mid+\mid p_{ji}\mid)G_i + m_i\mid J_i\mid + 2\sigma_i - 2l_i - 2\kappa_{2i})\bar{e}_i^2(s) \\ &\quad + \sum_{j=1}^{m}(2\sigma_i^2+2\sigma_il_i - 2\kappa_{1i})\mid\bar{e}_i(s)e_i(s)\mid].\end{aligned}$$

From (7)-(11), the following conclusion is given:

$$\frac{dV(s)}{ds} \leq -\sum_{i=1}^{m}e_i^2(s) - \sum_{i=1}^{m}\bar{e}_i^2(s) \leq -\sum_{i=1}^{m}e_i^2(s).$$

Hence, $\frac{dV(s)}{ds} = 0$ is equivalent to $e_i(s) = 0$, for $i = 1,2,\ldots,m$. According to LaSalle invariant principle and Definition 1, we can obtain the drive-response ICGNNs (1) and (2) are asymptotically synchronized.

Remark 1. To deduce the asymptotic synchronization of the models (1) and (2), we take into account inequalities (7), (9) and equality (10) in this theorem. However, in [32], equalities are considered. Compared with the values of the parameters which are required in the proof, inequalities (7) and (9) are more nonconservative.

In ICGNN model (1), the discrete delays are not taken into consideration. Next, the ICGNN model containing discrete and DD terms are considered as below:

For $i = 1,2,\ldots,m$,

$$\frac{d^2x_i(s)}{ds^2} = -l_i\frac{dx_i(s)}{ds} - a_i(x_i(s))[h_i(x_i(s)) - \sum_{j=1}^{m}b_{ij}g_j(x_j(s)) - \sum_{j=1}^{m}d_{ij}g_j(x_j(s-\tau_j(s))) - \sum_{j=1}^{m}p_{ij}\int_{-\infty}^{s}\chi_{ij}(s-z)g_j(x_j(z))dz + J_i], \tag{12}$$

where $\tau_i(s)$ is a discrete delay of the ICGNN, satisfying $0<\tau_i(s)<\tau^*$, $0<\dot{\tau}_i(s)<\alpha<1$ as in [37–39].

The response model for ICGNN (12) is given as:

For $i = 1,2,\ldots,m$,

$$\frac{d^2\bar{x}_i(s)}{ds^2} = -l_i\frac{d\bar{x}_i(s)}{ds} - a_i(\bar{x}_i(s))[h_i(\bar{x}_i(s))$$

$$
\begin{aligned}
&-\sum_{j=1}^{m} b_{ij} g_j(\bar{x}_j(s)) - \sum_{j=1}^{m} d_{ij} g_j(\bar{x}_j(s-\tau_j(s))) \\
&-\sum_{j=1}^{m} p_{ij} \int_{-\infty}^{s} \chi_{ij}(s-z) g_j(\bar{x}_j(z)) dz \\
&+J_i] + v_i(s). \qquad (13)
\end{aligned}
$$

Taking the same variable substitution in Section II and the similar process of proof for Theorem 1, the following conclusion is gained:

Corollary 1. *When the assumptions (H_1)-(H_4) are true, drive-response ICGNNs (12) and (13) are asymptotically synchronized, when the external control terms are designed as (6), and σ_i are offered as below:*

$$
\begin{aligned}
\sigma_i \geq\ & 1 + \frac{1}{2}(\gamma_i + \bar{a}_i \sum_{j=1}^{m} (\mid b_{ji} \mid + G_i \mid d_{ji} \mid) L_i \\
& + m_i \sum_{j=1}^{m} (\mid b_{ji} \mid + \mid d_{ji} \mid + \mid p_{ji} \mid) G_i + m_i \mid J_i \mid).
\end{aligned}
$$

Remark 2. So as to testify Corollary 1, we provide a LF as below:

$$
\begin{aligned}
V(s) =\ & \sum_{i=1}^{m} [e_i^2(s) + \bar{e}_i^2(s) \\
& + \bar{a}_i \sum_{j=1}^{m} G_j \mid d_{ij} \mid \int_{s-\tau_j(s)}^{s} e_j^2(z) dz \\
& + \bar{a}_i \sum_{j=1}^{m} \mid p_{ij} \mid \int_{0}^{+\infty} \chi_{ij}(z) \int_{s-z}^{s} \mid g_j(\bar{x}_j(t)) \\
& - g_j(x_j(t)) \mid^2 dt dz + \frac{(\omega_i(s) + \kappa_{1i})^2}{r_i} \\
& + \frac{(\nu_i(s) + \kappa_{2i})^2}{s_i}].
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa_{1i} =\ & \sigma_i(\sigma_i + l_i), \\
\kappa_{2i} \geq\ & \frac{1}{2}(1 + \gamma_i + \bar{a}_i \sum_{j=1}^{m} (\mid b_{ji} \mid + G_i \mid d_{ji} \mid) L_i \\
& + 2\bar{a}_i \sum_{j=1}^{m} \mid p_{ji} \mid \\
& + m_i \sum_{j=1}^{m} (\mid b_{ji} \mid + \mid d_{ji} \mid + \mid p_{ji} \mid) G_i \\
& + m_i \mid J_i \mid) + \sigma_i - l_i,\ i = 1, 2, \ldots, m.
\end{aligned}
$$

