A High-Order L1-2 Scheme Based on Compact Finite Difference Method for the Nonlinear Time-Fractional Schrödinger Equation

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Abstract—**In this paper, a high-order L1-2 scheme based on the compact finite difference method for the nonlinear time-fractional Schrödinger equation with homogeneous Dirichlet boundary condition is derived. Firstly, a standard fully discrete numerical scheme is constructed by adopting the L1-2 formula to approximate the Caputo fractional derivative for the time discretization and the compact finite difference method for the space discretization. In addition to proving the unique solvability of the numerical solution, we also established the convergence analysis of the fully discrete numerical scheme based on the discrete Grönwall inequality. Furthermore, the global convergence order $O(\tau^{3-\alpha}+h^4)$ in discrete L^2-norm of the numerical scheme is proved rigorously. A variety of numerical results are carried out to confirm the theoretical analysis.**

Index Terms—**Nonlinear time-fractional Schrödinger equation, Caputo fractional derivative, Compact finite difference method, Convergence.**

I. Introduction

In this paper, we aim at developing the high-order numerical scheme for the following nonlinear time-fractional Schrödinger (NTFS) equation

$$\begin{cases} i\,{}_0^C\mathcal{D}_t^\alpha u = \Delta u + f\left(|u|^2\right)u, & \text{in } \Omega\times(0,T], \\ u(x,0) = u_0(x), & \text{in } \Omega\times\{0\}, \\ u(x,t) = 0, & \text{on } \partial\Omega\times(0,T], \end{cases} \tag{1}$$

where $i = \sqrt{-1}$ is the imaginary unit, $u(x,t)$ represents the wave function, $f \in C^2(R)$ is a given function, $u_0(x)$ is a given smooth function. The opetator ${}_0^C\mathcal{D}_t^\alpha$ with $\alpha \in (0,1)$ denotes the Caputo fractional derivative, which is defined by

$${}_0^C\mathcal{D}_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,s)}{\partial s}\frac{1}{(t-s)^\alpha}ds, \tag{2}$$

where $\Gamma(\cdot)$ is the standard Gamma function.

Recently, more and more scholars are interested in fractional differential equations which simulate many problems in physics, biology, geology, mechanical engineering, signal processing and control theory [6], [15]. The fractional Schrödinger equation is a general extension of the Schrödinger equation which describes the evolution of the position-space wave function of a particle in physics. And due to its extensive applications in fractional quantum mechanics and condensed matter physics [8], [14], [18], [19], it has attracted the attention of a large number of researchers. Further, Nabel [17] constructed the time-fractional Schrödinger equation based on the conception of fractional Brownian motion and proposed that the model can describe the non-Markovian evolution of free particles in quantum mechanics. Meanwhile, we have found that in some physical phenomena, such as the propagation of solitary waves in optical fibers, deep water turbulence, and laser beams, the time-fractional Schrödinger equation is also widely applied and of great significance [11].

In recent decades, a large number of efficient numerical methods have been proposed to solve the NTFS equation. For example, Mohebbi et al. [16] designed a numerical scheme for the NTFS equation by selecting the meshless method. Chen et al. [1] investigated the spectral approximation for the NTFS equation on graded meshes. Wang et al. [20] combined the L2-1_σ formula with the Galerkin finite element method to discretize the NTFS equation for constructing the effective numerical scheme. Their achievement is to construct two effective second-order linear numerical schemes and establish the optimal error estimates. Ding et al. [3] constructed a linear scheme for the one-dimensional NTFS equation by using the L1 formula in time and the quintic non-polynomial spline in space, respectively. Their theoretical achievements include analysis of unique solvability, stability and convergence. And a linearized L1-Galerkin finite element method was adopted by Li et al. [10] to solve the NTFS equation. They strictly proved the optimal error estimate with the convergence order of $O(\tau^{2-\alpha} + h^{r+1})$ for the numerical scheme, where r was the degree of the polynomial in space. For more numerical methods of the NTFS equation, we could refer the literatures to [2], [7], [13], [21].

In this paper, we improve the numerical accuracy in the temporal direction by using the L1-2 formula [5] to approximate the Caputo fractional derivative. The fundamental principle of this method is to use higher-order interpolation instead of linear interpolation to achieve higher-order accuracy. In order to improve the spatial convergence order, the compact finite difference approach [4] was applied to approximate the second-order derivatives in space. It has been proven that the our numerical scheme has a unique solution. Based on the discrete Grönwall inequality, the convergence analysis for the fully discrete numerical scheme is established. It has been theoretically proven that the numerical scheme achieves (3-α) order accuracy in the temporal direction and fourth order accuracy in the spatial direction.

The framework of the remaining part of this paper is

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presented as follows. In Section II, we use the classical compact finite difference method for spatial discretization and the L1-2 scheme on uniform meshes for temporal discretization. The unique solvability and convergence of the proposed scheme are established in Section III. In Section IV, numerical results are given to verify the theoretical analysis. Finally, conclusions are drawn in Section V.

