Robust Dual G-Frames

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Abstract—**In this paper, we give some characteristics of g-frames for finite dimensional version which are counterpart the frame theory. Moreover, we investigate their dual g-frames that are optimal for erasures. For a given g-frame, we will give some conditions under which the canonical dual is the unique optimal dual g-frame for the erasure problem. We also discuss some special conditions under which the canonical dual g-frame is either not optimal or it is optimal dual but not unique one.**

Index Terms—**G-frames, Optimal dual g-frame, Linear bounded operator, Erasure.**

I. Introduction

FRAMES, which was first introduced by Duffin and Schaeffer [1] in 1952 to study the nonharmonic Fourier series. After the fundamental paper by Daubechies, et.al [2] in 1986 , frames were popularized from then on. Nowadays, frames have been widely used in many fields, the readers are referred to some references, e.g.[3], [4], [5], [6], [7], [8].

Let $\mathcal{H}$ be a Hilbert space. A sequence $\Phi = \{\phi_i\}_{i=1}^{\infty} \subset \mathcal{H}$ is called a frame, if there exist positive constants A and B such that,

$$A\|f\|^2 \le \sum_{i=1}^{\infty} |\langle f, \phi_i\rangle|^2 \le B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

We call A, B the lower frame bound and upper frame bound of frame Φ, respectively. If Φ is a frame, then for any $f \in \mathcal{H}$ can be expressed as $f = \sum_{i=1}^{\infty} \langle f, \psi_i\rangle \phi_i$, where $\{\psi_i\}_{i=1}^{\infty}$ denotes the dual frame of $\{\phi_i\}_{i=1}^{\infty}$. For the frame study, researchers are very interested in searching for optimal dual frame from the coding theory viewpoint. For example, in [3], the authors showed that uniform (length) tight frames are optimal for one erasure and equiangular frames are optimal for two erasures. In [4], [5], Han, Leng and Lopez considered the question of finding optimal dual frame for decoding when a frame has been preselected for encoding.

Recently, various generalizations of the frame have been proposed and studied, such as fusion frames [9], oblique frames [10] and pseudo-frames [11]. In [12], Sun introduced a more general frame (g-frame), showed that g-frame include the frames mentioned above and proved that many basic properties of g-frame can be shared with classical frame (see [13], [8]), but there also exist some different properties between frames and g-frames. For example, exact frames are equivalent to Riesz bases, but exact g-frames are not equivalent to g-Riesz bases [12]. Whether the way of finding optimal dual frame for any frames may be extended to any g-frames for the erasure problem? In this paper, we investigate this problem. Firstly, we focus on some characteristics of g-frames in finite dimensional space. Then, by using the worst case error [4], we set the problem studied in the context of g-frames, i.e., the largest packet-lost operator norm among all possible erasures gives some conditions under which the canonical dual g-frame is the unique optimal dual g-frame for the erasure problem.

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Throughout this paper, let $\mathcal{H}$ be finite dimensional (real or complex) Hilbert space, and $\{\mathcal{H}_i : i \in I\}$ a sequence of closed subspaces of $\mathcal{H}$, where $I = \{1, 2, \cdots, m\}$ is a subset of integer set Z. $B(\mathcal{H}, \mathcal{H}_i)$ is denoted by the set of all the linear bounded operators from $\mathcal{H}$ to $\mathcal{H}_i$, if $\mathcal{H} = \mathcal{H}_i$ then $B(\mathcal{H}, \mathcal{H}_i)$ is abbreviated to $B(\mathcal{H})$. $I_{\mathcal{H}}$ is the identical operator of $\mathcal{H}$. If $\Lambda = \{\Lambda_i\}_{i=1}^m$ is a finite g-frame, that is, Λ with $dim(\mathcal{H}) = n < \infty$, $dim(\mathcal{H}_i) = l_i < \infty$, $i \in I$, where dim denotes the dimension of a Hilbert space. Also we always let $l = \sum_{i=1}^{m} l_i$. The analysis operator T_Λ for $\{\Lambda_i\}_{i=1}^m$ is an $l \times n$ matrix. For vectors on $\mathcal{C}^n$ we shall use the Euclidean norm, but for matrix $T \in \mathcal{C}^{n\times n}$, we shall use the Frobenius norm $\|T\|_F^2 = \text{tr } (T^*T) = \sum_{i,j\in I_n} |T_{ij}|^2$ which is induced by the inner product $\langle A, B\rangle = \text{tr } B^*A$, for $A, B \in \mathcal{C}^{n\times n}$.

II. Preliminaries

We first recall the definitions and results of g-frames in Hilbert spaces as formulated in [13], [8].