IV. NUMERICAL EXAMPLE

Example1: Take into account the below ICGNN:

$$
\begin{cases}
\frac{d^2 x_1(s)}{ds^2} = -\frac{dx_1(s)}{ds} - a_1(x_1(s))[x_1(s) \\
-\sum_{j=1}^{2} b_{1j} \tanh(x_j(s)) \\
-\sum_{j=1}^{2} p_{1j} \int_{-\infty}^{s} e^{-(s-z)} \tanh(x_j(z)) dz + 0.2] \\
\frac{d^2 x_2(s^2)}{ds} = -\frac{dx_2(s)}{ds} - a_2(x_2(s))[x_2(s) \\
-\sum_{j=1}^{2} b_{2j} \tanh(x_j(s)) \\
-\sum_{j=1}^{2} p_{2j} \int_{-\infty}^{s} e^{-(s-z)} \tanh(x_j(z)) dz + 0.1],
\end{cases} \qquad (14)
$$

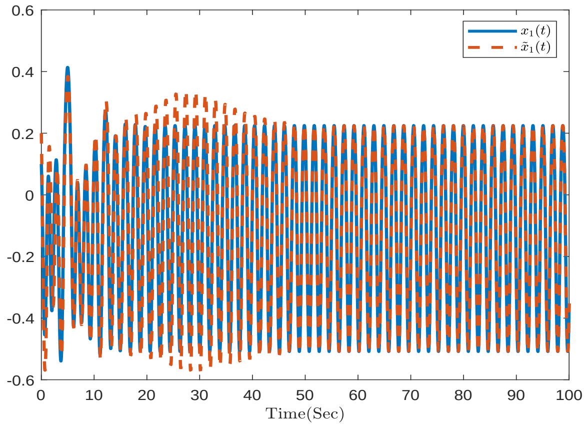

Fig. 1. Transient behavior of $x_1(s)$ and $\bar{x}_1(s)$.

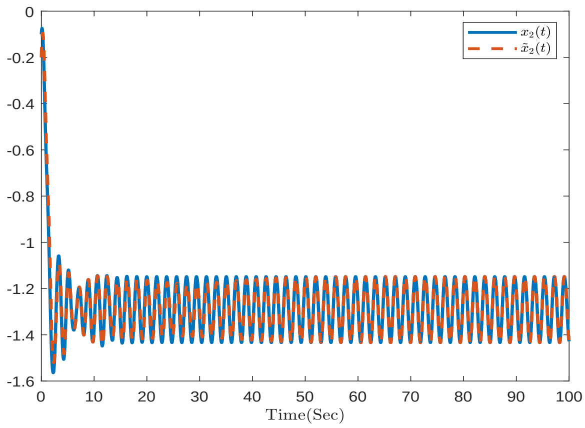

Fig. 2. Transient behavior of $x_2(s)$ and $\bar{x}_2(s)$.

where $a_1(x_1) = 3 + \frac{1}{1+x_1^2}, a_2(x_2) = 3 - \frac{1}{1+x_1^2}$,

$$
\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} -0.5 & 0.2 \\ 0.3 & -0.1 \end{pmatrix},
$$
$$
\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 1.5 & -1 \\ -0.3 & -2 \end{pmatrix}.
$$

On the basis of the values that are given in the above model, the hypothesizes $(H1)$-$(H4)$ come true, and the parameters in the assumptions are: $\bar{a}_1 = 4, \bar{a}_2 = 3, G_1 = G_2 = m_1 = m_2 = l_1 = l_2 = L_1 = L_2 = 1, J_1 = 0.2, J_2 = 0.1, \gamma_1 = \gamma_2 = 4$.