II. Derivation of the high-order L1-2 scheme based on compact finite difference method

We consider the NTFS equation in one-dimensional case with the computational domain $\Omega = (a,b)$. Let $x_j = a + jh, t_n = n\tau, j = 0,...,M, n = 0,1,...,N$, where M and N are two positive integers, $h = \frac{b-a}{M}$ is spatial step size and $\tau = \frac{T}{N}$ denotes temporal step size. Denote $\Omega_h = \{x_j | j = 0,1,...,M\}$ and $\Omega_\tau = \{t_n | n = 0,1,...,N\}$. Let $W = \{\omega_j^n | \omega_0^n = \omega_M^n = 0, j = 0,1,...,M, n = 0,1,...,N\}$ be the grid function space defined on $\Omega_h \times \Omega_\tau$. For a grid function $u^n \in W$, we introduce the L1-2 formula given in [5] to approximate the Caputo fractional derivative, i.e.

$$ {}^{L}D_t^{\alpha} u^n = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[c_0^{(\alpha)} u^n - \sum_{k=1}^{n-1} \left(c_{n-k-1}^{(\alpha)} - c_{n-k}^{(\alpha)} \right) u^k - c_{n-1}^{(\alpha)} u^0 \right], \tag{3} $$

where $c_0^{(\alpha)} = a_0^{(\alpha)} = 1$ for $n = 1$; and for $n \geq 2$

$$ c_k^{(\alpha)} = \begin{cases} a_0^{(\alpha)} + b_0^{(\alpha)}, & k = 0, \\ a_k^{(\alpha)} + b_k^{(\alpha)} - b_{k-1}^{(\alpha)}, & k = 1,2,...,n-2, \\ a_k^{(\alpha)} - b_{k-1}^{(\alpha)}, & k = n-1, \end{cases} \tag{4} $$

with

$$ a_k^{(\alpha)} = (k+1)^{1-\alpha} - k^{1-\alpha}, k \geq 0, \tag{5} $$

$$ b_k^{(\alpha)} = \frac{1}{2-\alpha} \left[(k+1)^{2-\alpha} - k^{2-\alpha} \right] - \frac{1}{2} \left[(k+1)^{1-\alpha} + k^{1-\alpha} \right], k \geq 0. \tag{6} $$

The following lemma shows the truncation error estimation of the L1-2 formula.

Lemma 2.1: [5] Suppose that $u \in C^3[0, t_n]$. For any $\alpha \in (0,1)$, ${}^{L}D_t^{\alpha}(\upsilon(t_n))$ is the approximation of ${}_0^C\mathcal{D}_t^{\alpha}(\upsilon(t_n))$. Denote $R(\upsilon(t_n)) = {}_0^C\mathcal{D}_t^{\alpha}(\upsilon(t_n)) - {}^{L}D_t^{\alpha}(\upsilon(t_n))$. Then we have

$$ |R(\upsilon(t_1))| \leq \frac{\alpha}{2\Gamma(3-\alpha)} \max_{t_0 \leq t \leq t_1} \left| \upsilon''(t) \right| \Delta t^{2-\alpha} = O(\tau^{2-\alpha}), \tag{7} $$

$$ \begin{aligned} |R(\upsilon(t_n))| \leq & \frac{1}{\Gamma(1-\alpha)} \left\{ \frac{\alpha}{12} \max_{t_0 \leq t \leq t_1} \left| \upsilon''(t) \right| (t_n - t_1)^{-\alpha-1} \Delta t^3 \right. \\ & \left. + \left[\frac{1}{12} + \frac{\alpha}{3(1-\alpha)(2-\alpha)} \left(\frac{1}{2} + \frac{1}{3-\alpha} \right) \right] \max_{t_0 \leq t \leq t_n} \left| \upsilon'''(t) \right| \Delta t^{3-\alpha} \right\} \\ & = O(\tau^{3-\alpha}), n \geq 2. \end{aligned} \tag{8} $$

For any grid functional $\omega^n, \upsilon^n \in W$, we introduce the following finite difference quotient operators and define discrete inner products and discrete norms over W as:

$$ \delta_x \omega_{j-\frac{1}{2}}^n = \frac{1}{h}(\omega_j^n - \omega_{j-1}^n), \quad \delta_x^2 \omega_j^n = \frac{1}{h^2}(\omega_{j-1}^n - 2\omega_j^n + \omega_{j+1}^n), $$

$$ (\omega, \upsilon) = h \sum_{j=1}^{M-1} \omega_j \overline{\upsilon_j}, \qquad \|\omega\| = \sqrt{(\omega, \omega)}, $$

where the $\overline{\upsilon}$ is the complex conjugate of υ.