Definition 1.1.[13] A collection of the vector $\Lambda = \{\Lambda_i\}_{i\in I} \subseteq B(\mathcal{H}, \mathcal{H}_i)$ is called a *g-frame* for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, if there exist two positive constants A and B such that

$$A\|f\|^2 \le \sum_{i\in I} \|\Lambda_i f\|^2 \le B\|f\|^2, \quad \forall f \in \mathcal{H}. \qquad (1)$$

We call A, B the lower frame bound and upper frame bound of g-frame Λ, respectively.

We call Λ the tight g-frame if $A = B$ and it is the Parseval g-frame if $A = B = 1$.

We call Λ an exact g-frame if it ceases to be a g-frame whenever any single element is removed from Λ.

We call Λ a g-frame sequence, if it is a g-frame for $\overline{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in I}$.

We call Λ g-complete, if $\{f : \Lambda_i f = 0\}_{i\in I} = \{0\}$. It is easy to see that every g-frame is a g-complete family.

In the study of frame theory, analysis operator and frame operator are the main tools. We give the operator theory of g-frames as follows, for details see [8].

The analysis and the synthesis operators in g-frames are defined by

$$T_\Lambda : \mathcal{H} \longrightarrow \oplus_{i\in I}\mathcal{K}_i, \quad T_\Lambda(f) = \{\Lambda_i f\}_{i\in I}, \quad f \in \mathcal{H},$$

$$T_\Lambda^* : \oplus_{i\in I}\mathcal{K}_i \longrightarrow \mathcal{H}, \quad T_\Lambda^*(f) = \sum_{i\in I}\Lambda_i^* f_i, \quad f \in \mathcal{H}.$$

The g-frame operator $S : \mathcal{H} \longrightarrow \mathcal{H}$ defined by

$$S(f) = \sum_{i\in I}\Lambda_i^*\Lambda_i f, \quad f \in \mathcal{H},$$

which is a bounded, positive and invertible operator. The canonical dual g-frame for Λ, defined by

$$\tilde{\Lambda} = \{\tilde{\Lambda}_i = \Lambda_i S^{-1}\}_{i\in I},$$

is also a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ with B^{-1} and A^{-1} as its lower and upper g-frame bounds, respectively. This leads to the generalized reconstruction formula

$$f = \sum_{i\in I}\Lambda_i^*\tilde{\Lambda}_i f = \sum_{i\in I}\tilde{\Lambda}_i^*\Lambda_i f, \qquad f \in \mathcal{H}. \qquad (2)$$

Next, we give the notion of dual g-frames:

Definition 1.2.[8] Two g-Bessel sequences $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Gamma = \{\Gamma_i\}_{i\in I}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$.

1) We say that Γ is an *alternate dual g-frame* for g-frame Λ if $T_\Gamma^* T_\Lambda = I_\mathcal{H}$, or equivalently if $f = \sum_{i\in I}\Gamma_i^*\Lambda_i f$, for every $f \in \mathcal{H}$.
2) We denote by

$$D(\Lambda) \overset{\text{def}}{=} \{\Gamma \subseteq B(\mathcal{H}, \mathcal{H}_i) : T_\Gamma^* T_\Lambda = I_\mathcal{H}\},$$

the set of all dual g-frames for a fixed $\Lambda \subseteq B(\mathcal{H}, \mathcal{H}_i)$. Observe that $D(\Lambda) \neq \emptyset$, since $\tilde{\Lambda} \in D(\Lambda)$.

Remark 1.1. Let $\Lambda \subseteq B(\mathcal{H}, \mathcal{H}_i)$. Then $\Gamma \in D(\Lambda)$ if and only if its synthesis operator T_Γ^* is a pseudo-inverse of T_Λ. Indeed,

$$\Gamma \in D(\Lambda) \Leftrightarrow T_\Gamma^* T_\Lambda = I_\mathcal{H}.$$

It is easy to obtain that each g-frame has many dual g-frames. For any g-frame $\{\Lambda_i\}_{i=1}^m$, we also have the following results:

1) $\{\Lambda_i\}_{i=1}^m$ is a g-frame if and only if T_Λ is full column rank.
2) $\{\Lambda_i\}_{i=1}^m$ is a parseval g-frame if and only if T_Λ is column orthogonal ($T_\Lambda^* T_\Lambda = I_\mathcal{H}$).
3) $\{\Lambda_i\}_{i=1}^m$ is an orthonormal g-frame if and only if T_Λ is an unitary matrix.

The first question arising is: How do you use all dual g-frames? A comprehensive answer is provided by the following result, which is particular case of classical frames in the book [14].

