Optimality and Duality for Multiobjective Programming with Generalized Univexity

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***Abstract*—This paper presents the introduction of d-ρ-(η,θ)-univex functions, which are built upon the concepts of d-ρ-(η,θ)-invex functions and univex functions. By utilizing these concepts, the paper derives various optimality results for feasible solutions to be considered as efficient or weak efficient solutions. Additionally, the paper establishes several duality theorems for Mond-Weir and Wolf duality.**

***Index Terms*—Generalized d-ρ-(η,θ)-univexity; Duality; Optimality; Multiobjective programming**

I. Introduction

CONVEXITY plays a key role in optimization theory as it allows for the relaxation of the assumption of standard convexity. Over the years, many researchers have proposed different types of generalized convex functions. One example is η-invexity (1981), which established that the KT conditions are sufficient for optimality. Other types of generalized convexity, such as ρ-(η,θ)-invexity and d-ρ-(η,θ)-invexity, were proposed by Nahak and Nanda (2005) and Nahak and Mohapatra (2009) respectively. These concepts have been extensively studied, and various researchers, including Ben-Isreal and Mond (1986) and Das and Nanda (1995), have explored their properties, extensions, and applications. Additionally, Nahak and Mohapatra (2009) have investigated duality for Mond-Weir and Wolf type duality using d-ρ-(η,θ)-invexity. Overall, these studies have contributed to a better understanding for generalized convex in optimization problems. In 2001, Aghezzaf and Hachimi introduced several concepts related to convexity and invexity of functions. These concepts include weakly strictly pseudoinvex, strongly pseudoinvex, weakly quasi-invex, weakly pseudoinvex, and strongly quasi-invex functions. For nondifferentiable multiobjective programming, Antczak (2002) proposed the concept of d-invexity and obtained optimization conditions and duality results by using Mond-Weir and Wolf duality. Building upon these concepts, Zhang and Wang (2022) introduced the concepts of d-ρ-(η,θ)-quadiinvex, and strictly d-ρ-(η,θ)-psedoinvex functions based on the existing d-ρ-(η,θ)-invexity concept. They further derived some duality results. These developments have expanded the understanding of convexity and invexity concepts and their applications in optimization problems.

This paper introduces the concepts of d-ρ-(η,θ)-univex functions, which are based on the previous works of Rueda et al. (1995), Nahak et al. (2009). It then proceeds to establish several sufficient optimality results for feasible solutions to be considered efficient or weak efficient solutions. Additionally, the paper presents duality results for Mond-Weir and Wolf type, expanding upon the results of Antczak (2002) and Nahak et al. (2009).

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II. Preliminaries

The following conventions are used for $x, y \in \mathbb{R}^n$ in the paper:

$$\begin{aligned} x \leqq y &\text{ is equivalent to } x_i \leqq y_i,\ i=1,\cdots,n,\\ x \leq y &\text{ is equivalent to } x_i \leqq y_i,\ x \neq y,\\ x < y &\text{ is equivalent to } x_i < y_i,\ i=1,\cdots,n. \end{aligned}$$

Let $X \subseteq \mathbb{R}^n$ be a nonempty set, and let $\eta : X \times X \to \mathbb{R}^n$ and $\theta : X \times X \to \mathbb{R}^n$ be vector functions.

The problem that will be examined in this paper is shown below:

$$(VP)\begin{cases} \min & f(x) = (f_1(x), \cdots, f_k(x)) \\ \text{s.t.} & g(x) = (g_1(x), \cdots, g_m(x)) \leqq 0, \\ & x \in X. \end{cases}$$

Let $M = \{1, 2, \cdots, m\}$, $F = \{x \in X \mid g(x) \leqq 0\}$ be the feasible region of (VP), $J = \{j \in M | g_j(\bar{x}) = 0\}$, $\tilde{J} = \{j \in M | g_j(\bar{x}) < 0\}$. It is clear that $J \bigcup \tilde{J} = M$. Along the direction $\eta(x,u)$, the directional derivative $f^{'}(u, \eta(x,u))$ of f is given below:

$$f^{'}(u, \eta(x,u)) = \lim_{\lambda \to 0^+} \frac{f(u + \lambda\eta(x,u)) - f(u)}{\lambda}.$$

Definition 1. For all x and some $\bar{x}$ in F, if $f(x) \nless f(\bar{x})$, then $\bar{x}$ is named a weak efficient solution for (VP).

Definition 2. For all x and some $\bar{x}$ in F, if $f(x) \nleq f(\bar{x})$, then $\bar{x}$ is named an efficient solution for (VP).

Definition 3. (Nahak and Mohapatra, 2009) For all x in F, we define d-ρ-(η,θ)-invexity at $u \in X$ for a nonempty set $X \subseteq \mathbb{R}^n$ and a directionally differentiable function $h : X \to \mathbb{R}$ as follows: $h(x) - h(u) \geq h'(u, \eta(x,u)) + \rho\|\theta(x,u)\|^2$.

Definition 4. For all $x \in X$, a nonempty set $X \subseteq R^n$ and a directionally differentiable function $h : X \to R$ at $u \in X$, $h(x)$ is d-ρ-(η,θ)-uninvex at u: if $h^{'}(u, \eta(x,u)) + \rho \parallel \theta(x,u) \parallel^2 \leqq b(x,u)\phi(h(x) - h(u))$ holds.

Remark 1. If $\rho > 0$, $h(x)$ is named strongly d-ρ-(η,θ)-uninvex. On the other hand, if $\rho < 0$, $h(x)$ is referred to as weakly d-ρ-(η,θ)-uninvex.

Remark 2. If we set $b = 1$ and ϕ as the identity function, d-ρ-(η,θ)-uninvexity reduces to the concept of d-ρ-(η,θ)-invexity as defined in Nahak and Mohapatra (2009).

Remark 3. When $b = 1$, $\rho = 0$, and ϕ is the identity function, h is referred to as d-invex as defined by Antczak (2002).

Remark 4. Note that d-ρ-(η,θ)-invexity is a special case of d-ρ-(η,θ)-uninvexity. In other words, every function that is d-ρ-(η,θ)-invex is also d-ρ-(η,θ)-uninvex. However, there exist functions that are d-ρ-(η,θ)-uninvex but not d-ρ-(η,θ)-invex.