The compact operator ℓ_x used in the compact finite difference method is defined as follows:

$$ \ell_x u_j^n = \begin{cases} \frac{1}{12}(u_{j-1}^n + 10u_j^n + u_{j+1}^n), & 1 \leq j \leq M-1, \\ u_j^n, & j = 0, M. \end{cases} \tag{9} $$

The following lemma will be useful for constructing the compact finite difference scheme.

Lemma 2.2: [12] Denote $\xi(s) = (1-s)^3 \left[5 - 3(1-s)^2 \right]$. If $f(x) \in C^6[x_{j-1}, x_{j+1}], 1 \leq j \leq M-1$, then it holds that

$$ \begin{aligned} & \frac{1}{12} \left[f''(x_{j-1}) + 10f''(x_j) + f''(x_{j+1}) \right] \\ = & \frac{f(x_{j-1}) - 2f(x_j) + f(x_{j+1})}{h^2} \\ & + \frac{h^4}{360} \int_0^1 \left[f^{(6)}(x_j - sh) + f^{(6)}(x_j + sh) \right] \xi(s) ds. \end{aligned} \tag{10} $$

We denote by U_j^n and u_j^n as the exact value and the numerical approximation of $u(x_j, t_n)$, respectively. Considering the problem (1) at the point (x_j, t_n), we have

$$ i \, {}_0^C\mathcal{D}_t^{\alpha} U_j^n = \frac{\partial^2 U_j^n}{\partial x^2} + f\left(\left| U_j^n \right|^2 \right) U_j^n. \tag{11} $$

After using the L1-2 operator ${}^{L}D_t^{\alpha}$ to approximate the Caputo fractional derivative ${}_0^C\mathcal{D}_t^{\alpha}$, we get the time semi-discrete numerical scheme:

$$ i \, {}^{L}D_t^{\alpha} U_j^n = \frac{\partial^2 U_j^n}{\partial x^2} + f\left(\left| U_j^n \right|^2 \right) U_j^n - iR_{1j}^n, \tag{12} $$

where R_{1j}^n is the truncation error of the temporal discretization.

After applying the compact operator ℓ_x on both sides of the above equation, we obtain

$$ i\ell_x {}^{L}D_t^{\alpha} U_j^n = \ell_x \frac{\partial^2 U_j^n}{\partial x^2} + \ell_x f\left(\left| U_j^n \right|^2 \right) U_j^n - i\ell_x R_{1j}^n. \tag{13} $$

For $1 \leq j \leq M-1$, using Lemma 2.2, we have

$$ i\ell_x {}^{L}D_t^{\alpha} U_j^n = \delta_x^2 U_j^n + \ell_x f\left(\left| U_j^{n-1} \right|^2 \right) U_j^{n-1} + R_{2j}^n, \tag{14} $$

where

$$ \begin{aligned} R_{2j}^n = & - i\ell_x R_{1j}^n + \ell_x f\left(\left| U_j^n \right|^2 \right) U_j^n - \ell_x f\left(\left| U_j^{n-1} \right|^2 \right) U_j^{n-1} \\ & + \frac{h^4}{360} \int_0^1 \left[\frac{\partial^6 u}{\partial x^6}\left(x_j - sh, t_n \right) + \frac{\partial^6 u}{\partial x^6}\left(x_j + sh, t_n \right) \right] \xi(s) ds. \end{aligned} \tag{15} $$

By Lemma 2.1, Taylor expansion and the fact that $\ell_x R_{1j}^n = O(\tau^{3-\alpha})$, we obtain

$$ \left| R_{2j}^n \right| = O(\tau^{3-\alpha} + h^4). \tag{16} $$

Omitting the truncation error R_{2j}^n in (14) and replacing U_j^n with u_j^n, we obtain the following high-order L1-2 scheme:

$$ i\ell_x {}^{L}D_t^{\alpha} u_j^n = \delta_x^2 u_j^n + \ell_x f\left(\left| u_j^{n-1} \right|^2 \right) u_j^{n-1}, 1 \leq j \leq M-1, 1 \leq n \leq N, \tag{17} $$

$$ u_j^0 = u_0(x_j), 0 \leq j \leq M, \tag{18} $$

$$ u_0^n = u_M^n = 0, 1 \leq n \leq N. \tag{19} $$

By integrating the terms of the scheme (17), we can obtain the equivalent form, i.e.

$$\begin{aligned}
&\left(i\ c_0^{(\alpha)}-12w\right)u_{j-1}^n+\left(i\ 10c_0^{(\alpha)}+24w\right)u_j^n+\left(i\ c_0^{(\alpha)}-12w\right)u_{j+1}^n\\
=&i\sum_{k=1}^{n-1}\left(c_{n-k-1}^{(\alpha)}-c_{n-k}^{(\alpha)}\right)\left(u_{j-1}^k+10u_j^k+u_{j+1}^k\right)\\
&+i\ c_{n-1}^{(\alpha)}\left(u_{j-1}^0+10u_j^0+u_{j+1}^0\right) \qquad (20)\\
&+\mu\left[f\left(\left|u_{j-1}^{n-1}\right|^2\right)u_{j-1}^{n-1}+10f\left(\left|u_j^{n-1}\right|^2\right)u_j^{n-1}+f\left(\left|u_{j+1}^{n-1}\right|^2\right)u_{j+1}^{n-1}\right],
\end{aligned}$$

where $\mu=\tau^\alpha\Gamma(2-\alpha)$, $w=\frac{\mu}{h^2}$.