Theorem 1.1. Let $\{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ with analysis operator T_Λ and g-frame operator S. Then, the following conditions are equivalent:

(1) $\{\Gamma_i\}_{i\in I}$ is a dual g-frame for $\{\Lambda_i\}_{i\in I}$.

(2) The analysis operator T_1 of the sequence $(\Gamma_i - \Lambda_i S^{-1})_{i\in I}$ satisfies $\operatorname{ran} T_\Lambda \perp \operatorname{ran} T_1$.

Proof: We set $h_i := \Gamma_i - \Lambda_i S^{-1}$ for all $i \in I$, and note that

$$\begin{aligned}\sum_{i\in I}\Lambda_i^*\Gamma_i f &= \sum_{i\in I}\Lambda_i^*(h_i + \Lambda_i S^{-1})f \\ &= \sum_{i\in I}\Lambda_i^* h_i f + f = T_\Lambda^* T_1 f + f, \quad f \in \mathcal{H}.\end{aligned}$$

Hence, $\{\Gamma_i\}_{i\in I}$ is a dual g-frame for $\{\Lambda_i\}_{i\in I}$ if and only if $T_\Lambda^* T_1 = 0$, which is equivalent to (2). The conclusion holds.

We also obtain the following result, which gives a characterization for all of the alternate dual g-frames of the given g-frame.

Theorem 1.2. Every dual g-frame of $\{\Lambda_i\}_{i\in I}$ is of the form $\Gamma_i = \Lambda_i S^{-1} + h_i$, where

$$\sum_{i=1}^n h_i^*\Lambda_i f = \sum_{i=1}^n \Lambda_i^* h_i f = 0, \quad f \in \mathcal{H}.$$

Proof: Let $\{\Gamma_i\}_{i\in I}$ be a dual g-frame for $\{\Lambda_i\}_{i\in I}$, and define $h_i = \Gamma_i - \Lambda_i S^{-1}$.

$$\begin{aligned}\sum_{i\in I} h_i^*\Lambda_i f &= \sum_{i\in I}(\Gamma_i - \Lambda_i S^{-1})^*\Lambda_i f \\ &= \sum_{i\in I}\Gamma_i^*\Lambda_i f - S^{-1}\Lambda_i^*\Lambda_i f \\ &= f - f = 0.\end{aligned}$$

Conversely, assume $\{\Gamma_i\}_{i\in I}$ is a set of vectors and $\Gamma_i = \Lambda_i S^{-1} + h_i$, where $\sum_{i\in I} h_i^*\Lambda_i f = 0$. Then

$$\begin{aligned}\sum_{i\in I}\Gamma_i^*\Lambda_i f &= \sum_{i\in I}(\Lambda_i S^{-1} + h_i)^*\Lambda_i f \\ &= \sum_{i\in I} S^{-1}\Lambda_i^*\Lambda_i f + h_i^*\Lambda_i f \\ &= f.\end{aligned}$$

Thus, we obtain that $\{\Gamma_i\}_{i\in I}$ satisfies the equation (2), and it is a dual g-frame for $\{\Lambda_i\}_{i\in I}$.

The proof for $\sum_{i\in I}\Lambda_i^* h_i f = 0$ is similar.

The next result summarizes some basic, yet useful, properties of the g-frames.

Lemma 1.1. Let $S : \mathcal{H} \to \mathcal{H}$ be a linear operator, and the set of vectors $\{\Lambda_i S^{-1}\}_{i\in I}$ has analysis operator $T_{\Lambda S^{-1}}$, then $T_{\Lambda S^{-1}} f = T_\Lambda S^{-1} f$.

Proof: $T_{\Lambda S^{-1}} f = \{\Lambda_i S^{-1} f\}_{i\in I} = T_\Lambda S^{-1} f$.

Proposition 1.1. Let $\Lambda = \{\Lambda_i\}_{i=1}^m \subseteq B(\mathcal{H}, \mathcal{H}_i)$ be a parseval g-frame and P be an orthogonal projection on $\mathcal{H}$. Then $\{\Lambda_i P^k\}_{i=1}^m$ is a parseval g-frame for $P(\mathcal{H})$.

Proof: The result follows from

$$\begin{aligned}\sum_{i=1}^m(\Lambda_i P^k)^*(\Lambda_i P^k) &= \sum_{i=1}^m (P^k)^*\Lambda_i^*\Lambda_i P^k \\ &= (P^*)^k\sum_{i=1}^m\Lambda_i^*\Lambda_i P^k = PI_\mathcal{H}.\end{aligned}$$

Definiton 1.3. Let $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ be two g-frames. If there is a onto invertible operator T such that $\Lambda_i = \Gamma_i T$ $(i \in I)$, then we say $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are *similar*. If T is unitary, then they are *unitarily equivalent*.