Example. Define $h: R^2 \to R$ as follows:

$$h(x_1,x_2) = \begin{cases} \frac{x_1^2}{\sqrt{x_1x_2}}, & x_1 \neq 0, x_2 \neq 0, \\ 0, & x_1 = x_2 = 0. \end{cases}$$

Clearly, h is not continuous at the point $(0,0)$. We define $\eta: R^2 \times R^2 \to R^2$ and $\theta: R^2 \times R^2 \to R^2$ as follows:

$$\eta(x,y) = (x_1,x_2),\ \theta(x,y) = (x_1,0)$$

where $x = (x_1,x_2)$ and $y = (y_1,y_2)$. Let $b(x,y) = 2\sqrt{x_1x_2}+1$, $\rho = 1$ and ϕ be the identity function.

Using the concept of the directional derivative, for $x_1 \neq 0$ and $x_2 \neq 0$, we have that

$$\lim_{\lambda\to 0^+} \frac{\frac{\lambda^2x_1^2}{\lambda\sqrt{x_1x_2}}}{\lambda} = \frac{x_1^2}{\sqrt{x_1x_2}} = h^{'}(0,\eta(x,0)).$$

We first show that $h(x)$ is d-ρ-(η,θ)-uninvex at $(0,0)$. To do so, we need to prove that the inequality below holds:

$$b(x,0)\phi(h(x)-h(0)) \geqq h^{'}(0,\eta(x,0)) + \rho \parallel \theta(x,0) \parallel^2 .$$

Suppose it is not true, that is,

$$(2\sqrt{x_1x_2}+1)\frac{x_1^2}{\sqrt{x_1x_2}} < \frac{x_1^2}{\sqrt{x_1x_2}} + x_1^2$$
$$\Rightarrow 2x_1^2 < x_1^2$$
$$\Rightarrow x_1^2 < 0,$$

which contradicts the fact that $x_1 \neq 0$ and $x_1 \in R$. Therefore, at $(0,0)$, $h(x)$ is d-ρ-(η,θ)-uninvex. Next, the aim is to show that $h(x)$ is not d-ρ-(η,θ)-invex at $(0,0)$. Suppose for contradiction. Then,

$$\frac{x_1^2}{\sqrt{x_1x_2}} \geqq \frac{x_1^2}{\sqrt{x_1x_2}} + x_1^2 \Rightarrow x_1^2 \leqq 0,$$

which contradicts the fact that $x_1 \neq 0$ and $x_1 \in R$. Hence, the conclusion is completed.

In the following definitions, let $\rho = (\rho_1,\rho_2,\cdots,\rho_k)^T \in R^k$, and $\rho^1 = (\rho_1^1,\cdots,\rho_k^1)^T \in R^k$, $\rho^2 \in R$, $\phi_0: R^k \to R^k$, $\phi_1: R^m \to R^m$, $b_0, b_1: X \times X \to R^+$.

Definition 5. For x in F, if $0 \geq \phi_0\left(f(x)-f(\bar{x})\right) b_0(x,\bar{x})$ implies $0 > \rho \parallel \theta(x,\bar{x}) \parallel^2 + f^{'}(\bar{x},\eta(x,\bar{x}))$, then $f: F \to R$ is named weakly strictly pseudo d-ρ-(η,θ)-univex at $\bar{x}$.

Definition 6. For x in F, if $0 \geq \phi_0\left(f(x)-f(\bar{x})\right) b_0(x,\bar{x})$ implies $0 \geq \rho \parallel \theta(x,\bar{x}) \parallel^2 + f^{'}(\bar{x},\eta(x,\bar{x}))$, then $f: F \to R$ is named strongly pseudo d-ρ-(η,θ)-univex at $\bar{x}$.

Definition 7. For x in F, if $0 > \phi_0\left(f(x)-f(\bar{x})\right) b_0(x,\bar{x})$ implies $0 \geq \rho \parallel \theta(x,\bar{x}) \parallel^2 + f^{'}(\bar{x},\eta(x,\bar{x}))$, then $f: F \to R$ is named weakly pseudo d-ρ-(η,θ)-univex at $\bar{x}$.

Definition 8. For x in F, if $0 \geq \phi_0\left(f(x)-f(\bar{x})\right) b_0(x,\bar{x})$ implies $0 \geqq \rho \parallel \theta(x,\bar{x}) \parallel^2 + f^{'}(\bar{x},\eta(x,\bar{x}))$, then $f: F \to R$ is named weakly quasi d-ρ-(η,θ)-univex at $\bar{x}$.

Definition 9. For x in F, if $0 \geqq \phi_0\left(f(x)-f(\bar{x})\right) b_0(x,\bar{x})$ implies $0 \geq \rho \parallel \theta(x,\bar{x}) \parallel^2 + f^{'}(\bar{x},\eta(x,\bar{x}))$, then $f: F \to R$ is named strongly quasi d-ρ-(η,θ)-univex at $\bar{x}$.

Lemma 1. (KKT Necessary Optimality Condition, Nahak and Mohapatra, 2009) For (VP), let $\bar{x}$ be its a weakly efficient solution. If g_j $(j \in \tilde{J})$ is continuous, and at $\bar{x}$, both f and g are directionally differentiable, $f^{'}(\bar{x},\eta(x,\bar{x}))$ and $g_J^{'}(\bar{x},\eta(x,\bar{x}))$ being preinvex functions on X, and g satisfies the generalized Slater constraint qualification at $\bar{x}$, then there exists a non-negative vector $\bar{v} \in R_+^m$ such that $(\bar{x},\bar{v})$ satisfies:

$$\bar{v}^T g(\bar{x}) = 0, \tag{1}$$

$$g(\bar{x}) \leqq 0. \tag{2}$$

$$f^{'}(\bar{x},\eta(x,\bar{x})) + \bar{v}^T g^{'}(\bar{x},\eta(x,\bar{x})) \geqq 0,\ \forall x \in X, \tag{3}$$

III. Sufficient optimality conditions

Theorem 1. For (VP), let $\bar{x}$ be its a feasible point, and the conditions below hold:

$$\bar{v}^T g(\bar{x}) = 0, \quad (\bar{u}^T f^{'} + \bar{v}^T g^{'})(\bar{x},\eta(x,\bar{x})) \geqq 0,$$

where $\bar{u} \in \mathbb{R}_{++}^k$ and $\bar{v} \in \mathbb{R}_+^m$. Additionally, $\bar{u}^T\rho^1 + \rho^2 \geqq 0$, and one of (i) and (ii) below holds:

(i) f is strongly pseudo d-ρ^1-(η,θ)-univex at $\bar{x}$, where, $b_0 > 0$ and $a < 0$ implies $\phi_0(a) \leq 0$, while $\bar{v}^T g$ is strongly quasi d-ρ^2-(η,θ)-univex at $\bar{x}$, where $a \leqq 0$ implies $\phi_1(a) \leqq 0$.

