Some Results on Generalized Inverse of Picture Fuzzy Matrix

V. Kamalakannan, P. Murugadas

***Abstract*—This article presents a definition of the pseudo-similar picture fuzzy matrix (PicFM) along with its associated properties. Detailed exploration is undertaken concerning both pseudo-similar and semi-similar picture fuzzy matrices. Furthermore, we analyze relationships pertaining to pseudo-similarity, the preservation of idempotency, and regularity within picture fuzzy matrices. The paper delves deeper into the characterizations of symmetric PicFMs and the group inverse of PicFMs. Finally, we introduce 2x2 Centrosymmetric and K-Centrosymmetric PicFMs, illustrating their attributes with pertinent examples.**

***Index Terms*—Picture Fuzzy Set(PicFS), Picture Fuzzy Matrices(PicFMs), Pseudo-similar PicFM, Semi-similar, Symmetric.**

I. Introduction

Zadeh proposed fuzzy set theory as a way to mathematically represent imprecise or vague systems of information in the real world [1]. The matrix is essential in many fields of research and engineering. Unfortunately, we cannot be successful with classical matrices because of the diverse forms of uncertainties inherent in real-world issues. The classical matrix problems can be solved using fuzzy matrix [2]. The idea of Fuzzy Matrix was first introduced by Hashimoto in 1983. When dealing with uncertainties, there are some restrictions in fuzzy sets. These are overcome by the concept of Intuitionistic Fuzzy Set(IFS) proposed by Atanassov in 1986 [3] which is a generalization of Fuzzy sets [4], [5], [6], [7], [8], [9], [10]. After that M.Pal, Pradhan studied many outcomes based on Intuitionistic Fuzzy Matrices (IFMs) [11] and S. Sriram and P. Murugadas examined the Moore-Penrose inverse of IFMs [12]. Only two factors including membership and Non membership are scrutinized under IFM.

Two factors are insufficient to denote certain types of data in various sectors of social and medical sciences. In such cases, one additional element is required to fully represent the data.

Thus, Cuong and Kreinovich introduced the concept of PicFS in 2013 [13], [14] as a generalization of IFS. Shovan Dogra and Madhumangal Pal then investigated the PicFMs and its application in 2020 [15]. Regular matrices and generalized inverse play a significant role in many fields of sciences. Some results based on the generalized inverse of IFMs were studied by M.Pal and Pradhan [16]. Also, Khan S.K and A.Pal discussed the generalized inverse of the IFMs [17]. Furthermore, Rajkumar Pradhan and Madhumangal Pal defined the generalized inverse of the Atanassovs IFMs [18]. Murugadas.P investigated implication operations on PicFMs(2021) [19]. Elumalai.N and Arthi.B investigated Properties of k- Centrosymmetric and k-Skew Centrosymmetric Matrices [20]. After that Punithavalli.G investigated Symmetric-Centro Symmetric Fuzzy Matrices [21]. This paper introduces a pseudo-similar PicFM along with certain properties. It explores the notions of pseudo-similarity and semi-similarity in PicFMs. Additionally, we showcase that the pseudo-similarity relation maintains idempotency and regularity for PicFMs. Furthermore, the characteristics of symmetric PicFMs are examined and supported by relevant examples.

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II. Preliminaries

Throughout the entire work $\mathscr{P}_{kl}$ denotes PicFMs of order $k \times l$ and $\mathscr{P}_k$ denotes PicFMs of order $k \times k$. For a comprehensive introduction to PicFS and PicFM see [13,15]

In arithmetic operations, the values of the positive membership, neutral membership and negative membership are only needed. So all elements of PicFM are the members of $< P >$, where $< P >= \{\langle a_\mu, a_\eta, a_\nu\rangle | a_\mu, a_\eta, a_\nu \in [0,1]$ and $0 \leq a_\mu + a_\eta + a_\nu \leq 1\}$

Definition II.1. For $a = \langle a_1', a_1'', a_1'''\rangle, b = \langle b_1', b_1'', b_1'''\rangle \in PicFS$, we define Joint ($\vee$) and meet ($\wedge$) operations as,
1) $\langle a_1', a_1'', a_1'''\rangle \vee \langle b_1', b_1'', b_1'''\rangle = \langle \max(a_1', b_1'),\ \max(a_1'', b_1''),$ $\min(a_1''', b_1''')\rangle = \langle (c_1', c_1'', c_1''')\rangle$ if $c_1' + c_1'' + c_1''' \leq 1$, otherwise find $\max\{c_1', c_1'', c_1'''\}$ and replace $\max\{c_1', c_1'', c_1'''\}$ by 1- (sum of the rest of the components)
2) $\langle a_1', a_1'', a_1'''\rangle \wedge \langle b_1', b_1'', b_1'''\rangle = \langle \min(a_1', b_1'),\ \min(a_1'', b_1''),$ $\max(a_1''', b_1''')\rangle$
3) $a_1^c = \langle a_1''', a_1'', a_1'\rangle$