Theorem 1.3. Let $\{\Lambda_i\}_{i=1}^m$, $\{\Gamma_i\}_{i=1}^m$ be two g-frames, which are similar. If $\{\Gamma_i\}_{i=1}^m$ is an equal-norm tight frame, then $\{\Lambda_i S^{-\frac{1}{2}}\}_{i=1}^m$ is an equal-norm Parseval g-frame.

Proof: Given $\{\Lambda_i S^{-\frac{1}{2}}\}_{i=1}^m$ is Parseval g-frame [12]. Since $\{\Gamma_i\}_{i=1}^m$ is similar to $\{\Lambda_i\}_{i=1}^m$, obviously, $\{\Gamma_i\}_{i=1}^m$ is similar to $\{\Lambda_i S^{-\frac{1}{2}}\}_{i=1}^m$. By Definition 1.3 we obtain $\{\Gamma_i\}_{i=1}^m = \{\Lambda_i S^{-\frac{1}{2}}T\}_{i=1}^m$. Suppose A be a frame bound for $\{\Gamma_i\}_{i=1}^m$, we have

$$\begin{aligned}AI_\mathcal{H} &= \sum_{i=1}^m(\Lambda_i S^{-\frac{1}{2}}T)^*(\Lambda_i S^{-\frac{1}{2}}T) \\ &= \sum_{i=1}^m T^* S^{-\frac{1}{2}}\Lambda_i^*\Lambda_i S^{-\frac{1}{2}}T = T^*T.\end{aligned}$$

It implies that T is onto and $\frac{T}{\sqrt{A}}$ is unitary.

Next, we show that $\{\Lambda_i S^{-\frac{1}{2}}\}_{i=1}^m$ is an equal-norm Parseval g-frame. Since

$$\begin{aligned}\|\Lambda_i S^{-\frac{1}{2}}\|_F^2 &= \|\Gamma_i T^{-1}\|_F^2 = tr[(\Gamma_i T^{-1})^*(\Gamma_i T^{-1})] \\ &= tr[(T^{-1})^*\Gamma_i^*\Gamma_i T^{-1}] = tr(\tfrac{1}{A}T\Gamma_i^*\Gamma_i T^{-1}) \\ &= tr(\tfrac{1}{A}\Gamma_i^*\Gamma_i) = \tfrac{1}{A}\|\Gamma_i\|_F^2.\end{aligned}$$

Hence we obtain that $\{\Lambda_i S^{-\frac{1}{2}}\}_{i=1}^m$ is an equal-norm Parseval g-frame.

III. Optimal Dual G-Frames

In this section we mainly study dual g-frames which can be used decode the signal from the receiver. Given a g-frame $\{\Lambda_i\}_{i=1}^m$, we can compute the frame coefficients $\{\Lambda_i(f)\}_{i=1}^m$ of a signal f, some frame coefficients could get lost during the data transmission, then we reconstruct (decoding) the original signal f with the dual g-frames $\{\Gamma_i\}_{i=1}^m$ of g-frame $\{\Lambda_i\}_{i=1}^m$. Obviously, we have the following reconstruction formula $f = \sum\limits_{i=1}^m \Gamma_i^*\Lambda_i(f)$.

As it was mentioned in the introduction, our purpose is to set the problem studied in the context of g-frames. That is, for a fixed g-frame, the goal is to give some conditions under which the canonical dual is the unique optimal dual g-frame for the erasure problem. In [4], Han and Lopez gave some results which imply the existence of the unique dual frame that is optimal for the erasure of 1-packet of coefficients. In [3], [4], the authors obtained optimal alternate dual for r erasures among those dual frames which are optimal for $r-1$ erasures. Therefore, we draw the following results that the optimal dual for r erasures coincide with optimal dual for one erasure. So, in this paper, we only consider one erasure problem.

In order to describe the reconstruction error when an arbitrary packet of coefficients of the g-frames is erased, we consider the following notions which were adopted from reference [15] for reconstruction systems. Let $j \in I$ and $M_j \in B(\mathcal{H})$ defined by

$$M_j((y_i)_{i\in I}) = (\mathbf{1}_j(i)\cdot y_i)_{i\in I},$$

where $\mathbf{1}_j : I \to \{0,1\}$ denotes the characteristic function of the set $\{j\} \subset I$. Similarly, we consider the packet-lost operator