III. Theoretical analysis of the numerical scheme

In this section, we will be dedicated to studying the unique solvability and convergence of the numerical scheme (17)-(19). To begin with, we introduce some lemmas, which play a great role in theoretical analysis.

Lemma 3.1: [22] For any grid function $\omega\in W$, it holds that $\frac{4}{9}\|\omega\|\le\|\ell_x\omega\|\le\|\omega\|$.

Lemma 3.2: [9] Suppose that the nonnegative sequences $\{\omega^n,g^n|n=0,1,2...\}$ satisfy

$$^LD_t^\alpha\omega^n\le\lambda_1\omega^n+\lambda_2\omega^{n-1}+g^n,n\ge1,$$

where $\lambda_1\ge0$ and $\lambda_2\ge0$ are given constant independent of τ. Then there exists a positive constant τ^* such that, when $\tau\le\tau^*$,

$$\omega^n\le\left(6\omega^0+\frac{12t_n^\alpha}{\Gamma(1+\alpha)}\max_{0\le l\le n}g^l\right)E_\alpha(2\lambda t_n^\alpha),1\le n\le N,$$

where $E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k\alpha)}$ is the Mittag-Leffler function and $\lambda=6\lambda_1+\frac{c_0^{(\alpha)}}{c_0^{(\alpha)}-c_1^{(\alpha)}}\lambda_2$.

Now, we present the unique solvability and the convergence results in the following theorems.

Theorem 3.1: The high-order $L1$-2 scheme (17)-(19) is uniquely solvable.

Proof. It is not difficult to find that the numerical scheme (17) is a linear tridiagonal system at each time level, and its coefficient matrix is strictly diagonally dominant. Therefore, the numerical solution exists and is unique.

Theorem 3.2: Suppose that $\frac{\tau^{3-\alpha}}{\sqrt{h}}$ is sufficiently small. Let $U^n=u(t_n)$ be the solution of the problem (1), u^n be the solution of fully discrete numerical scheme (17)-(19). Then there exists a constant C independent on τ and h such that

$$\|U^n-u^n\|\le C(\tau^{3-\alpha}+h^4),n\ge1. \qquad (21)$$

Proof. Let $E^n=U^n-u^n$. Subtracting (17) from (14), we have

$$i\ell_x{}^LD_t^\alpha E^n=\delta_x^2E^n+\ell_x\left[f(|U_j^{n-1}|^2)U_j^{n-1}-f(|u_j^{n-1}|^2)u_j^{n-1}\right]+R_{2j}^n. \qquad (22)$$

Taking the inner product with respect to ℓ_xE^n and considering the imaginary part of the resulting equation, we obtain

$$\begin{aligned}
&\mathrm{Im}\left(i\ell_x{}^LD_t^\alpha E^n,\ell_xE^n\right)=\mathrm{Im}\left(\delta_x^2E^n,\ell_xE^n\right)\\
&+\mathrm{Im}\left(\ell_x\left[f\left(|U_j^{n-1}|^2\right)U_j^{n-1}-f\left(|u_j^{n-1}|^2\right)u_j^{n-1}\right],\ell_xE^n\right) \qquad (23)\\
&+\mathrm{Im}\left(R_{2j}^n,\ell_xE^n\right).
\end{aligned}$$