Definition II.2. Suppose a $n \times n$ PicFM, I has diagonal entries $\langle \epsilon_1, \epsilon_2, 0\rangle$ and non diagonal entries as $\langle 0, 0, \epsilon_3\rangle$ where $\epsilon_1 + \epsilon_2 = 1, \epsilon_2 + \epsilon_3 = 1$ and $A = (\langle p_{nm\mu}, , p_{nm\eta}, p_{nm\nu})$ an $n \times n$ PicFM such that $p_{nm\mu} \in [0, \epsilon_1], p_{nm\eta} \in [0, \epsilon_2]$ and $p_{nm\nu} \in [0, \epsilon_3]$, then $IA = AI = A$. Further
$\langle 0, 0, \epsilon_3\rangle \vee \langle p_{nm\mu}, p_{nm\eta}, p_{nm\nu}\rangle = \langle p_{nm\mu}, p_{nm\eta}, p_{nm\nu}\rangle = \langle p_{nm\mu}, p_{nm\eta}, p_{nm\nu}\rangle \vee \langle 0, 0, \epsilon_3\rangle$ and
$\langle \epsilon_1, \epsilon_2, 0\rangle \wedge \langle p_{nm\mu}, p_{nm\eta}, p_{nm\nu}\rangle = \langle p_{nm\mu}, p_{nm\eta}, p_{nm\nu}\rangle = \langle p_{nm\mu}, p_{nm\eta}, p_{nm\nu}\rangle \wedge \langle \epsilon_1, \epsilon_2, 0\rangle$ for $p_{nm\mu} \in [0, \epsilon_1], p_{nm\eta} \in [0, \epsilon_2]$ and $p_{nm\nu} \in [0, \epsilon_3]$

Definition II.3. A restricted PicFM, P is defined as $P = (\langle p_\mu, p_\eta, p_\nu\rangle)$ Where $p_\mu \in [0, \epsilon_1], p_\eta \in [0, \epsilon_2]$ and $p_\nu \in [0, \epsilon_3]$ such that $\epsilon_1 + \epsilon_2 = 1, \epsilon_2 + \epsilon_3 = 1$.

Definition II.4. A PicFM $P \in \mathscr{P}_k$ is said to be a Picture fuzzy Permutation Matrix if it has exactly one entry $\langle \epsilon_1, \epsilon_2, 0\rangle$ in each row and column, as well as all other entries are $\langle 0, 0, \epsilon_3\rangle$ such that $\epsilon_1 + \epsilon_2 = 1, \epsilon_2 + \epsilon_3 = 1$

Definition II.5. Let $P = (\langle p_{\mu}, p_{\eta}, p_{\nu}\rangle)$ be a PicFM, then multiplication by Picture fuzzy element (Scalar multiplication), $c = \langle c', c'', c'''\rangle$is defined as
$c.P = \langle c' \wedge p_{\mu}, c'' \wedge p_{\eta}, c''' \vee p_{\nu}\rangle$.

Definition II.6. Let $P = (p_{ij}) = (\langle p_{ij\mu}, p_{ij\eta}, p_{ij\nu}\rangle) \in \mathscr{P}_{k\times m}$. Then the element $\langle p_{ij\mu}, p_{ij\eta}, p_{ij\nu}\rangle$ is the ij-th entry of P. Let $P_{i*}(P_{*j})$ denote the i-th row (j-th coloumn) of P. The row space $R(P)$ of P is the subspace of V_m generated by rows P_{i*} of P. The column space, denoted as $C(P)$, of the PicFM P represents the subspace within V_k that is spanned by the columns P_{*j} of P.

Ranking is one of the primary theme for the establishment of PicFM theory. There are two distinct PicFM rank concepts namely row rank and column rank, which are described as follows.

Definition II.7. The row rank, denoted as $\rho_r(P)$, of a PicFM P corresponds to the count of linearly independent rows that form the row space $R(P)$ of P. Similarly, the column rank, denoted as $\rho_c(P)$, of a PicFM P signifies the count of linearly independent columns that compose the column space $C(P)$ of P.

Nevertheless, within a PicFM, the row rank and column rank may not necessarily be equal. However, there are certain PicFMs for which they are indeed equal. This can be illustrated through the following example.

Example II.1. Let $P = \begin{bmatrix} \langle 0.3, 0.2, 0.4\rangle & \langle 0.4, 0.2, 0.3\rangle \\ \langle 0.4, 0.2, 0.1\rangle & \langle 0.5, 0.1, 0.3\rangle \end{bmatrix}$
The row vectors are $R_1 = (\langle 0.3, 0.2, 0.4\rangle\langle 0.4, 0.2, 0.3\rangle)$ and $R_2 = (\langle 0.4, 0.2, 0.1\rangle, \langle 0.5, 0.1, 0.3\rangle)$ are linearly dependent since $R_1 = cR_2$ where $c = \langle 0.4, 0.3, 0.0\rangle \Rightarrow \rho_r(P) = 1$
The column vectors are $C_1 = (\langle 0.3, 0.2, 0.4\rangle, \langle 0.4, 0.2, 0.1\rangle)$ and $C_2 = (\langle 0.4, 0.2, 0.3\rangle, \langle 0.5, 0.1, 0.3\rangle)$ are linearly independent since $C_1 \neq cC_2$ for any c . $\Rightarrow \rho_c(P) = 2$
Now let us consider the PicFM,
Let $Q = \begin{bmatrix} \langle 0.5, 0.3, 0.2\rangle & \langle 0.4, 0.1, 0.3\rangle \\ \langle 0.3, 0.5, 0.2\rangle & \langle 0.1, 0.5, 0.2\rangle \end{bmatrix}$
Here set of row and column vectors of Q are linearly independent. Hence, $\rho_r(Q) = \rho_c(Q) = 2$.

Definition II.8. A PicFM $P \in \mathscr{P}_{k\times m}$ is said to be regular if there exists another PicFM, $U \in \mathscr{P}_{m\times k}$ such that $PUP = P$. In this instance, U is referred to as a generalized inverse (g-inverse) of P and is denoted by P^-.

For a regular PicFM, the row rank and column rank are equal. The regular PicFMs are generalization of the invertible matrices. The generalised inverse of a PicFM comes in various forms.

Definition II.9. For a PicFM $P \in \mathscr{P}_{k\times m}$ and $H \in \mathscr{P}_{m\times k}$ is said to be outer inverse of P if $HPH = H$ and it is denoted by $P\{2\}$.