$$L_j \overset{\text{def}}{=} M_{I\setminus\{j\}} = I_{\mathcal{H}} - M_j.$$

In coding theory, a signal vector $f \in \mathbf{R}^n$ is encoded as $T_\Lambda f = \{\Lambda_i f\}_{i\in I}$ against a g-frame Λ and then $T_\Lambda f$ is sent to a receiver for decoding the original signal f. However, some of the coefficients in the encoded data $T_\Lambda f$ may be lost in the transmission process. If the encoded information $T_\Lambda f \in \mathcal{H}$ is altered according to the packet-lost operator L_j, our reconstructed vector will be $\hat{f} = T_\Gamma^* L_j T_\Lambda f$, where $\Gamma = \{\Gamma_i\}_{i\in I} \in D(\Lambda)$ is dual g-frame for Λ. Then the reconstruction error will be

$$f - \hat{f} = f - T_\Gamma^* L_j T_\Lambda f = T_\Gamma^* M_j T_\Lambda f = \Gamma_j^*\Lambda_j f.$$

In this case, we will use the Frobenius norm $\|\cdot\|_F$ to perform the measure of the operator $\Gamma_j^*\Lambda_j$. Consider the m-tuple

$$\begin{aligned}E_1(\Lambda,\Gamma) &= (\|I - T_\Gamma^* L_j T_\Lambda\|_F)_{j\in I} \\ &= (\|T_\Gamma^* M_j T_\Lambda\|_F)_{j\in I} = (\|\Gamma_j^*\Lambda_j\|_F)_{j\in I}.\end{aligned}$$

Notice that we can bound uniformly the reconstruction error in terms of the entries of this vector for the erasure of one packet of coefficients (for all m possible choices). In what follows we shall consider the reconstruction error based on $E_1(\Lambda,\Gamma)$, namely the (normalized) worst-case error.

Let $\Lambda = \{\Lambda_i\}_{i=1}^m$ be a g-frame. For $\Gamma = \{\Gamma_i\}_{i=1}^m \in D(\Lambda)$, we introduce the worst-case reconstruction error when one packet is lost with respect to the Frobenius norm. Now, we measure the error vector $E_1(\Lambda,\Gamma)$ with the maximum of its entries.

$$\begin{aligned}e_1(\Lambda) &= \inf_{\Gamma\in D(\Lambda)} \|E_1(\Lambda,\Gamma)\|_\infty \\ &= \inf_{\Gamma\in D(\Lambda)} \max_{i\in I} \|T_\Gamma^* M_i T_\Lambda\|_F \\ &= \inf_{\Gamma\in D(\Lambda)} \max_{i\in I} \|\Gamma_i^*\Lambda_i\|_F.\end{aligned}$$

We define the set of *one loss optimal dual* g-frame for Λ as

$$D_1(\Lambda) \overset{\text{def}}{=} \{\Gamma \in D(\Lambda) : \|E_1(\Lambda,\Gamma)\|_\infty = e_1(\Lambda)\}.$$

According to the Proposition 12 in [15], we have the following lemma:

Lemma 2.1. Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame with $\Lambda_i \neq 0$ for all i. Then the set $D_1(\Lambda)$ of one loss optimal dual g-frame for Λ is non-empty, compact and convex.

Theorem 2.1. Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame with frame operator S. If

$$\|S^{-1}\Lambda_i^*\Lambda_i\|_F = c, \quad i \in I,$$

then the canonical dual g-frame $\tilde{\Lambda}$ of Λ, is the unique one loss optimal dual g-frame (and hence the r-loss optimal dual g-frame for any r).

Proof: By Lemma 2.1, we obtain $\Gamma = \{\Gamma_i\}_{i\in I} \in D_1(\Lambda)$. Then

$$\max_{i\in I} \|\Gamma_i^*\Lambda_i\|_F \le \max_{i\in I} \|S^{-1}\Lambda_i^*\Lambda_i\|_F = c.$$

Denote $\Lambda_i S^{-1} = C_i$, then $\|\Gamma_i^*\Lambda_i\|_F \le c = \|C_i^*\Lambda_i\|_F$ for every $i \in I$. It is obvious that $\|\Gamma_i^*\|_F \le \|C_i^*\|_F$ for every $i \in I$. Note that

$$\begin{aligned}\|\Gamma_i\|_F^2 &= \|C_i^* + (\Gamma_i^* - C_i^*)\|_F^2 \\ &= \|C_i^*\|_F^2 + \|\Gamma_i^* - C_i^*\|_F^2 + 2Re(tr[(\Gamma_i^* - C_i^*)C_i]).\end{aligned}$$

Then, $\|\Gamma_i^* - C_i^*\|_F^2 + 2Re(tr[(\Gamma_i^* - C_i^*)C_i]) \le 0$ for all $i \in I$. Consequently,

$$\sum_{i\in I} tr[(\Gamma_i^* - C_i^*)C_i] = tr[(T_\Gamma^* - T_{S^{-1}\Lambda})T_\Lambda S^{-1}] = 0.$$