By the definition of $^LD_t^\alpha$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\mathrm{Im}\left(i\ell_x{}^LD_t^\alpha E^n,\ell_xE^n\right)\\
=&\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\mathrm{Re}\left(\ell_x\left[c_0^{(\alpha)}E^n-\sum_{s=1}^{n-1}\left(c_{n-s-1}^{(\alpha)}-c_{n-s}^{(\alpha)}\right)E^s\right.\right.\\
&\left.\left.-\ c_{n-1}^{(\alpha)}E^0\right],\ell_xE^n\right)\\
=&\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}c_0^{(\alpha)}\|\ell_xE^n\|^2\\
&-\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\mathrm{Re}\left(\ell_x\left[\sum_{s=1}^{n-1}\left(c_{n-s-1}^{(\alpha)}-c_{n-s}^{(\alpha)}\right)E^s+c_{n-1}^{(\alpha)}E^0\right],\ell_xE^n\right)\\
\ge&\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}c_0^{(\alpha)}\|\ell_xE^n\|^2\\
&-\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{s=1}^{n-1}\left(c_{n-s-1}^{(\alpha)}-c_{n-s}^{(\alpha)}\right)\frac{\|\ell_xE^s\|^2+\|\ell_xE^n\|^2}{2}\\
&-\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}c_{n-1}^{(\alpha)}\frac{\left\|\ell_xE^0\right\|^2+\|\ell_xE^n\|^2}{2}\\
=&\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[c_0^{(\alpha)}-\frac{1}{2}\sum_{s=1}^{n-1}\left(c_{n-s-1}^{(\alpha)}-c_{n-s}^{(\alpha)}\right)-\frac{1}{2}c_{n-1}^{(\alpha)}\right]\|\ell_xE^n\|^2\\
&-\frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\sum_{s=1}^{n-1}\left(c_{n-s-1}^{(\alpha)}-c_{n-s}^{(\alpha)}\right)\|\ell_xE^s\|^2\\
&-\frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}c_{n-1}^{(\alpha)}\left\|\ell_xE^0\right\|^2\\
=&\frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\left[c_0^{(\alpha)}\|\ell_xE^n\|^2-\sum_{s=1}^{n-1}\left(c_{n-s-1}^{(\alpha)}-c_{n-s}^{(\alpha)}\right)\|\ell_xE^s\|^2\right.\\
&\left.-\ c_{n-1}^{(\alpha)}\left\|\ell_xE^0\right\|^2\right]\\
=&\frac{1}{2}\ {}^LD_t^\alpha\|\ell_xE^n\|^2. \qquad (24)
\end{aligned}$$

Using integration by part formula, we obtain

$$\mathrm{Im}\left(\delta_x^2E^n,\ell_xE^n\right)=0. \qquad (25)$$

Together with (23), (24) and (25), we have

$$\begin{aligned}
\frac{1}{2}{}^LD_t^\alpha\|\ell_xE^n\|^2\le&\mathrm{Im}\left(\ell_x\left[f(|U_j^{n-1}|^2)U_j^{n-1}-f(|u_j^{n-1}|^2)u_j^{n-1}\right],\ell_xE^n\right)\\
&+\mathrm{Im}\left(R_{2j}^n,\ell_xE^n\right). \qquad (26)
\end{aligned}$$

Next, we will use the mathematical induction to prove inequality (21) holds. Firstly, it is not difficult to see that (21) holds when $n\equiv0$. Secondly, assuming that (21) holds for $0\le n\le l-1$, we will then prove that it holds for $n=l$.

According to the inequality $\left\|E^{l-1}\right\|_\infty\le\frac{1}{\sqrt{h}}\left\|E^{l-1}\right\|$, we have

$$\begin{aligned}
\left\|u^{l-1}\right\|_\infty\le&\left\|U^{l-1}\right\|_\infty+\left\|E^{l-1}\right\|_\infty\le\left\|U^{l-1}\right\|_\infty+\frac{C}{\sqrt{h}}(\tau^{3-\alpha}+h^4)\\
\le&\left\|U^{l-1}\right\|_\infty+1, \qquad (27)
\end{aligned}$$

whenever $\frac{\tau^{3-\alpha}}{\sqrt{h}}$ is sufficiently small.

Since $f \in C^2$ and u^{l-1} is bounded, we have

$$\begin{aligned}
&\left\| f\left(|U^{l-1}|^2\right) U^{l-1} - f\left(|u^{l-1}|^2\right) u^{l-1} \right\| \\
=&\left\| f(|U^{l-1}|^2) E^{l-1} + \left(f(|U^{l-1}|^2) - f(|u^{l-1}|^2) \right) u^{l-1} \right\| \\
\le&\left\| f(|U^{l-1}|^2) \right\|_\infty \left\| E^{l-1} \right\| \\
&+ \left\| f'(\xi) \right\|_\infty \left(\left\| U^{l-1} \right\|_\infty + \left\| u^{l-1} \right\|_\infty \right) \left\| u^{l-1} \right\|_\infty \left\| E^{l-1} \right\| \\
\le&\left\| f(|U^{l-1}|^2) \right\|_\infty \left\| E^{l-1} \right\| \\
&+ \left\| f'(\xi) \right\|_\infty \left(2\left\| U^{l-1} \right\|_\infty + 1 \right) \left(\left\| U^{l-1} \right\|_\infty + 1 \right) \left\| E^{l-1} \right\| \\
=&\sqrt{C_1} \left\| E^{l-1} \right\|,
\end{aligned} \tag{28}$$

where the symbol C_1 can be express as $C_1 = \left(\left\| f(|U^{l-1}|^2) \right\|_\infty + \| f'(\xi) \|_\infty \left(2\left\| U^{l-1} \right\|_\infty + 1 \right) \left(\left\| U^{l-1} \right\|_\infty + 1 \right) \right)^2$.