H is claimed as $\{1, 2\}$ inverse or semi inverse of P if $PHP = P$ and $HPH = H$ and it is denoted by $P\{1, 2\}$.

The PicFM H is claimed as $\{1, 3\}$ inverse or a least square g-inverse of P if $PHP = P$ and $(PH)^T = PH$ and it is denoted by $P\{1, 3\}$.

Again, H is said to be $\{1, 4\}$ inverse or a minimum norm g-inverse of P if $PHP = P$ and $(HP)^T = HP$ and it is denoted by $P\{1, 4\}$.

Definition II.10. For a PicFM $P \in \mathscr{P}_{k\times m}$ and $H \in \mathscr{P}_{m\times k}$ is said to be a Moore-Penrose inverse of P if $PHP = P, HPH = H, (PH)^T = PH$ and $(HP)^T = HP$.

The Moore-Penrose inverse of P is denoted by P^+.

III. Pseudo-similar Picture Fuzzy Matrices

Within this section, we establish the definition of pseudo-similarity for PicFMs and demonstrate that the pseudo-similarity relation preserves the idempotency and regularity of PicFMs P and Q.

Definition III.1. The PicFMs $P \in \mathscr{P}_k$ and $Q \in \mathscr{P}_m$ are said to be pseudo-similar and is denoted by $P \tilde{\approx} Q$ if there exists an idempotent PicFMs $U \in \mathscr{P}_{k\times m}$ and $V \in \mathscr{P}_{m\times k}$ such that $P = UQV; Q = VPU$ and $U = UVU$.

The PicFMs $P = \begin{bmatrix} \langle 0.4, 0.4, 0.0\rangle & \langle 0.4, 0.4, 0.0\rangle \\ \langle 0.0, 0.0, 0.5\rangle & \langle 0.0, 0.0, 0.5\rangle \end{bmatrix}$
and $Q = \begin{bmatrix} \langle 0.4, 0.4, 0.0\rangle & \langle 0.0, 0.0, 0.5\rangle \\ \langle 0.4, 0.4, 0.0\rangle & \langle 0.0, 0.0, 0.5\rangle \end{bmatrix}$
are pseudo-similar with respect to the idempotent PicFM, $U = \begin{bmatrix} \langle 0.5, 0.5, 0.0\rangle & \langle 0.0, 0.0, 0.5\rangle \\ \langle 0.0, 0.0, 0.5\rangle & \langle 0.0, 0.0, 0.5\rangle \end{bmatrix}$.

Definition III.2. The PicFMs $P \in \mathscr{P}_k$ and $Q \in \mathscr{P}_m$ are said to be semi-similar and is denoted by $P \approx Q$ if there exists PicFMs $U \in \mathscr{P}_{k\times m}$ and $V \in \mathscr{P}_{m\times k}$ such that $P = UQV$ and $Q = VPU$.
Following the definition, any pair of pseudo-similar PicFMs are also semi-similar. Hence, the mentioned two PicFMs are semi-similar.

Theorem III.1. Let the PicFMs $P \in \mathscr{P}_k$ and $Q \in \mathscr{P}_m$. Consequently, these are equivalent:
(i) $P \tilde{\approx} Q$
(ii) There exists an idempotent PicFMs $U \in \mathscr{P}_{k\times m}$ and $V \in \mathscr{P}_{m\times k}$ such that $P = UQV; Q = VPU$ and $UV \in \mathscr{P}_k$ is idempotent.
(iii) There exists an idempotent PicFMs $U \in \mathscr{P}_{k\times m}$ and $V \in \mathscr{P}_{m\times k}$ such that $P = UQV; Q = VPU$ and $VU \in \mathscr{P}_m$ is idempotent.

Proof: (i) $\Rightarrow$ (ii)
From the definition of pseudo-similarity, $P \tilde{\approx} Q$ implies,
$P = UQV$
$Q = VPU$ and
$U = UVU$
From the third relation,
$U = UVU$
or $UV = UVUV$
or $UV = (UV)^2$.
$\therefore UV$ is idempotent.
(i) $\Rightarrow$ (iii)
Let $U = UVU$
or $VU = VUVU$ or $VU = (VU)^2$.
$\therefore VU$ is idempotent.
(ii) $\Rightarrow$ (i)
Now, $P = UQV = UVPUV = (UVU)Q(VUV)$.
Similarly, $Q = VPU = VUQVU = (VUV)P(UVU)$.
Putting $U' = UVU$ and $V' = VUV$ we get,
$P = U'QV'$ and $Q = V'PU'$.
Now , $U'V' = (UVU)(VUV) = (UV)(UV)(UV) = UV$, as UV is idempotent and
$(U'V')(U'V') = (UV)(UV) = UV$.

$\therefore U'V'$ is also idempotent.
Let us consider, $U'' = U'V'U'$ and $V'' = V'U'V'$, then
$P = U'QV'$
$= U'V'PU'V'$
$= (U'V'U')Q(V'U'V')$
$= U''QV''$.
Similarly, $Q = V'$ P U'
$= V'U'$ Q $V'U'$
$= V'U'V'PU'V'U'$
$= U'V'U'$
$= U''$ as $(U'V')$ is idempotent.
$\therefore P\tilde{\sim}Q$
(ii) $\Rightarrow$ (i)
The evidence supports the case mentioned above.