(ii) f is weakly pseudo d-ρ^1-(η,θ)-univex at $\bar{x}$. Here, $b_0 > 0$ and $a < 0$ implies $\phi_0(a) < 0$, while $\bar{v}^T g$ is strongly quasi d-ρ^2-(η,θ)-univex at $\bar{x}$, where $a \leqq 0$ implies $\phi_1(a) \leqq 0$.

Under these conditions, we can conclude that $\bar{x}$ is a weakly efficient solution for (VP).

Proof. Assuming the conclusion is not true, it implies the existence of x in the set F with $f(x) < f(\bar{x})$.

(i) Using the inequality $b_0 > 0$ and $a < 0$ implies $0 \geq \phi_0(a)$, we can obtain

$$0 \geq \phi_0(f(x)-f(\bar{x}))b_0(x,\bar{x}).$$

Given that f is strongly pseudo d-ρ^1-(η,θ)-convex at $\bar{x}$, we can conclude that:

$$0 \geq f^{'}(\bar{x},\eta(x,\bar{x})) + \rho^1 \parallel \theta(x,\bar{x}) \parallel^2 .$$

As $\bar{u} > 0$, we can deduce that

$$0 > \bar{u}^T f^{'}(\bar{x},\eta(x,\bar{x})) + \bar{u}^T\rho^1 \parallel \theta(x,\bar{x}) \parallel^2 . \tag{7}$$

Given $\bar{v} \geqq 0$ and the feasibility of x, it can be inferred that $0 \geqq \bar{v}^T g(x)$. When combined with $\bar{v}^T g(\bar{x}) = 0$, we are easy to get

$$0 \geq \bar{v}^T g(x) - \bar{v}^T g(\bar{x}).$$

Using the fact that $a \leqq 0$ implies $\phi_1(a) \geqq 0$, and $b_1 \geqq 0$, we obtain

$$0 \geqq \phi_1(\bar{v}^T g(x) - \bar{v}^T g(\bar{x}))b_1(x,\bar{x}).$$

By the strong quasi d-ρ^2-(η,θ)-univexity of $\bar{v}^T g$ at $\bar{x}$, we get

$$0 > \bar{v}^T g'(\bar{x},\eta(x,\bar{x})) + \rho^2\|\theta(x,\bar{x})\|^2. \tag{8}$$

The expression obtained by adding (7) and (8) is:

$$(\bar{u}^T f^{'} + \bar{v}^T g^{'})(\bar{x},\eta(x,\bar{x})) + (\bar{u}^T\rho^1 + \rho^2) \parallel \theta(x,\bar{x}) \parallel^2 \leq 0$$

As $\bar{u}^T\rho^1 + \rho^2 \geqq 0$, we have:

$$0 > (\bar{u}^T f^{'} + \bar{v}^T g^{'})(\bar{x},\eta(x,\bar{x}))$$

This contradiction invalidates equation (1) and concludes the proof.

(ii) Given that $b_0 > 0$ and $a < 0$, we can conclude that $\phi_0(a) < 0$, which implies:

$$0 > \phi_0(f(x) - f(\bar{x}))b_0(x, \bar{x}).$$

Using the weak pseudo d-ρ^1-(η, θ)-univexity of f, we can derive:

$$0 > f^{'}(\bar{x}, \eta(x, \bar{x})) + \rho^1 \parallel \theta(x, \bar{x}) \parallel^2,$$

Since $\bar{u} > 0$, we can get that:

$$0 > \bar{u}^T f^{'}(\bar{x}, \eta(x, \bar{x})) + \bar{u}^T \rho^1 \parallel \theta(x, \bar{x}) \parallel^2 .$$

The rest follows a similar logic as in (i) and is therefore omitted for brevity.

Theorem 2. For (VP), suppose $\bar{x}$ is its a feasible point, and $\bar{u} \in R^k$, $\bar{v} \in R^m$ satisfy:

$$\bar{v}^T g(\bar{x}) = 0 \quad (9)$$

$$\bar{u} \geq 0, \; \bar{v} \geqq 0 \quad (10)$$

$$0 \leqq (\bar{u}^T f^{'} + \bar{v}^T g^{'})(\bar{x}, \eta(x, \bar{x})) \quad (11)$$

Additionally, $\bar{u}^T \rho^1 + \rho^2 \geqq 0$ and f is weakly strictly pseudo d-ρ^1-(η, θ)-univex at $\bar{x}$, where $a < 0$ implies $\phi_0(a) \leq 0$ and $b_0 > 0$. Also assume that $\bar{v}^T g$ is strongly quasi d-ρ^2-(η, θ)-univex at $\bar{x}$, where $a \leqq 0$ implies $\phi_1(a) \leqq 0$. Then, $\bar{x}$ is a weakly efficient solution for (VP).