Applying the Cauchy-Schwarz inequality, (16) and Lemma 3.1, we obtain

$$\begin{aligned}
&\operatorname{Im}\left(\left(\ell_x [f(|U^{l-1}|^2) U^{l-1} - f(|u^{l-1}|^2) u^{l-1}], \ell_x E^l \right) + \left(R^l, \ell_x E^l \right) \right) \\
\le&\frac{C_1}{2} \left\| E^{l-1} \right\|^2 + \frac{1}{2} \left\| \ell_x E^l \right\|^2 + \frac{1}{2} \left\| R^l \right\|^2 + \frac{1}{2} \left\| \ell_x E^l \right\|^2 \\
=&\frac{C_1}{2} \left\| E^{l-1} \right\|^2 + \left\| \ell_x E^l \right\|^2 + \frac{1}{2} \left\| R^l \right\|^2 \\
\le&\frac{81 C_1}{32} \left\| \ell_x E^{l-1} \right\|^2 + \left\| \ell_x E^l \right\|^2 + C_2 (\tau^{3-\alpha} + h^4)^2,
\end{aligned} \tag{29}$$

where C_2 is a constant independent of the induction variable l.

Inserting (29) into (26) gives

$$^L D_t^\alpha \left\| \ell_x E^l \right\|^2 \le \frac{81 C_1}{16} \left\| \ell_x E^{l-1} \right\|^2 + 2 \left\| \ell_x E^l \right\|^2 + 2 C_2 (\tau^{3-\alpha} + h^4)^2. \tag{30}$$

Applying Lemma 3.2, we have

$$\left\| \ell_x E^l \right\| \le C_3 (\tau^{3-\alpha} + h^4), \tag{31}$$

where C_3 is a positive constant independent on τ and h.

Using Lemma 3.1 and letting $C = \frac{9}{4} C_3$, we obtain

$$\left\| E^l \right\| \le \frac{9}{4} \left\| \ell_x E^l \right\| = \frac{9}{4} C_3 (\tau^{3-\alpha} + h^4) = C (\tau^{3-\alpha} + h^4). \tag{32}$$

Therefore, (21) holds for $n = l$. ∎

IV. Numerical experiments

In this section, we validate our theoretical results through several numerical experiments and demonstrate the accuracy and effectiveness of the proposed numerical format.

A. Example 1

Considering the time-fractional Schrödinger equation with the exact solution

$$u(x,t) = (1+i) t^2 sin(\pi x), x \in [0,2], t \in [0,1]. \tag{33}$$

Firstly, in order to test the temporal errors and convergence orders, we fix a sufficiently small spatial step $h = \frac{1}{1000}$, so that the spatial error is negligible compared to the temporal error and does not affect the estimation in time. The corresponding numerical experimental results at time $T = 1$ for difference $\alpha = 0.25$, 0.5 and 0.75 are presented in Table I. Secondly, the spatial errors at time $T = 1$ and convergence orders are listed in Table II for the same α in temporal direction, where the time step size is sufficiently small. From these numerical results, it is not difficult to find that our numerical scheme is $(3-\alpha)$ order in time and fourth order in space.

B. Example 2

Considering the following nonlinear time-fractional Schrödinger equation

$$i\, {}_0^C\mathcal{D}_t^\alpha u + \Delta u + |u|^2 u = 0, \ x \in [-10, 10], \ t \in (0, 1], \tag{34}$$

with initial condition

$$u(x, 0) = e^{-x^2}, \ x \in [-10, 10], \tag{35}$$

and boundary condition

$$u(-10, t) = u(10, t) = 0, \ t \in (0, 1]. \tag{36}$$

By solving the above NTFS equation, we tested the temporal and spatial errors and convergence orders of our numerical scheme, respectively. The temporal and spatial L^2-errors at time $T = 1$ and convergence orders with different $\alpha = 0.25$, 0.5 and 0.75 are listed in Tables III and IV, respectively. To test whether changes in the value of α have an impact on the evolution of the wave function, we draw the approximation of $|u|$ with different $\alpha = 0.1$, 0.5 and 0.9 in Fig. 1. From Fig. 1, It is not difficult to observe that as the value of α increases, the wave disperse faster.

V. Conclusions

In this paper, an effective high-order L1-2 scheme based on compact finite difference method is constructed for solving the nonlinear time-fractional Schrödinger equation with homogeneous Dirichlet boundary condition. We combine L1-2 approximation for discretizing the temporal variable with fourth order compact finite difference scheme for discretizing the spatial variable to solve the problem. The numerical scheme achieves (3-α) order and fourth order accuracy in temporal and spatial directions, respectively. Based on analysis of the fully discrete scheme and the discrete Grönwall inequality, the unique solvability and the global convergence in discrete L^2-norm with the convergence order of $O(\tau^{3-\alpha} + h^4)$ are proved rigorously. Finally, a number of numerical results are carried out to verify the accuracy and effectiveness of the proposed scheme.

References

[1] L. Chen, S. Lü, Spectral approximation for the nonlinear time fractional Schrödinger equation on graded meshes, Int. J. Comput. Math. 99 (2022), pp. 2524-2541.