Example III.1. Let us consider the PicFMs,
Let $P = \begin{bmatrix}\langle 0.4,0.4,0.0\rangle & \langle 0.4,0.4,0.0\rangle\\ \langle 0.0,0.0,0.5\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$ and
$Q = \begin{bmatrix}\langle 0.4,0.4,0.0\rangle & \langle 0.0,0.0,0.5\rangle\\ \langle 0.4,0.4,0.0\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$ with repsect to
$U = \begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\\ \langle 0.0,0.0,0.5\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$ and
$V = \begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\\ \langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix} \in U\{1\}$,

$UQV = \begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\\ \langle 0.0,0.0,0.5\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$
$\begin{bmatrix}\langle 0.4,0.4,0.0\rangle & \langle 0.0,0.0,0.5\rangle\\ \langle 0.4,0.4,0.0\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$
$\begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\\ \langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix} = P$ and

$VPU = \begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\\ \langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$
$\begin{bmatrix}\langle 0.4,0.4,0.0\rangle & \langle 0.4,0.4,0.0\rangle\\ \langle 0.0,0.0,0.5\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$
$\begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\\ \langle 0.0,0.0,0.5\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix} = Q$

So $P \in \mathscr{P}_2$ and $Q \in \mathscr{P}_2$ are pseudo-similar, that is $P\tilde{\sim}Q$

Again, $UV = \begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\\ \langle 0.0,0.0,0.5\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$
$\begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\\ \langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$
$= \begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\\ \langle 0.0,0.0,0.5\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$.

Now, $(UV)^2 = \begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\\ \langle 0.0,0.0,0.5\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$
$\begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\\ \langle 0.0,0.0,0.5\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$ = UV and

$VU = \begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\\ \langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$
$\begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\\ \langle 0.0,0.0,0.5\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$
$= \begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\\ \langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$.

Then, $(VU)^2 = \begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\\ \langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$
$\begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\\ \langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix} = VU$.

Theorem III.2. Let $P \in \mathscr{P}_k$ and $Q \in \mathscr{P}_m$ be two PicFMs such that $P\tilde{\sim}Q$. Then, P is idempotent $\Leftrightarrow$ Q is idempotent.

Proof: Since $P\tilde{\sim}Q$ then according to definition, there exists idempotent PicFMs
$U \in \mathscr{P}_{k\times m}$ and $V \in \mathscr{P}_{m\times k}$ such that
$Q = VPU; P = UQV$ and $UVU = U$.
That is, $UVP = UVUQV = UQV = P$.
Suppose P is idempotent, then $P^2 = P$.
Now, $Q^2 = (VPU)(VPU) = VP(UVP)U = VP^2U = VPU = Q$.
That is, Q is idempotent.
The converse can be demonstrated using a similar approach.

Theorem III.3. Let $P \in \mathscr{P}_k$ and $Q \in \mathscr{P}_m$ be two PicFMs such that $P\tilde{\sim}Q$. Then, for the idempotent PicFMs $U \in \mathscr{P}_{k\times m}$ and $V \in \mathscr{P}_{m\times k}$, P is regular $\Leftrightarrow$ Q is regular.

Proof: Since $P\tilde{\sim}Q$, then from the definition there exists idempotent PicFMs
$U \in \mathscr{P}_{k\times m}$ and $V \in \mathscr{P}_{m\times k}$ such that $Q = VPU; P = UQV$ and $U = UVU$.
Now V is idempotent, that is $V^2 = V$ and P is regular,then there exists $G \in \mathscr{P}_k$ such that $PGP = P$.
Let us define, $W = VU$, clearly $W \in \mathscr{P}_m$.Then
$QWQ = (VPU)VU(VPU)$
$= VP(UVU)VPU$
$= VP(UVP)U$
$= V(PGP)(UVP)U$ (Since $PGP = P$)
$= VPGPPU$ (Since $UVP = P$)
$= VPGPU$
$= VPU$
$= Q$.
Hence, Q is regular.
The converse can be demonstrated using a similar approach.

Example III.2. Let us consider the PicFMs,
Let $P = \begin{bmatrix}\langle 0.4,0.4,0.0\rangle & \langle 0.4,0.4,0.0\rangle\\ \langle 0.0,0.0,0.5\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$ and
$Q = \begin{bmatrix}\langle 0.4,0.4,0.0\rangle & \langle 0.0,0.0,0.5\rangle\\ \langle 0.4,0.4,0.0\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$ with repsect to
$U = \begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\\ \langle 0.0,0.0,0.5\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$ and
$V = \begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\\ \langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$

P and Q are pseudo-similar, that is $P\tilde{\sim}Q$.
Again, $P^2 = P$ and $Q^2 = Q$
Hence P and Q are idempotent.
Now, P is regular as,
$G = \begin{bmatrix}\langle 0.6,0.4,0.0\rangle & \langle 0.5,0.4,0.1\rangle\\ \langle 0.0,0.0,0.5\rangle & \langle 0.6,0.3,0.1\rangle\end{bmatrix} \in P\{1\}$. For,
$W = VU = \begin{bmatrix}\langle 0.5,0.5,0.0\rangle & \langle 0.5,0.5,0.0\rangle\\ \langle 0.5,0.5,0.0\rangle & \langle 0.0,0.0,0.5\rangle\end{bmatrix}$
and $QWQ = Q$.
$\Rightarrow Q$ is also regular.

IV. Characterization of Symmetric PicFMs

Theorem IV.1. Let $P \in \mathscr{P}_k$, then the following claims are identical to one another:
(i) P is symmetric and $\rho(P) = r_1$.
(ii) $C(P) = C(P^T)$
(iii) $P^T = PM = NP$ for some PicFMs M and N.

(iv) SPS^T is symmetric PicFM of rank r_1 for some Picture fuzzy permutation matrix S.

Proof: (i) $\Rightarrow (ii)$
P is symmetric, $\therefore P^T = P$, So $R(P) = R(P^T)$.
Again, for any symmetric PicFm, $C(A) = R(P^T)$ and $C(P^T) = R(P)$.
$\therefore C(P) = C(P^T)$
(ii) $\Rightarrow$ (iii)
We know that, $R(P) = R(P^T)$ then each row of P^T is a linear combination of the rows of P.
Hence, $P_i^T = \sum_j x_{ij} P_j$ and from which it follows that $P^T = NP$ for some $N \in \mathscr{P}_k$.