Proof. Assuming the conclusion is false. Then, for some x in F, we have $f(x) < f(\bar{x})$. Since $a < 0$ implies $\phi_0(a) \leq 0$, and $b_0 > 0$, we can conclude that:

$$0 \geq b_0(x, \bar{x})\phi_0(f(x) - f(\bar{x})).$$

By the weakly strictly pseudo d-ρ^1-(η, θ)-univex of f at $\bar{x}$, we are easy to get:

$$0 > f^{'}(\bar{x}, \eta(x, \bar{x})) + \rho^1 \parallel \theta(x, \bar{x}) \parallel^2 .$$

As $\bar{u} \geq 0$, we can derive:

$$0 > \bar{u}^T f^{'}(\bar{x}, \eta(x, \bar{x})) + \bar{u}^T \rho^1 \parallel \theta(x, \bar{x}) \parallel^2 . \quad (12)$$

As x is feasible for (VP), and $\bar{v} \geqq 0$, we can conclude that $0 \geqq \bar{v}^T g(x)$. Combining this with $\bar{v}^T g(\bar{x}) = 0$, we can derive:

$$0 \geqq \bar{v}^T g(x) - \bar{v}^T g(\bar{x}).$$

By $a \leqq 0$ implies $\phi_1(a) \leqq 0$, and $b_1 \geqq 0$, we can derive the following inequality:

$$0 \geqq \phi_1(\bar{v}^T g(x) - \bar{v}^T g(\bar{x}))b_1(x, \bar{x}).$$

Using the fact that $\bar{v}^T g$ is strongly quasi d-ρ^2-(η, θ)-univex at $\bar{x}$, we can obtain:

$$0 \geq \bar{v}^T g^{'}(\bar{x}, \eta(x, \bar{x})) + \rho^2 \parallel \theta(x, \bar{x}) \parallel^2 . \quad (13)$$

By combining (12) and (13), we get:

$$0 > (\bar{u}^T f^{'} + \bar{v}^T g^{'})(\bar{x}, \eta(x, \bar{x})) + (\bar{u}^T \rho^1 + \rho^2) \parallel \theta(x, \bar{x}) \parallel^2 .$$

Since $\bar{u}^T \rho^1 + \rho^2 \geqq 0$, it implies that:

$$0 > (\bar{u}^T f^{'} + \bar{v}^T g^{'})(\bar{x}, \eta(x, \bar{x})),$$

which contradicts (9). Thus, the proof is completed.

Theorem 3. For (VP), suppose $\bar{x}$ is its a feasible point, and $\bar{u} \in R^k$ and $\bar{v} \in R^m$ satisfy:

$$\bar{v}^T g(\bar{x}) = 0 \quad (14)$$

$$\bar{u} > 0, \; \bar{v} \geqq 0 \quad (15)$$

$$0 \leqq (\bar{u}^T f^{'} + \bar{v}^T g^{'})(\bar{x}, \eta(x, \bar{x})) \quad (16)$$

Additionally, suppose that $\bar{u}^T \rho^1 + \rho^2 \geqq 0$, and (i) or (ii) is met: at $\bar{x}$

(i) f is strong pseudo d-ρ^1-(η, θ)-univex with $a \leq 0$ implies $\phi_0(a) \leq 0$ and $b_0 > 0$; $\bar{v}^T g$ is strong quasi d-ρ^2-(η, θ)-univex with $a \leqq 0$ implies $\phi_1(a) \leqq 0$.

(ii) f is weak pseudo d-ρ^1-(η, θ)-univex with $a \leq 0$ implies $\phi_0(a) < 0$, and $b_0 > 0$; $\bar{v}^T g$ is strong quasi d-ρ^1-(η, θ)-univex with $a \leqq 0$ implies $\phi_1(a) \leqq 0$.

Then we have that $\bar{x}$ is an efficient solution for (VP).

Proof. Assume the conclusion is not true. Then, for some x in F, we have $f(x) \leq f(\bar{x})$.

(i) Using the inequality $b_0 > 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, we can derive:

$$0 \geq \phi_0(f(x) - f(\bar{x}))b_0(x, \bar{x}).$$

At $\bar{x}$, since f is strongly strictly pseudo d-ρ^1-(η, θ)-univex, we can obtain:

$$0 \geq f'(\bar{x}, \eta(x, \bar{x})) + \rho^1 \|\theta(x, \bar{x})\|^2.$$

Multiplying both sides by $\bar{u}^T$, and by $\bar{u} > 0$, we get:

$$0 > \bar{u}^T f'(\bar{x}, \eta(x, \bar{x})) + \bar{u}^T \rho^1 \|\theta(x, \bar{x})\|^2. \quad (17)$$

Since x is feasible for (VP) and $\bar{v} \geqq 0$, we have $0 \geqq \bar{v}^T g(x)$. Combining this with $\bar{v}^T g(\bar{x}) = 0$, we derive:

$$\bar{v}^T g(x) - \bar{v}^T g(\bar{x}) \leqq 0.$$

This is the desired inequality.

Using the fact that $b_1 \geqq 0$, and $a \leqq 0$ implies $\phi_1(a) \leqq 0$, we can write:

$$\phi_1(\bar{v}^T g(x) - \bar{v}^T g(\bar{x}))b_1(x, \bar{x}) \leqq 0.$$

At $\bar{x}$, since $\bar{v}^T g$ is strongly quasi d-ρ^2-(η, θ)-univex, we obtain:

$$\bar{v}^T g'(\bar{x}, \eta(x, \bar{x})) + \rho^2 \|\theta(x, \bar{x})\|^2 \leq 0. \quad (18)$$

Adding equations (17) and (18), we get:

$$0 > (\bar{u}^T f' + \bar{v}^T g')(\bar{x}, \eta(x, \bar{x})) + (\bar{u}^T \rho^1 + \rho^2)\|\theta(x, \bar{x})\|^2.$$

Since $\bar{u}^T \rho^1 + \rho^2 \geq 0$, we can conclude that:

$$0 > (\bar{u}^T f' + \bar{v}^T g')(\bar{x}, \eta(x, \bar{x})),$$

which contradicts equation (14). Therefore, we have proved that our assumption that $\bar{x}$ is not an efficient solution for (VP) leads to a contradiction, and the proof is completed.