Since both Γ and $S^{-1}\Lambda$ are dual g-frames for Λ, we have

$$\begin{aligned}0 &\le \sum_{i\in I} \|\Gamma_i^* - C_i^*\|_F^2 \\ &= \sum_{i\in I} \|\Gamma_i^* - C_i^*\|_F^2 + \sum_{i\in I} 2Re(tr[(\Gamma_i^* - C_i^*)C_i]) \le 0,\end{aligned}$$

which implies that $\Gamma = \{\Gamma_i\}_{i\in I} = \{C_i\}_{i\in I}$. Thus, we conclude that the canonical dual is the unique optimal dual g-frame of Λ for one erasure.

As a special case we have the following result:

Corollary 2.1. Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a uniform tight g-frame, then the canonical g-dual is the unique optimal dual frame of Λ for r-erasures.

Proof: Assume that $\{\Lambda_i\}_{i\in I}$ is a tight g-frame with frame bound A. Then the frame operator of $\{\Lambda_i\}_{i\in I}$ is $S = AI_{\mathcal{H}}$, and $\|S^{-1}\Lambda_i^*\Lambda_i\|_F = \frac{1}{A}\|\Lambda_i^*\Lambda_i\|_F$. Therefore, $\|S^{-1}\Lambda_i^*\Lambda_i\|_F$

is a constant for all i when $\{\Lambda_i\}_{i\in I}$ is also uniform. The consequence is implied from the above Theorem 2.1.

Next, we give a more general result than Theorem 2.1.

For a g-frame $\Lambda = \{\Lambda_i\}_{i=1}^m$, let $c = \max\{\|\Lambda_i S^{-1}\|\|\Lambda_i\| : 1 \le i \le m\}$, set $I_1 = \{i : \|\Lambda_i S^{-1}\|\|\Lambda_i\| = c\}$ and $I_2 = I\backslash I_1$, let $H_j = span\{\Lambda_i, i \in I_j\}, \ (j = 1, 2)$. We prove the following results:

Theorem 2.2. Let $\Lambda = \{\Lambda_i\}_{i=1}^m$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. Then the following are equivalent:

(1) The canonical dual $\{\Lambda_i S^{-1}\}_{i=1}^m$ is the unique optimal dual for one erasure.

(2) $H_1 \cap H_2 = \{0\}$ and $\{\Lambda_i\}_{i\in I_2}$ is linearly independent.

Proof: (1) $\Rightarrow$ (2): If $\{\Lambda_i\}_{i\in I_2}$ is linearly dependent, there exist u_i ($i \in I_2$ not all zero) in $\mathcal{H}_2$ such that

$$\sum_{i=1}^m u_i^* \Lambda_i f = 0, f \in \mathcal{H},$$

when $u_i = 0, \ i \in I_1$ and $U = \{u_i\}_{i\in I}$ is not zero sequence, $\{\Lambda_i S^{-1} + tu_i\}_{i\in I}, \ t \neq 0$ is the dual g-frame of Λ, then

$$\sum_{i\in I} (tu_i)^* \Lambda_i f = \sum_{i\in I} tu_i^* \Lambda_i f = 0,$$

and

$$\|\Lambda_i S^{-1} + tu_i\|\|\Lambda_i\| = \|\Lambda_i S^{-1}\|\|\Lambda_i\| = c, \ i \in I_1,$$

$$\|\Lambda_i S^{-1}\|\|\Lambda_i\| < c, \ i \in I_2,$$

$\exists \delta > 0$ and $|t| < \delta$, $\|\Lambda_i S^{-1} + tu_i\|\|f_i\| < c, \ \forall i \in I_2$. Then $\{\Lambda_i S^{-1} + tu_i\}_{i=1}^m$ ($t \neq 0$) is also an optimal dual for Λ, which is a contradiction. Hence $\{\Lambda_i\}_{i\in I_2}$ is linearly independent.

Next, we show $H_1 \cap H_2 = \{0\}$. If not, there exist linear independent set $\Lambda_{i_1}, \cdots, \Lambda_{i_l}, \ i_j \in I_1$ and nonzero constants $c_{i_1}, \cdots, c_{i_l}$ such that

$$\sum_{j=1}^l c_{i_j} \Lambda_{i_j} + \sum_{i\in I_2} c_i \Lambda_i = 0, \quad \text{for some } c_i \ (i \in I_2).$$

Since $\{\Lambda_{i_j}\}_{j=1}^l$ is also linearly independent, we can find $h \in \mathcal{H}$ such that

$$\langle \Lambda_{i_j} S^{-1}, \bar{c}_{i_j} h\rangle = \langle c_{i_j} \Lambda_{i_j} S^{-1}, h\rangle < 0.$$

Define $u_i = \bar{c}_i h, \ i \in \{i_1, \cdots, i_l\} \cup I_2$, and $u_i = 0, \ i \in I_1 \backslash \{i_1, \cdots, i_l\}$. Then $T_{tU}^* T_\Lambda = 0$ for all scalars t.