Similarly, using $(P) = C(P^T)$ and $(NP)^T = P^T N^T$, we can show that $P^T = PM$ for some $M \in \mathscr{P}_k$.
(i) $\Rightarrow$ (iv)

$(SPS^T)^T = (S^T)^T P^T S^T = SP^T S^T = SPS^T$.
So, SPS^T is symmetric for some PicFM S.
As the rank of P is r_1, so there exists r_1 independent rows of P.
Again SPS^T is the matrix whose columns and rows are rearrangements of matrix P.
So the rank of SPS^T will remain the same, (i.e) $\rho(SPS^T) = r_1$.

Example IV.1. Let us consider the PicFMs,
Let $P = \begin{bmatrix} \langle 0.5, 0.3, 0.1\rangle & \langle 0.6, 0.3, 0.1\rangle \\ \langle 0.6, 0.3, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$. Here $P^T = P$, (i.e) P is symmetric.
Here, $\rho(P) = 2$ and also $C(P) = C(P^T)$.
For, $H = \begin{bmatrix} \langle 0.6, 0.3, 0.1\rangle & \langle 0.4, 0.1, 0.4\rangle \\ \langle 0.6, 0.3, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$
$PH = \begin{bmatrix} \langle 0.5, 0.3, 0.1\rangle & \langle 0.6, 0.3, 0.1\rangle \\ \langle 0.6, 0.3, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$
$\begin{bmatrix} \langle 0.6, 0.3, 0.1\rangle & \langle 0.4, 0.1, 0.4\rangle \\ \langle 0.6, 0.3, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix} = P$
and for $K = \begin{bmatrix} \langle 0.6, 0.3, 0.1\rangle & \langle 0.5, 0.2, 0.2\rangle \\ \langle 0.4, 0.2, 0.3\rangle & \langle 0.6, 0.3, 0.1\rangle \end{bmatrix}$
$KP = \begin{bmatrix} \langle 0.6, 0.3, 0.1\rangle & \langle 0.5, 0.2, 0.2\rangle \\ \langle 0.4, 0.2, 0.3\rangle & \langle 0.6, 0.3, 0.1\rangle \end{bmatrix}$
$\begin{bmatrix} \langle 0.5, 0.3, 0.1\rangle & \langle 0.6, 0.3, 0.1\rangle \\ \langle 0.6, 0.3, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix} = P.$
For $S = \begin{bmatrix} \langle 0.0, 0.0, 0.5\rangle & \langle 0.5, 0.5, 0.0\rangle \\ \langle 0.5, 0.5, 0.0\rangle & \langle 0.0, 0.0, 0.5\rangle \end{bmatrix}$
SPS^T= $\begin{bmatrix} \langle 0.4, 0.3, 0.2\rangle & \langle 0.5, 0.3, 0.1\rangle \\ \langle 0.5, 0.3, 0.1\rangle & \langle 0.5, 0.3, 0.1\rangle \end{bmatrix}$ is also symmetric.
(i.e) $(SPS)^T = (SPS)$ and $\rho(SPS) = 2$.

Theorem IV.2. If $P \in \mathscr{P}_k$ is symmetricand idempotent PicFM, then
(i) P^+ exists and symmetric.
(ii) There is a PicFM that is symmetric as well as idempotent, D such that $PD = DP$ and $R(P) = R(D)$.

Proof: (i) Since P is symmetric so,
$R(P) = R(P^T) = C(P)$.
We have $p_{ij}P_i = p_{ji}P_j$ and $p_{ij} \leq P_i$.So $p_{ij}P_i \leq P_iP_j$.
Consider some k, $p_{ik}p_{jk} > p_{ij}p_{ik}$,
then $p_{ik}p_{jk} > p_{ij}$.Yet $p_{ij} \leq p_{ik}p_{kj} = p_{ik}p_{jk}$.
This contradiction proves $p_{ij}P_i = P_iP_j$.
So for the basis $\{v_1, v_2, ..., v_{r_1}\}$of $R(P)$, for any two basis vectors v_i, v_j there is an element $x \in \mathscr{P}$ satisfying $v_iv_j = xv_i = xv_j$, This indicates that P has a $\{1, 3\}$ inverse.
Similarly, for any two column basis vectors in the column space $C(P)$, $P_{*i}P_{*j} = xP_{*i} = xP_{*j}$ for $x \in \mathscr{P}$, which shows that P has a $\{1, 4\}$ inverse.

Putting these two together, we can conclude that P possesses a Moore-Penrose inverse P^+ and as P is symmetric so $P^+ = P^T$.
(ii) Since $P^+ = P^T$ and $D = P^+P$ is symmetric idempotent PicFM.
Now, $PD = PP^+P = P = PP = P^TP = P^+PP = DP$.
Again $PD = PP^+P = P$ imply, $R(D) \supseteq R((P)$ and $P^+P = D$ imply, $R(D) \subseteq R(P)$.
Hence, $R(P) = R(D)$.

Remark: There might be an idempotent PicFM D that satisfies the relation for PicFMs, $PD = DP$ but $R(P) \neq R(D)$.