(ii) From $b_0 > 0$, and $a \leq 0$ implies $\phi_0(a) < 0$, we can have

$$0 > \phi_0(f(x) - f(\bar{x}))b_0(x, \bar{x}).$$

At $\bar{x}$, since f is weak pseudo d-ρ^1-(η, θ)-univex, we can derive

$$0 \geq f^{'}(\bar{x}, \eta(x, \bar{x})) + \rho^1 \parallel \theta(x, \bar{x}) \parallel^2,$$

Multiplying both sides by $\bar{u}$, and using $\bar{u} > 0$, we can obtain that

$$0 > \bar{u}^T f^{'}(\bar{x}, \eta(x, \bar{x})) + \bar{u}^T \rho^1 \parallel \theta(x, \bar{x}) \parallel^2 .$$

The rest of the proof can be shown in a similar way as in case (i), and hence, we omit the details.

Theorem 4. For (VP), suppose $\bar{x}$ is its a feasible point, and $\bar{u} \in \mathbb{R}^k$ and $\bar{v} \in \mathbb{R}^m$ satisfy the following conditions:

$$\bar{v}^T g(\bar{x}) = 0 \quad (19)$$

$$\bar{u} \geq 0, \quad \bar{v} \geqq 0 \quad (20)$$

$$0 \geqq (\bar{u}^T f' + \bar{v}^T g')(\bar{x}, \eta(x, \bar{x})) \quad (21)$$

Assuming that $\bar{u}^T \rho^1 + \rho^2 \geq 0$, f is weakly strictly pseudo d-ρ^1-(η, θ)-quasiconvex at $\bar{x}$, where $a \leq 0$ implies $\phi_0(a) \leq 0$ and $b_0 > 0$, and $\bar{v}^T g$ is strongly quasi d-ρ^2-(η, θ)-quasiconvex at $\bar{x}$, where $a \leq 0$ implies $\phi_1(a) \leq 0$, we can conclude that $\bar{x}$ is an efficient solution for (VP).

Proof. Suppose the conclusion is not true. This means the existence of some $x \in F$ with $f(x) \leq f(\bar{x})$. By utilizing the assumption that $b_0 > 0$, and $a \leq 0$ implies $\phi_0(a) \leq 0$, we can get

$$0 \geq \phi_0(f(x) - f(\bar{x}))b_0(x, \bar{x}).$$

Since f is weakly strictly pseudo d-ρ^1-(η, θ)-quasiconvex at $\bar{x}$, it implies

$$0 > f'(\bar{x}, \eta(x, \bar{x})) + \rho^1 ||\theta(x, \bar{x})||^2.$$

Using the fact that $\bar{u} \geq 0$, we obtain

$$0 > \bar{u}^T f'(\bar{x}, \eta(x, \bar{x})) + \bar{u}^T \rho^1 ||\theta(x, \bar{x})||^2. \quad (22)$$

By the feasibility of x and the fact that $\bar{v} \geqq 0$, we have $0 \geqq \bar{v}^T g(x)$. Combining this with $\bar{v}^T g(\bar{x}) = 0$, we get

$$0 \geqq \bar{v}^T g(x) - \bar{v}^T g(\bar{x}).$$

Using the assumption that $a \leqq 0$ implies $\phi_1(a) \leqq 0$, and $b_1 \geqq 0$, we have

$$0 \geqq \phi_1(\bar{v}^T g(x) - \bar{v}^T g(\bar{x}))b_1(x, \bar{x}).$$

At $\bar{x}$, since $\bar{v}^T g$ is strongly quasi d-ρ^2-(η, θ)-quasiconvex, we can get

$$0 \geqq \bar{v}^T g'(\bar{x}, \eta(x, \bar{x})) + \rho^2 ||\theta(x, \bar{x})||^2. \quad (23)$$

Adding (22) and (23), we obtain

$$0 > (\bar{u}^T f' + \bar{v}^T g')(\bar{x}, \eta(x, \bar{x})) + (\bar{u}^T \rho^1 + \rho^2)||\theta(x, \bar{x})||^2.$$

By $\bar{u}^T \rho^1 + \rho^2 \geq 0$, we can get

$$0 > (\bar{u}^T f' + \bar{v}^T g')(\bar{x}, \eta(x, \bar{x})),$$

which contradicts (19), and the proof is completed.

IV. Mond-Weir type duality

This section will examine the dual of Mond-Weir type given by

$$(VD) \begin{cases} \max & f(y) = (f_1(y), \cdots, f_k(y)) \\ \text{s.t.} & (u^T f' + v^T g')(y, \eta(x, y)) \geqq 0, \ \forall x \in F, \\ & v^T g(y) \geqq 0, \ u^T e = 1, \\ & u \in \mathbb{R}^k_+, \ v \in \mathbb{R}^m_+. \end{cases}$$

Here, $e = (1, \cdots, 1)^T \in \mathbb{R}^k$. We denote the feasible region of (VD) by W, and $Pr_X W$ represents the projection of the set W onto X.

Theorem 5. (Weak duality) Suppose x and (y, u, v) are feasible points of (VP) and (VD), respectively. Additionally, assume $u^T \rho^1 + \rho^2 \geqq 0$, and one of (i)-(iii) holds: at y

(i) f is strongly pseudo d-ρ^1-(η, θ)-univex on $F \bigcup Pr_X W$ under the conditions $u > 0$, $b_0 > 0$, and $a \leq 0$ implies $\phi_0(a) \leq 0$. Moreover, $v^T g$ is strongly quasi d-ρ^2-(η, θ)-univex under $a \leqq 0$ implies $\phi_1(a) \leqq 0$.

(ii) f is weakly pseudo d-ρ^1-(η, θ)-univex on $F \bigcup Pr_X W$ under the conditions $u > 0$, $b_0 > 0$, and $a \leq 0$ implies $\phi_0(a) < 0$. Moreover, $v^T g$ is strongly quasi d-ρ^2-(η, θ)-univex under $a \leqq 0$ implies $\phi_1(a) \leqq 0$.

(iii) f is weakly strictly pseudo d-ρ^1-(η, θ)-univex on $F \bigcup Pr_X W$ under the condition $b_0 > 0$ and $a \leq 0$ implies $\phi_0(a) \leq 0$. Moreover, $v^T g$ is strongly quasi d-ρ^2-(η, θ)-univex under the condition $a \leqq 0$ implies $\phi_1(a) \leqq 0$.

Then, we have $f(x) \nleq f(y)$.