[2] X. Chen, Y. Di, J. Duan, D. Li, Linearized compact ADI schemes for nonlinear time-fractional Schrödinger equations, Appl. Math. Lett. 84 (2018), pp. 160-167.

[3] Q. Ding, P. Wong, Quintic non-polynomial spline for time-fractional nonlinear Schrödinger equation, Adv. Difference Equ. 577 (2020), pp. 1-27.

[4] G. Gao, Z. Sun, A compact finite difference scheme for the fractional sub-diffusion equation, J. Comput. Phys. 230 (2011), pp. 586-595.

[5] G. Gao, Z. Sun, H. Zhang, A new fractional numerical differentiation formula to approximate the Caputo fractional derivative and its applications, J. Comput. Phys. 259 (2014), pp. 33-50.

[6] R. Gorenflo, F. Mainardi, D. Moretti, G. Pagnini, P. Paradisi, Discrete random walk models for space-time fractional diffusion, Chem. Phys. 284 (2002), pp. 521-541.

[7] R. Grande, Space-time fractional nonlinear Schrödinger equation, SIAM J. Math. Anal. 51 (2019), pp. 4172-4212.

[8] X. Guo, M. Xu, Some physical applications of fractional Schrödinger equation, J. Math. Phys. 47 (2006), pp. 082104.

[9] D. Li, H. Liao, W. Sun, J. Wang, J. Zhang, Analysis of L1-Galerkin FEMs for time-fractional nonlinear parabolic problems, Commun. Comput. Phys. 24 (2018), pp. 86-103.

[10] D. Li, J. Wang, J. Zhang, Unconditionally convergent L1-Galerkin FEMs for nonlinear time-fractional Schrödinger equations, SIAM J. Sci. Comput. 39 (2017), pp. A3067-A3088.

Table I. Numerical errors and convergence orders in temporal direction with $h = \frac{1}{1000}$.

	α=0.25		α=0.5		α=0.75	
τ	Error	Order	Error	Order	Error	Order
$\frac{1}{10}$	5.4693E-06	-	1.0361E-05	-	5.2157E-05	-
$\frac{1}{20}$	6.7400E-07	3.02	1.5093E-06	2.78	6.4234E-06	3.02
$\frac{1}{40}$	8.5081E-08	2.99	2.3978E-07	2.65	6.1155E-07	3.39
$\frac{1}{80}$	1.0924E-08	2.96	4.0378E-08	2.57	1.3617E-07	2.17

Table II. Numerical errors and convergence orders in spatial direction with $\tau = \frac{1}{1000}$.

	α=0.25		α=0.5		α=0.75	
h	Error	Order	Error	Order	Error	Order
$\frac{1}{2}$	0.0394	-	0.0393	-	0.0394	-
$\frac{1}{4}$	0.0023	4.11	0.0023	4.11	0.0023	4.11
$\frac{1}{8}$	1.3995E-04	4.03	1.3974E-04	4.03	1.3998E-04	4.03
$\frac{1}{16}$	8.7065E-06	4.01	8.6937E-06	4.01	8.7079E-06	4.01

Table III. Numerical errors and convergence orders in temporal direction with $h = \frac{1}{200}$.

	α=0.25		α=0.5		α=0.75	
τ	Error	Order	Error	Order	Error	Order
$\frac{1}{10}$	0.0172	-	0.0348	-	0.0487	-
$\frac{1}{20}$	0.0073	1.24	0.0147	1.24	0.0206	1.24
$\frac{1}{40}$	0.0027	1.45	0.0054	1.45	0.0076	1.44
$\frac{1}{80}$	4.3858E-04	2.61	8.8291E-04	2.61	0.0013	2.59

Table IV. Numerical errors and convergence orders in spatial direction with $\tau = \frac{1}{100}$.

	α=0.25		α=0.5		α=0.75	
h	Error	Order	Error	Order	Error	Order
$\frac{1}{10}$	9.4534E-07	-	9.9262E-07	-	1.8771E-06	-
$\frac{1}{20}$	5.8993E-08	4.00	6.1948E-08	4.00	1.1716E-07	4.00
$\frac{1}{40}$	3.6841E-09	4.00	3.8623E-09	4.00	7.3092E-09	4.00
$\frac{1}{100}$	9.2968E-11	4.02	8.9752E-11	4.11	1.7536E-10	4.07

[11] X. Liang, A. Khaliq, Y. Xing, Fourth order exponential time differencing method with local discontinuous Galerkin approximation for coupled nonlinear Schrödinger equations, Commun. Comput. Phys. 17 (2015), pp. 510-541.

[12] H. Liao, Z. Sun, Maximum norm error bounds of ADI and compact ADI methods for solving parabolic equations, Numer. Methods Partial Differ. Equ. 26 (2010), pp. 37-60.