Example IV.2. Let us consider the PicFMs,
$P = \begin{bmatrix} \langle 0.4, 0.4, 0.1\rangle & \langle 0.3, 0.4, 0.2\rangle \\ \langle 0.3, 0.4, 0.2\rangle & \langle 0.4, 0.4, 0.2\rangle \end{bmatrix}$ be an idempotent and symmetric PicFM,
(i.e) $P^2 = P$ and $P^T = P$.
Here $P^+ = P^T = P$ itself.
Now for the symmetric idempotent PicFM, $D = \begin{bmatrix} \langle 0.4, 0.3, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$
$PD = \begin{bmatrix} \langle 0.4, 0.4, 0.1\rangle & \langle 0.3, 0.4, 0.2\rangle \\ \langle 0.3, 0.4, 0.2\rangle & \langle 0.4, 0.4, 0.2\rangle \end{bmatrix}$
$\begin{bmatrix} \langle 0.4, 0.3, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$
$PD = \begin{bmatrix} \langle 0.4, 0.3, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$
and $DP = \begin{bmatrix} \langle 0.4, 0.3, 0.1\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$

But $R(P) \neq R(D)$.

Theorem IV.3. Let $P \in \mathscr{P}_k$ be a PicFM and $U \in P\{1, 2\}P$ is symmetric if and only if U is symmetric, such that PU, UP are symmetric.

Proof: Since $U \in P\{1, 2\}$,
we have $PUP = P, UPU = U$.
also PU and UP are symmetric.
So $R(PU) = R((PU)^T)$ and $R(UP) = R((UP)^T)$.
Now,
$R(U) = R(UP)$ (Since $U \in P\{1\}$)
= $R((UP)^T)$ (Since UP is symmetric)
= $R(P^TU^T)$
= $R(U^T)$
$R(P^T) = R(U^TP^T)$ (Since $U^T \in P\{1\}$)
= $R((PU)^T)$
= $R(PU)$ (Since PU is symmetric)
= $R(P)$
and P is symmetric $\Rightarrow R(P) = R(P^T)$
$(i.e)R(U) = R(U^T)$.
Hence, U is symmetric.
U is symmetric, on the other hand $\Rightarrow R(U) = R(U^T)$
$(i.e)R(P) = R(P^T)$.
Hence P is symmetric.

V. Group Inverse

Let $B \in \mathscr{P}_k$ be a PicFM, then $B^\#$ denote the group inverse of B if $BB^\#B = B$, $B^\#BB^\# = B^\#$ and $BB^\# = B^\#B$

Theorem V.1. For an idempotent PicFM $B \in \mathscr{P}_k$, the subsequent claims are identical:
(i) $B^\#$ exists.
(ii) pair of equations $BX = B$ and $YB = B$ is solvable for X and Y.

Proof: (i) $\Rightarrow$ (ii)
B is idempotent $\Rightarrow B^2 = B$.
If $B^\#$ exists, then $B = BB^\#B = B^\#B^2 = B^\#B$ and $B = BB^\#B = B^2B^\# = BB^\#$
Now, $BB^\#B = B \Rightarrow BB^\#$ is a solution to the matrix equation $YB = B$ and similarly, $B^\#B$ is a solution to the matrix equation $BX = B$.
(ii) $\Rightarrow$ (i)
Let M and N be the solutions to the equations $BX = B$ and $YB = B$ respectively.
For $U = NBM$, we see that
$BMB = (NB)MB = N(BM)B = NBB = NB = B$
and
$BNB = BN(BM) = B(NB)M = BBM = BM = B$.
Now,
$BU = B(NBM) = (BNB)M = BM = NBM = NB$
$= N(BMB) = (NBM)B = UB$,
$UBU = (NBM)B(NBM) = N(BMB)NBM$
$= N(BNB)M = NBM = U$.
and
$BUB = B(NBM)B = (BNB)MB = BMB = B$.
Hence, $U = B^\#$ is the group inverse of B.

Example V.1. Let us consider the PicFM,
$B = \begin{bmatrix} \langle 0.4, 0.4, 0.0\rangle & \langle 0.5, 0.4, 0.1\rangle \\ \langle 0.0, 0.0, 0.5\rangle & \langle 0.6, 0.3, 0.1\rangle \end{bmatrix}$ such that $B^2 = B$.
Let $M = \begin{bmatrix} \langle 0.6, 0.4, 0.0\rangle & \langle 0.5, 0.4, 0.1\rangle \\ \langle 0.0, 0.0, 0.5\rangle & \langle 0.6, 0.3, 0.1\rangle \end{bmatrix}$ be a solution of the relation $BX = B$ and
$N = \begin{bmatrix} \langle 0.5, 0.4, 0.0\rangle & \langle 0.5, 0.3, 0.2\rangle \\ \langle 0.0, 0.0, 0.5\rangle & \langle 0.6, 0.4, 0.0\rangle \end{bmatrix}$ be a solution of the relation YB=B.
Then
$U = NBM = \begin{bmatrix} \langle 0.4, 0.4, 0.0\rangle & \langle 0.5, 0.4, 0.1\rangle \\ \langle 0.0, 0.0, 0.5\rangle & \langle 0.6, 0.3, 0.1\rangle \end{bmatrix} = B$.
Again, $BU = UB, BUB = B$ and $UBU = U$ all holds.
Thus, B itself a group inverse.

Theorem V.2. If the PicFM $B \in \mathscr{P}_k$ is symmetric and B^+ exists $\Leftrightarrow B^\#$ exists and $B^\# = B^+$.

Proof: B is symmetric implies, $R(B) = R(B^T)$. B^+ exists means, $BB^+B = B$ and $B^+BB^+ = B^+$.
Now, $BB^+ = BB^T = B^TB = B^+B$ (as $B^+ = B^T$ and $B^T = B$).
Thus $B^+ \in B\{1, 2\}$ and $BB^+ = B^+B \Rightarrow B^T = B^+ = B^\#$.
Conversely, if $B^\#$, then $B^\#BB^\# = B^\#$ or $B^\#B^\#B = B^\#$ or $(B^\#)^2B = B^\#$
Therefore, $R(B^\#) = R((B^\#)^2B) \subseteq R(B)$ ....(1)
Also, $B = BB^\#B = BBB^\# = B^2B^\#$.
$\Rightarrow R(B) = R(B^2B^\#) \subseteq R(B^\#)$ ....(2)
Again $B^\# = B^T \Rightarrow R(B^\#) = R(B^T)$ ....(3)
From (1), (2) and (3), $R(B) = R(B^\#) = R(B^T)$,
Which shows that $B = B^T$ that is, B is symmetric.
Now, $B^\# \in B\{1, 2\}$ and $B^\# = B^T \Rightarrow B^+ = B^\#$.
Thus B^+ exists.