Proof. Assume the opposite holds, namely $f(x) \leq f(y)$.

(i) Given that $a \leq 0$ implies $\phi_0(a) \leq 0$ and $b_0 > 0$, we can derive

$$\phi_0(f(x) - f(y))b_0(x, y) \leq 0.$$

From f is strongly pseudo d-ρ^1-(η, θ)-univex, we get

$$f^{'}(y, \eta(x, y)) + \rho^1 \parallel \theta(x, y) \parallel^2 \leq 0,$$

By the positivity of u, we can deduce that

$$u^T f^{'}(y, \eta(x, y)) + u^T \rho^1 \parallel \theta(x, y) \parallel^2 < 0. \quad (24)$$

According to the feasibility of x and $v \geqq 0$, we have $v^T g(x) \leqq 0$. Combining this with $v^T g(y) \geqq 0$, it is easy to derive

$$v^T g(x) - v^T g(y) \leqq 0.$$

Given that $a \leqq 0$ implies $\phi_1(a) \leqq 0$ and $b_1 \geqq 0$, it implies that

$$\phi_1(v^T g(x) - v^T g(y))b_1(x, y) \leqq 0.$$

By the strong quasi d-ρ^2-(η, θ)-univexity of $v^T g$ at y, we have

$$v^T g^{'}(y, \eta(x, y)) + \rho^2 \parallel \theta(x, y) \parallel^2 \leq 0. \quad (25)$$

By adding (24) and (25), we can derive the following inequality:

$$0 > (u^T f^{'} + v^T g^{'})(y, \eta(x, y)) + (u^T \rho^1 + \rho^2) \parallel \theta(x, y) \parallel^2 .$$

Since $u^T \rho^1 + \rho^2 \geqq 0$, it implies that

$$0 > (u^T f^{'} + v^T g^{'})(y, \eta(x, y)),$$

which contradicts

$$(u^T f^{'} + v^T g^{'})(y, \eta(x, y)) \geqq 0.$$

Therefore, the assumption that $f(x) \leq f(y)$ is false, and the process is completed.

(ii) Given $a \leq 0$ implies $\phi_0(a) < 0$, and $b_0 > 0$, we can derive

$$\phi_0(f(x) - f(y))b_0(x, y) < 0.$$

At y, as f is weakly pseudo d-ρ^1-(η, θ)-univex, it is not difficult to get

$$0 \geq f^{'}(y, \eta(x, y)) + \rho^1 \parallel \theta(x, y) \parallel^2,$$

Since $u > 0$, we can deduce

$$0 > u^T f^{'}(y, \eta(x, y)) + u^T \rho^1 \parallel \theta(x, y) \parallel^2 .$$

The remainder follows similarly to (i) and is omitted.

(iii) Given $a \leq 0$ implies $\phi_0(a) \leq 0$, and $b_0 > 0$, it is easy to derive

$$\phi_0(f(x) - f(y))b_0(x, y) \leq 0.$$

By the weakly strictly pseudo d-ρ^1-(η, θ)-univexity of f at y, we obtain

$$f^{'}(y, \eta(x, y)) + \rho^1 \parallel \theta(x, y) \parallel^2 < 0,$$

Based on $u \geq 0$ and $u^T e = 1$, we can easily derive

$$u^T f^{'}(y, \eta(x, y)) + u^T \rho^1 \parallel \theta(x, y) \parallel^2 < 0.$$

The rest similars to (i).

Theorem 6. (Strong duality) For (VP), suppose $\bar{x}$ is its a weakly efficient solution, and the generalized Slater's constraint qualification is met. Let f and g be preinvex functions on X and directionally differentiable at $\bar{x}$. Additionally, for all $j \in J$, assume that g_j is continuous. Then, for (VD), there exists a vector $\bar{v} \in R^m_+$ such that $(\bar{x}, 1, \bar{v})$ is its a feasible point. Furthermore, if the weak duality between (VP) and (VD) holds as given in Theorem 5, then, for (VD), $(\bar{x}, 1, \bar{v})$ is its a weakly efficient solution.

Proof. Lemma 1 confirms that $\bar{x}$ meets all the necessary conditions, allowing us to find a vector $\bar{v} \in R^m_+$ that satisfies conditions (1) to (3). As a result of these conditions, $(\bar{x}, 1, \bar{v})$ is feasible for (VD). Theorem 5 then implies that $(\bar{x}, 1, \bar{v})$ is a weakly efficient solution for (VD).

Theorem 7. (Converse duality) Suppose $(\bar{y}, \bar{u}, \bar{v})$ is a weakly efficient solution for (VD), and $\bar{u}^T \rho^1 + \rho^2 \geqq 0$ and $\bar{u} > 0$. Additionally, one of (i)-(iii) is met:

(i) f is strongly pseudo d-ρ^1-(η, θ)-univex at $\bar{y}$, where $a < 0$ implies $\phi_0(a) \leq 0$ and $b_0 > 0$, and $\bar{v}^T g$ is strongly quasi d-ρ^2-(η, θ)-univex at $\bar{y}$, where $a \leqq 0$ implies $\phi_1(a) \leqq 0$.

(ii) f is weakly pseudo d-ρ^1-(η, θ)-univex at $\bar{y}$, where $a < 0$ implies $\phi_0(a) < 0$ and $b_0 > 0$, and $\bar{v}^T g$ is strongly quasi d-ρ^1-(η, θ)-univex at $\bar{x}$, where $a \leqq 0$ implies $\phi_1(a) \leqq 0$.

Under these conditions, $\bar{y}$ is a weakly efficient solution of (VP).

Proof. Suppose the conclusion is incorrect. This implies that there is some $\bar{x}$ in F with $f(\bar{x}) < f(\bar{y})$.