Obviously, for all $i \in I_1 \backslash \{i_1, \cdots, i_l\}$, we have

$$\|\Lambda_i S^{-1} + tu_i\|\|\Lambda_i\| = \|\Lambda_i S^{-1}\|\|\Lambda_i\| = c,$$

then $\exists \delta > 0$, $|t| < \delta$, such that

$$\|\Lambda_i S^{-1} + tu_i\|\|\Lambda_i\| < c, \ i \in I_2,$$

and

$$\begin{aligned} &\|\Lambda_i S^{-1} + tu_i\|^2 \|\Lambda_i\|^2 \\ &= [\|\Lambda_i S^{-1}\|^2 + \|tu_i\|^2 + 2t\langle \Lambda_i S^{-1}, u_i\rangle]\|\Lambda_i\|^2 \\ &< c^2, \quad i \in \{i_1, \cdots, i_l\}. \end{aligned}$$

Thus the canonical dual is not the only optimal dual, which leads to contradiction. Therefore $H_1 \cap H_2 = \{0\}$.

(2) $\Rightarrow$ (1): Suppose $\{\Lambda_i S^{-1} + u_i\}_{i=1}^m$ is optimal dual g-frame for one erasure, by Theorem 1.2, we know that $\sum\limits_{i\in I}^n \Lambda_i^* u_i f = 0, f \in \mathcal{H}$, that is

$$\sum_{i\in I_1}^n \Lambda_i^* u_i f + \sum_{i\in I_2}^n \Lambda_i^* u_i f = 0,$$

by the assumption (2), which implies that

$$\sum_{i\in I_1}^n \Lambda_i^* u_i f = 0, \qquad \sum_{i\in I_2}^n \Lambda_i^* u_i f = 0.$$

Since $\{\Lambda_i\}_{i\in I_2}$ is linear independent, then

$$u_i f = 0, \quad i \in I_2, \ \text{i.e.,} \ u_i = 0, \ i \in I_2.$$

Also, by the previous lemma, we get

$$\sum_{i\in I_1}^n (\Lambda_i^* S^{-1}) u_i f = 0,$$

we only need to show that $u_i = 0$ for all $i \in I_1$. In fact, according to

$$\|\Lambda_i S^{-1} + u_i\|\|\Lambda_i\| \le c = \|\Lambda_i S^{-1}\|\|u_i\|, \quad i \in I_1,$$

we get

$$2\langle \Lambda_i S^{-1}, u_i\rangle + \|u_i\|^2 \le 0, \ \text{for } i \in I_1.$$

Summing up the right hand side, we get

$$2\sum_{i\in I_1} \langle \Lambda_i S^{-1}, u_i\rangle + \sum_{i\in I_1} \|u_i\|^2 = 0 + \sum_{I_1} \|u_i\|^2 \le 0,$$

and hence $\sum\limits_{I_1} \|u_i\|^2 = 0$, since $\sum\limits_{i\in I_1} \langle \Lambda_i S^{-1}, u_i\rangle = 0$. This implies that $u_i = 0$ for all $i \in I_1$, and therefore $\{\Lambda_i S^{-1} + u_i\}_{i=1}^m$ is the canonical dual.

We also give the result that the canonical dual is an optimal dual g-frame but not the only optimal dual for one erasure.

Theorem 2.3. Let $\Lambda = \{\Lambda_i\}_{i=1}^m$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. Assume that $H_1 \cap H_2 = \{0\}$ and $\{\Lambda_i\}_{i\in I_1}$ is linearly independent. Especially, $\Lambda = \{\Lambda_i\}_{i=1}^m$ is linearly dependent Parseval g-frame. Then $\{\Lambda_i S^{-1}\}_{i=1}^m$ is an optimal dual g-frame but not the only optimal for one erasure.