The following equations are employed to describe corresponding response model:

$$
\begin{cases}
\frac{d^2 \bar{x}_1(s)}{ds^2} = -\frac{d\bar{x}_1(s)}{ds} - a_1(\bar{x}_1(s))[\bar{x}_1(s) \\
-\sum_{j=1}^{2} b_{1j} \tanh(\bar{x}_j(s)) \\
-\sum_{j=1}^{2} p_{1j} \int_{-\infty}^{s} e^{-(s-z)} \tanh(\bar{x}_j(z)) dz + 0.2] + v_1(s) \\
\frac{d^2 \bar{x}_2(s^2)}{ds} = -\frac{d\bar{x}_2(s)}{ds} - a_2(\bar{x}_2(s))[\bar{x}_2(s) \\
-\sum_{j=1}^{2} b_{2j} \tanh(\bar{x}_j(s)) \\
-\sum_{j=1}^{2} c_{2j} \int_{-\infty}^{s} e^{-(s-z)} \tanh(\bar{x}_j(z)) dz + 0.1] + v_2(s).
\end{cases} \qquad (15)
$$

According to the variable substitution in Section II, the

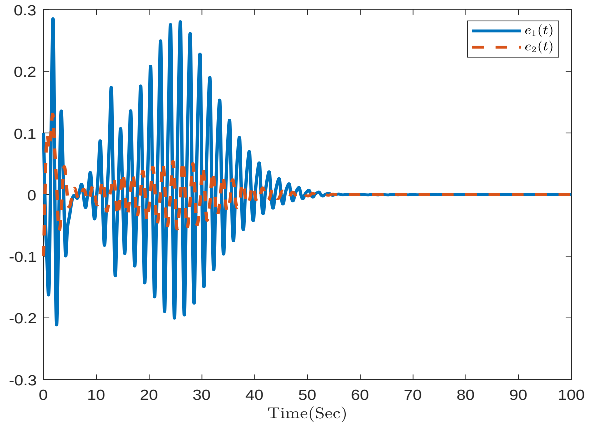

Fig. 3. Transient state for $e_1(s) = \bar{x}_1(s) - x_1(s), e_2(s) = \bar{x}_2(s) - x_2(s)$.

equations are given as follows:

$$\begin{cases} \frac{dx_i(s)}{ds} = u_i(s) - \sigma_i x_i(s), i = 1, 2, \\ \frac{du_i(s)}{ds} = -\sigma_i(\sigma_i - 1)x_i(s) + (\sigma_i - 1)u_i(s) \\ \quad -a_i(x_i(s))[x_i(s) - \sum_{j=1}^{2} b_{ij} \tanh(x_j(s)) \\ \quad - \sum_{j=1}^{2} p_{ij} \int_{-\infty}^{s} e^{-(s-z)} \tanh(x_j(z))dz + J_i], \\ \quad i = 1, 2, \end{cases}$$

$$\begin{cases} \frac{d\bar{x}_i(s)}{ds} = \bar{u}_i(s) - \sigma_i \bar{x}_i(s), i = 1, 2, \\ \frac{d\bar{u}_i(s)}{ds} = -\sigma_i(\sigma_i - 1)\bar{x}_i(s) + (\sigma_i - 1)\bar{u}_i(s) \\ \quad -a_i(\bar{x}_i(s))[\bar{x}_i(s) - \sum_{j=1}^{2} b_{ij} \tanh(\bar{x}_j(s)) \\ \quad - \sum_{j=1}^{2} p_{ij} \int_{-\infty}^{s} e^{-(s-z)} \tanh(\bar{x}_j(z))dz + J_i] \\ \quad +v_i(s), i = 1, 2. \end{cases}$$

By calculation, the parameter values are as follows:

$$\sigma_1 = 5.5,\ \sigma_2 = 3.925,\ \kappa_{11} = 35.75,$$
$$\kappa_{12} = 19.330625,\ \kappa_{21} = 11.2,\ \kappa_{22} = 6.85.$$

In line with Theorem 1, ICGNNs (14) and (15) are asymptotically synchronized. Fig. 1 describes the simulation results of the transient behavior for state variable x_1 and $\bar{x}_1$, Fig. 2 illustrates the simulation results of the transient behavior for the state variable x_2 and $\bar{x}_2$. Fig. 3 depicts the transient behavior of the error signals for drive-response ICGNNs (14) and (15), and the asymptotic synchronization is confirmed.

V. Conclusions

The adaptive synchronization for ICGNNs with DDs is focused on in this paper. With the help of the variable substitution method, the differential model which is a second-order equation is changed into a differential model which includes a first-order derivative. Then, by means of an appropriate LF and the LaSalle invariant principle, a criterion is provided which can ensure the adaptive asymptotical synchronization for the drive response model. A conclusion is also given for the ICGNN model with both discrete delays and DDs. At last, the validity of the obtained result is illustrated by an example.

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