(i) Using the conditions $a < 0$ implies $\phi_0(a) \leq 0$, and $b_0 > 0$, we can derive

$$0 \geq \phi_0(f(\bar{x}) - f(\bar{y}))b_0(\bar{x}, \bar{y}).$$

At $\bar{y}$, as f is strongly strictly pseudo d-ρ^1-(η, θ)-univex, it is easy to have

$$0 \geq f^{'}(\bar{y}, \eta(\bar{x}, \bar{y})) + \rho^1 \parallel \theta(\bar{x}, \bar{y}) \parallel^2 .$$

Given that $\bar{u} > 0$ and $\bar{v} \geqq 0$, we can conclude

$$0 > \bar{u}^T f^{'}(\bar{y}, \eta(\bar{x}, \bar{y})) + \bar{u}^T \rho^1 \parallel \theta(\bar{x}, \bar{y}) \parallel^2 . \tag{26}$$

Furthermore, using the feasibility of $\bar{x}$ and $\bar{v} \geqq 0$, we obtain $\bar{v}^T g(\bar{x}) \leqq 0$. This, combined with $\bar{v}^T g(\bar{y}) \geqq 0$, implies that

$$0 \geqq \bar{v}^T g(\bar{x}) - \bar{v}^T g(\bar{y}).$$

Since $a \leqq 0$ implies $\phi_1(a) \leqq 0$, and $b_1 \geqq 0$, we can deduce that

$$0 \geqq \phi_1(\bar{v}^T g(\bar{x}) - \bar{v}^T g(\bar{y}))b_1(\bar{x}, \bar{y}).$$

Given that $\bar{v}^T g$ is strongly quasi d-ρ^2-(η, θ)-univex, we obtain

$$0 \geq \bar{v}^T g^{'}(\bar{y}, \eta(\bar{x}, \bar{y})) + \rho^2 \parallel \theta(\bar{x}, \bar{y}) \parallel^2 . \tag{27}$$

By equations (26) and (27), we arrive at

$$0 > (\bar{u}^T f^{'} + \bar{v}^T g^{'})(\bar{y}, \eta(\bar{x}, \bar{y})) + (\bar{u}^T \rho^1 + \rho^2) \parallel \theta(\bar{x}, \bar{y}) \parallel^2 .$$

Since $\bar{u}^T \rho^1 + \rho^2 \geqq 0$, this implies that

$$0 > (\bar{u}^T f^{'} + \bar{v}^T g^{'})(\bar{y}, \eta(\bar{x}, \bar{y})),$$

which inconsistency in the equation below

$$0 \geqq (\bar{u}^T f^{'} + \bar{v}^T g^{'})(\bar{y}, \eta(\bar{x}, \bar{y})).$$

Thus, the process is completed.

(ii) Using the condition $a < 0$ implies $\phi_0(a) < 0$, and $b_0 > 0$, we can derive the inequality

$$0 > b_0(\bar{x}, \bar{y})\phi_0(f(\bar{x}) - f(\bar{y})).$$

Given that f is weakly pseudo d-ρ^1-(η, θ)-univex at $\bar{y}$, we obtain

$$0 \geq f^{'}(\bar{y}, \eta(\bar{x}, \bar{y})) + \rho^1 \parallel \theta(\bar{x}, \bar{y}) \parallel^2 .$$

Since $\bar{u} > 0$, we can conclude:

$$0 > \bar{u}^T f^{'}(\bar{y}, \eta(\bar{x}, \bar{y})) + \bar{u}^T \rho^1 \parallel \theta(\bar{x}, \bar{y}) \parallel^2 .$$

The remaining part proceeds in a similar manner as in (i), and the details are omitted here for brevity.

Theorem 8. (Converse duality) For (VD), given that $(\bar{y}, \bar{u}, \bar{v})$ is its a weak efficient solution, and f is weakly strictly pseudo d-ρ^1-(η, θ)-univex at $\bar{y}$, where $a < 0$ implies $0 \geq \phi_0(a)$. Additionally, assuming that $\bar{v}^T g$ is strongly quasi d-ρ^2-(η, θ)-univex at $\bar{y}$, where $a \leq 0$ implies $\phi_1(a) \leq 0$, and satisfying the constraint $\bar{u}^T \rho^1 + \rho^2 \geq 0$, we can conclude that: for (VP), $\bar{y}$ is its a weak efficient solution.

Proof. Suppose the opposite is correct, which indicates that there is some $\bar{x}$ in F such that $f(\bar{x}) < f(\bar{y})$. Using the condition $a < 0$ implies $\phi_0(a) \leq 0$, and $b_0 > 0$, we can derive the inequality

$$\phi_0(f(\bar{x}) - f(\bar{y}))b_0(\bar{x}, \bar{y}) \leq 0.$$

At $\bar{y}$, since f is weakly strictly pseudo d-ρ^1-(η, θ)-univex, we have the following inequalities:

$$0 > f'(\bar{y}, \eta(\bar{x}, \bar{y})) + \rho^1 \parallel \theta(\bar{x}, \bar{y}) \parallel^2 .$$

Using the fact that $\bar{u} \geq 0$, we can derive the inequality

$$0 > \bar{u}^T f'(\bar{y}, \eta(\bar{x}, \bar{y})) + \bar{u}^T \rho^1 \parallel \theta(\bar{x}, \bar{y}) \parallel^2 . \tag{28}$$

Based on the feasibility of $\bar{x}$ and the condition $\bar{v} \geqq 0$, we can get $\bar{v}^T g(\bar{x}) \leqq 0$. Combining this with $\bar{v}^T g(\bar{y}) \geqq 0$, the inequality below is easy to be got:

$$0 \geqq \bar{v}^T g(\bar{x}) - \bar{v}^T g(\bar{y}).$$

Using the assumption $a \geqq 0 \Rightarrow \phi_1(a) \leqq 0$, and $b_1 \geqq 0$, we can derive the inequality

$$0 \geqq \phi_1(\bar{v}^T g(\bar{x}) - \bar{v}^T g(\bar{y}))b_1(\bar{x}, \bar{y}).$$

At $\bar{y}$, Since $\bar{v}^T g$ is strongly quasi d-ρ^2-(η, θ)-univex, we have the inequality

$$0 \geq \bar{v}^T g'(\bar{y}, \eta(\bar{x}, \bar{y})) + \rho^2 \parallel \theta(\bar{x}, \bar{y}) \parallel^2 . \tag{29}$$

By adding equations (28) and (29), we obtain the following inequality:

$$0 > (\bar{u}^T f' + \bar{v}^T g')(\bar{y}, \eta(\bar{x}, \bar{y})) + (\bar{u}^T \rho^1 + \rho^2) \parallel \theta(\bar{x}, \bar{y}) \parallel^2 .$$

Since $\bar{u}^T \rho^1 + \rho^2 \geq 0$, we get

$$0 > (\bar{u}^T f' + \bar{v}^T g')(\bar{y}, \eta(\bar{x}, \bar{y})).$$

However, this contradicts the inequality

$$0 \leqq (\bar{u}^T f' + \bar{v}^T g')(\bar{y}, \eta(\bar{x}, \bar{y})),$$

and therefore the process is finished.