[13] J. Liu, T. Wang, T. Zhang, A second-order finite difference scheme for the multi-dimensional nonlinear time-fractional Schrödinger equation, Numer. Algorithms. 92 (2023), pp. 1153-1182.

[14] S. Longhi, Fractional Schrödinger equation in optics, Opt. Lett. 40 (2015), pp. 1117-1120.

[15] R. Metzler, J. Klafter, The random walks guide to anomalous diffusion: A fractional dynamics approach, Phys. Rep. 339 (2000), pp. 1-77.

[16] A. Mohebbi, M. Abbaszadeh, M. Dehghan, The use of a meshless technique based on collocation and radial basis functions for solving the time fractional nonlinear Schrödinger equation arising in quantum mechanics, Eng. Anal. Bound. Elem. 37 (2013), pp. 475-485.

[17] M. Naber, Time fractional Schrödinger equation, J. Math. Phys. 45 (2004), pp. 3339.

[18] N. Petroni, M. Pusterla, Levy processes and Schrödinger equation, Phys. A 388 (2009), pp. 824-836.

[19] B. Stickler, Potential condensed-matter realization of space-fractional quantum mechanics: the one-dimensional Levy crystal, Phys. Rev. E 88 (2013), pp. 012120.

[20] Y. Wang, G, Wang, L. Bu, L. Mei, Two second-order and linear numerical schemes for the muli-dimensional nonlinear time-fractional Schrödinger equation, Numer. Algorithms. 88(1) (2021), pp. 419-451.

[21] J. Zhang, D. Li, X. Antoine, Efficient numerical computation of the time-fractional nonlinear Schrödinger equations in unbounded domain, Commun. Comput. Phys. 25 (2019), pp. 218-243.

[22] Y. Zhang, Z. Sun, Error analysis of a compact ADI scheme for the 2D fractional subdiffusion equation, J. Sci. Comput. 59 (2014), pp. 104-128.

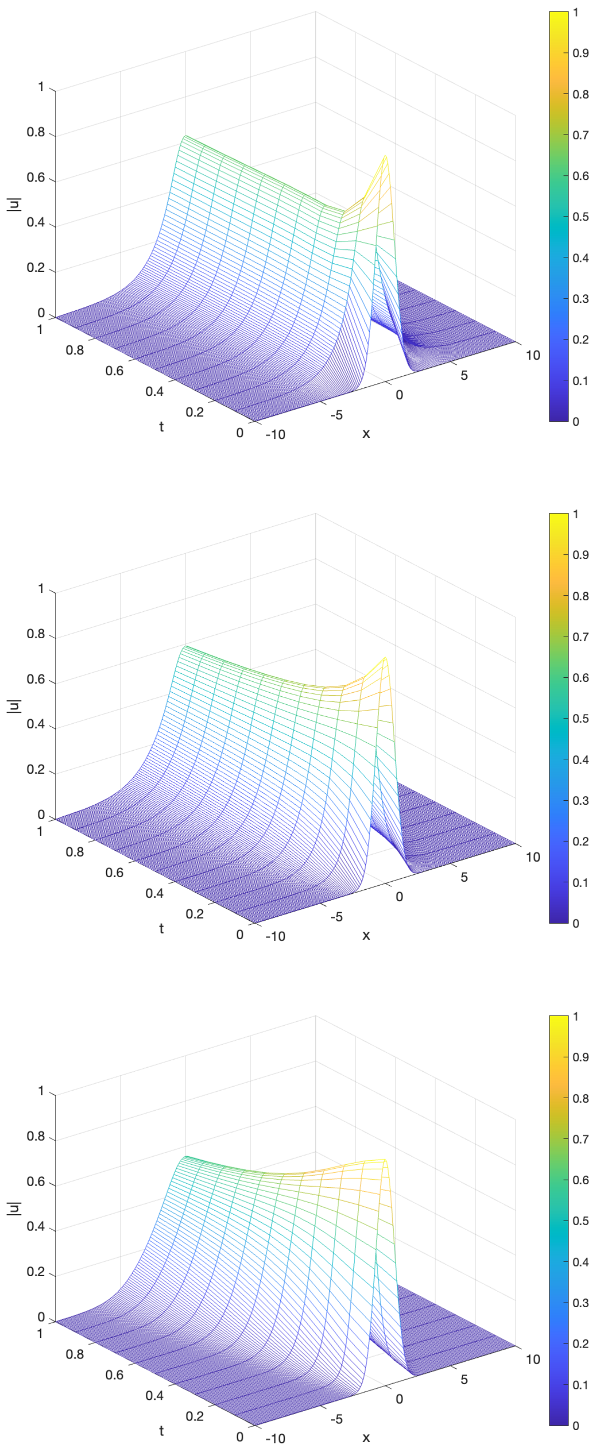

Fig. 1: Evolution of $|u|$ with different α: (a) α=0.1; (b) α=0.5; (c) α=0.9.

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