Theorem V.3. For any PicFM B if $(PBP^T)^\#$ exists $\Rightarrow B^\#$ exists for any PicFPM P.
$\Rightarrow (P^T(PBP^T)P)^\#$ exists.
$\Rightarrow ((P^TP)B(P^TP))^\#$ exists.
$\Rightarrow B^\#$ exists.

Example V.2. Let us consider the PicFM,
$B = \begin{bmatrix} \langle 0.4, 0.4, 0.1\rangle & \langle 0.3, 0.4, 0.2\rangle \\ \langle 0.3, 0.4, 0.2\rangle & \langle 0.4, 0.4, 0.2\rangle \end{bmatrix}$ and a Picture Fuzzy Permutation Matrix $P = \begin{bmatrix} \langle 0.0, 0.0, 0.5\rangle & \langle 0.5, 0.5, 0.0\rangle \\ \langle 0.5, 0.5, 0.0\rangle & \langle 0.0, 0.0, 0.5\rangle \end{bmatrix}$,
then $PBP^T = \begin{bmatrix} \langle 0.4, 0.4, 0.2\rangle & \langle 0.3, 0.4, 0.2\rangle \\ \langle 0.3, 0.4, 0.2\rangle & \langle 0.4, 0.4, 0.1\rangle \end{bmatrix}$.
PBP^T itself a commutating semi-inverse of it, that is, $(PBP^T)^\#$ exists and $(PBP^T)^\# = (PBP^T)$. Then $B^\#$ exists and it is verified that $B^\# = B$ also.

Theorem V.4. For any PicFMs $B, D \in \mathscr{P}_k^\#$ if $BD = DB$, then $(BD)^\#$ exists and $(BD)^\#$=$B^\#D^\#$=$D^\#B^\#$.

Proof: We first prove $B, D, B^\#, D^\#$ are mutully commute.
If $B^\#$ exists then,
$B^2B^\# = B = B^\#B^2; B(B^\#)^2 = B^\# = (B^\#)^2B$.
Similarly, if $B^\#$ exists then,
$D^2D^\# = D = D^\#D^2; D(D^\#)^2 = D^\# = (D^\#)^2D$.
$BD = DB \Rightarrow B^2D = BBD = BDB = DBB = DB^2$ and $BD^2 = BDD = DBD = DDB = D^2B$.
Now, $B^\#BD = B^\#DB = B^\#D(B^2B^\#)$
$= B^\#(B^2D)B^\# = BDB^\#.B^\#BD = BDB^\#$.
$\Rightarrow (B^\#)^2BD = B^\#(B^\#BD) = B^\#(BDB^\#)$
$= BD(B^\#)^2)$.
So $B^\#D = (B^\#)^2BD = BD(B^\#)^2 = DB(B^\#)^2$
$= DB^\#$. Thus $B^\#$ and D commutes.
Similarly, we can prove $D^\#B = BD^\#$ and
$B^\#D^\# = D^\#B^\#$.
Now let $B^\#D^\#$ be a group inverse of BD, then
$BD(B^\#D^\#)BD = B(DB^\#)D^\#BD = BB^\#DD^\#BD$
$= BB^\#DBB^\#D = (BB^\#B)(DD^\#D) = BD$
and $B^\#D^\#(BD)B^\#D^\# = B^\#B(D^\#DD^\#)B^\#$
$= BB^\#D^\#B^\# = B^\#D^\#$.
Finally, $BD(B^\#D^\#) = BB^\#DD^\# = B^\#BD^\#D$
$= (B^\#D^\#BD$. Hence, $B^\#D^\#$ is the group inverse of BD.
Since $B^\#$ and $D^\#$ commutes, $D^\#B^\#$ is also the group inverse of BD.
Thus, $(BD)^\# = B^\#D^\# = D^\#B^\#$.

Example V.3. Let us consider the PicFMs,

$B = \begin{bmatrix} \langle 0.4, 0.4, 0.1\rangle & \langle 0.3, 0.4, 0.2\rangle \\ \langle 0.3, 0.4, 0.2\rangle & \langle 0.4, 0.4, 0.2\rangle \end{bmatrix}$ and
$D = \begin{bmatrix} \langle 0.5, 0.5, 0.0\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.4, 0.3, 0.2\rangle & \langle 0.5, 0.4, 0.1\rangle \end{bmatrix}$

Here both are symmetric and idempotent, so it can be verifed that $B^\# = B$ and $D^\# = D$.
Now,
$BD = \begin{bmatrix} \langle 0.4, 0.4, 0.1\rangle & \langle 0.4, 0.4, 0.2\rangle \\ \langle 0.4, 0.4, 0.2\rangle & \langle 0.4, 0.4, 0.2\rangle \end{bmatrix} = DB$.

Again, BD is also symmetric and idempotent, so $(BD)^{\#} = BD$.
Thus, $(BD)^{\#} = BD = B^{\#}D^{\#}$.

VI. Centrosymmetric and K-Centrosymmetric of 2 x 2 PicFM's

Definition VI.1. A Square PicFM, $B \in \mathscr{P}_n$ which is symmetric about the centre of its array of elements is called Centrosymmetric, (i.e) $B = [b_{ij}]$ Centrosymmetric if $b_{ij} = b_{n-i+1,n-j+1}$. Equivalently, if $K \in \mathscr{P}_n$ be a Picture fuzzy Permutation Matrix, then B is Centrosymmetric PicFM iff $BK = KB$.