V. Wolf type duality

Consider the Wolf type dual problem given by

$$(VD) \begin{cases} \max & \phi(y,u,v) = f(y) + v^T g(y) e \\ \text{s.t.} & (u^T f' + v^T g')(y, \eta(x,y)) \geqq 0, \ \forall x \in F, \\ & u^T e = 1, \ u \in R^k_+, \ v \in R^m_+. \end{cases}$$

We denote the set of all feasible points of (VD) by W, and the projection of the set W on X by $Pr_X W$.

Theorem 9. (Weak duality) Suppose that x and (y,u,v) are feasible points for (VP) and (VD), respectively. Additionally, assume that $u^T f + v^T g$ is weakly strictly pseudo d-ρ^2-(η,θ)-univex at y on $F \cup Pr_X W$, where $a < 0$ implies $\phi(a) \leq 0$, and $\rho^2 \geqq 0, \ b > 0$. Then, $f(x) \not< \phi(y,u,v)$.

Proof. Suppose the opposite holds, i.e. $f(x) < \phi(y,u,v)$. By the feasibility of x, $u \in R^k_+, \ u^T e = 1$, and $v \in R^m_+$, we can derive the inequality

$$u^T f(x) + v^T g(x) < u^T f(y) + v^T g(y). \quad (30)$$

Using the conditions $a < 0$ implies $\phi(a) \leq 0$ and $b > 0$, we obtain:

$$0 \geq \phi(u^T f(x) + v^T g(x) - (u^T f(y) + v^T g(y)))b(x,y).$$

By assumption, $u^T f + v^T g$ is weakly strictly pseudo d-ρ^2-(η,θ)-univex at y on $F \cup Pr_X W$, and therefore we have the inequality

$$0 > (u^T f' + v^T g')(y, \eta(x,y)) + \rho^2 \parallel \theta(x,y) \parallel^2 .$$

Since $\rho^2 \geq 0$, it follows that

$$(u^T f' + v^T g')(y, \eta(x,y)) < 0,$$

which inconsistency in:

$$0 \leqq (u^T f' + v^T g')(y, \eta(x,y)).$$

This implies that $f(x) \not< \phi(y,u,v)$.

Theorem 10. (Strong duality) For (VP), assume that $\bar{x}$ is its a weak efficient solution, and the generalized Slater's constraint qualification is met. Moreover, at $\bar{x}$, g and f are directionally differentiable, $f'(\bar{x}, \eta(x,\bar{x}))$ and $g'(\bar{x}, \eta(x,\bar{x}))$ are preinvex functions on X, and $g_j (j \in J)$ is continuous. Under these conditions, there is a vector $\bar{v} \in R^m_+$ such that $(\bar{x}, 1, \bar{v})$ is a feasible solution for (VD). Additionally, if the weak duality between (VP) and (VD) holds as stated in Theorem 9, then $(\bar{x}, 1, \bar{v})$ is a weak efficient solution for (VD).

Proof. The proof follows a similar approach as in Theorem 6.

Theorem 11. (Converse duality) For (VD), assume that $(\bar{y}, \bar{u}, \bar{v})$ is its a weak efficient solution, and $\bar{u}^T f + \bar{v}^T g$ is weak strictly pseudo d-ρ^2-(η,θ)-uninvex at $\bar{y}$ on $F \bigcup Pr_X W$, and $a < 0$ implies $\phi(a) \leq 0$, $b > 0, \rho^2 \geqq 0$, then it follows that $\bar{y}$ is a weak efficient solution of (VP).

Proof. Suppose the statement is false, that means there is $\bar{x}$ in F with $f(\bar{x}) < f(\bar{y})$. Since $\bar{u} \in R^k_+$ and $\bar{u}^T e = 1$, we may have

$$\bar{u}^T f(\bar{x}) < \bar{u}^T f(\bar{y}). \quad (31)$$

Since $(\bar{y}, \bar{\lambda}, \bar{\mu}) \in W$, $\bar{x} \in D$ and $\bar{v} \in R^m_+$, it is easy to derive

$$\bar{v}^T g(\bar{x}) \leqq 0 \leqq \bar{v}^T g(\bar{y}). \quad (32)$$

Adding (31) and (32), we have the following inequality

$$\bar{u}^T f(\bar{x}) + \bar{v}^T g(\bar{x}) < \bar{u}^T f(\bar{y}) + \bar{v}^T g(\bar{y}). \quad (33)$$

By (33), and $a < 0$ implies $0 \geq \phi(a)$, and $b > 0$, we may derive

$$b(x,y)\phi(\bar{u}^T f(\bar{x}) + \bar{v}^T g(\bar{x}) - (\bar{u}^T f(\bar{y}) + \bar{v}^T g(\bar{y}))) \leq 0.$$

On $F \bigcup Pr_X W$, since $\bar{u}^T f + \bar{v}^T g$ is weak strictly pseudo d-ρ^2-(η,θ)-uninvex at y, we have

$$(\bar{u}^T f' + \bar{v}^T g')(\bar{y}, \eta(\bar{x},\bar{y})) + \rho^2 \parallel \theta(\bar{x},\bar{y}) \parallel^2 < 0.$$

Since $\rho^2 \geqq 0$, it implies that

$$0 > (\bar{u}^T f' + \bar{v}^T g')(\bar{y}, \eta(\bar{x},\bar{y})).$$

However, this contradicts the inequality below:

$$0 \leqq (u^T f' + v^T g')(\bar{y}, \eta(\bar{x},\bar{y})).$$

Therefore, the conclusion can be gotten.

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