Proof: Suppose $\{\Lambda_i S^{-1} + u_i\}_{i=1}^m$ is the dual g-frames of the g-frame Λ, by Theorem 1.1, we have

$$\sum_{i\in I}^n \Lambda_i^* u_i f = 0, \ \forall f \in \mathcal{H},$$

that is

$$\sum_{i\in I_1}^n \Lambda_i^* u_i f + \sum_{i\in I_2}^n \Lambda_i^* u_i f = 0.$$

Given $H_1 \cap H_2 = \{0\}$, thus

$$\sum_{i\in I_1}^n \Lambda_i^* u_i f = 0, \ \sum_{i\in I_2}^n \Lambda_i^* u_i f = 0,$$

since $\{\Lambda_i\}_{i\in I_2}$ is linear independent, we obtain

$$u_i f = 0, \quad i \in I_2, \ \text{i.e.,} \ u_i = 0, \ i \in I_2,$$

which implies that

$$\begin{aligned}&\max\{\|\Lambda_i S^{-1}+u_i\|\|\Lambda_i\|:1\le i\le m\}\\&\quad\ge\max\{\|\Lambda_i S^{-1}+u_i\|\|\Lambda_i\|:i\in I_1\}\\&\quad=\max\{\|\Lambda_i S^{-1}\|\|\Lambda_i\|:i\in I_1\}\\&\quad=\max\{\|\Lambda_i S^{-1}\|\|\Lambda_i\|:1\le i\le m\}.\end{aligned}$$

So the canonical dual is an optimal dual g-frame.

Because of $k>m$, we can find a dual g-frame $\Gamma=\{\Lambda_i S^{-1}+u_i\}_{i=1}^m$ with $u_i\neq 0$ for some $i\in I_2$. Let $t>0$ be small enough such that $\|\Lambda_i S^{-1}+tu_i\|\|\Lambda_i\|<c$ for $i\in I_2$. Then

$$\|\Lambda_i S^{-1}+tu_i\|\|\Lambda_i\|=c,\ \text{for}\ i\in I_1,$$

and

$$\|\Lambda_i S^{-1}+tu_i\|\|\Lambda_i\|<c,\ \text{for}\ i\in I_2,$$

when t is small enough. Thus, $\|\Lambda_i S^{-1}+tu_i\|_{i\in I}$ is also an optimal dual g-frame for one erasure.

Next, we investigate condition for the canonical dual is not optimal for one erasure.

Theorem 2.4. Let $\Lambda=\{\Lambda_i\}_{i=1}^m$ be a g-frame for $\mathcal{H}$. Assume that $\{\Lambda_i\}_{i\in I_1}$ is linearly independent, and there exist a sequence of scalar $\{c_i\}_{i\in I}$ such that $\sum_{i\in I}c_i\Lambda_i=0$, and $c_i\neq 0$ for all $i\in I_1$. Then $\{\Lambda_i S^{-1}\}_{i\in I}$ is not optimal dual g-frame for one erasure.

Proof: Since $\{\Lambda_i\}_{i\in I_1}$ is also linearly independent, we can find $h\in\mathcal{H}$ such that

$$\langle\Lambda_i S^{-1},\bar{c}_i h\rangle=\langle c_i\Lambda_i S^{-1},h\rangle<0,\qquad i\in I_1.$$

Define $u_i=\bar{c}_i h$, for all i. Then $T^*_{tU}T_\Lambda=0$ for all scalars t. Let $t>0$ be small enough such that

$$\begin{aligned}&\|\Lambda_i S^{-1}+tu_i\|^2\|\Lambda_i\|^2\\&\quad=[\|\Lambda_i S^{-1}\|^2+\|tu_i\|^2+2t\langle\Lambda_i S^{-1},u_i\rangle]\|\Lambda_i\|^2\\&\quad<c^2,\qquad i\in I_1,\end{aligned}$$

and

$$\|\Lambda_i S^{-1}+tu_i\|\|\Lambda_i\|<c,\ \text{for}\ i\in I_2.$$

Then we have

$$\max\{\|\Lambda_i S^{-1}+tu_i\|\|\Lambda_i\|:1\le i\le n\}<c,$$

i.e., the canonical dual is not optimal.

Corollary 2.2. Let $\Lambda=\{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. Assume that $k=n+1$, I_1 has only one element, and $\{\Lambda_i\}_{i\in I_2}$ is linearly independent. Then $\{\Lambda_i S^{-1}\}_{i\in I}$ is not optimal dual g-frame for one erasure.

Proof: We can assume that $I_1=\{1\}$. Since $\{\Lambda_i\}_{i=1}^{n+1}$ is linearly dependent, there exist c_i (not all zero) such that

$$c_1\Lambda_1+\sum_{i\in I_2}c_i\Lambda_i=0.$$

Given $c_1\neq 0$ and $\{\Lambda_i\}_{i\in I_2}$ is linearly independent. Therefore, by Theorem 2.4, $\|\Lambda_i S^{-1}+tu_i\|_{i\in I}$ is not optimal for one erasure.

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