Example VI.1. Let us consider the PicFMs,

$$B = \begin{bmatrix} \langle 0.3, 0.4, 0.2\rangle & \langle 0.1, 0.3, 0.4\rangle \\ \langle 0.1, 0.3, 0.4\rangle & \langle 0.3, 0.4, 0.2\rangle \end{bmatrix} \text{ and}$$
$$K = \begin{bmatrix} \langle 0.0, 0.0, 0.5\rangle & \langle 0.5, 0.5, 0.0\rangle \\ \langle 0.5, 0.5, 0.0\rangle & \langle 0.0, 0.0, 0.5\rangle \end{bmatrix}, \text{ then}$$
$$BK = KB = \begin{bmatrix} \langle 0.1, 0.3, 0.4\rangle & \langle 0.3, 0.4, 0.2\rangle \\ \langle 0.3, 0.4, 0.2\rangle & \langle 0.1, 0.3, 0.4\rangle \end{bmatrix}$$

Definition VI.2. A Centrosymmetric PicFM, $B \in \mathscr{P}_n$ is called K-Centrosymmertric PicFM if $B = KB^TK$.

Lemma VI.1. If $B, C \in \mathscr{P}_n$ are Centrosymmetric PicFMs, then $B + C, BC, cB$ (Scalar multiplication) are also Centrosymmetric PicFMs.

Proof: By Using definition VI.1

Example VI.2. Let us consider the Centrosymmetric PicFMs,

$$B = \begin{bmatrix} \langle 0.3, 0.4, 0.2\rangle & \langle 0.1, 0.3, 0.4\rangle \\ \langle 0.1, 0.3, 0.4\rangle & \langle 0.3, 0.4, 0.2\rangle \end{bmatrix} \text{ and}$$
$$C = \begin{bmatrix} \langle 0.4, 0.3, 0.2\rangle & \langle 0.3, 0.2, 0.4\rangle \\ \langle 0.3, 0.2, 0.4\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}, \text{ then}$$
$$B + C = B \vee C = \begin{bmatrix} \langle 0.4, 0.4, 0.2\rangle & \langle 0.3, 0.3, 0.4\rangle \\ \langle 0.3, 0.3, 0.4\rangle & \langle 0.4, 0.4, 0.2\rangle \end{bmatrix}$$

Therefore, $B + C$ is Centrosymmetric PicFM.

$$BC = B \wedge C = \begin{bmatrix} \langle 0.3, 0.3, 0.2\rangle & \langle 0.3, 0.3, 0.4\rangle \\ \langle 0.3, 0.3, 0.4\rangle & \langle 0.3, 0.3, 0.2\rangle \end{bmatrix}$$

Therefore, BC is Centrosymmetric PicFM.
Let us consider, $c = \langle 0.3, 0.1, 0.1\rangle$ then,

$$cB = \begin{bmatrix} \langle 0.3, 0.1, 0.2\rangle & \langle 0.3, 0.1, 0.4\rangle \\ \langle 0.3, 0.1, 0.4\rangle & \langle 0.3, 0.1, 0.2\rangle \end{bmatrix}$$

and $cC = \begin{bmatrix} \langle 0.3, 0.2, 0.2\rangle & \langle 0.1, 0.2, 0.4\rangle \\ \langle 0.1, 0.2, 0.4\rangle & \langle 0.3, 0.2, 0.2\rangle \end{bmatrix}$

Hence, cB and cC are Centrosymmetric PicFMs.

Theorem VI.1. If $B \in \mathscr{P}_2$ is K-Centrosymmetric PicFM then $B^T = KBK$.

Proof: Let $B \in \mathscr{P}_2$ is K-Centrosymmetric PicFM

$KBK = KB^TK$
$= B^TKK$
$= B^TK^2$
$= B^T$.

Example VI.3. Let us consider the PicFMs,

$$B = \begin{bmatrix} \langle 0.3, 0.4, 0.2\rangle & \langle 0.1, 0.3, 0.4\rangle \\ \langle 0.1, 0.3, 0.4\rangle & \langle 0.3, 0.4, 0.2\rangle \end{bmatrix} \text{ and}$$
$$K = \begin{bmatrix} \langle 0.0, 0.0, 0.5\rangle & \langle 0.5, 0.5, 0.0\rangle \\ \langle 0.5, 0.5, 0.0\rangle & \langle 0.0, 0.0, 0.5\rangle \end{bmatrix}, \text{ then}$$
$$KBK = \begin{bmatrix} \langle 0.3, 0.4, 0.2\rangle & \langle 0.1, 0.3, 0.4\rangle \\ \langle 0.1, 0.3, 0.4\rangle & \langle 0.3, 0.4, 0.2\rangle \end{bmatrix} = B^T.$$

Theorem VI.2. If B and C are K-Centrosymmetric PicFMs then $B + C$ is also K-Centrosymmetric PicFM.

Proof: If both B and C are K-Centrosymmetric PicFMs then, $B = KB^TK$ and $C = KC^TK$.
To Prove: $B + C$ is also K-Centrosymmetric.
(i.e) To prove: $B + C = K(B + C)^TK$.
Now, $K(B + C)^TK = K(B^T + C^T)K$
$= KB^TK + KC^TK$
$= B + C$

Example VI.4. Let us consider the PicFMs,

$$B = \begin{bmatrix} \langle 0.3, 0.4, 0.2\rangle & \langle 0.1, 0.3, 0.4\rangle \\ \langle 0.1, 0.3, 0.4\rangle & \langle 0.3, 0.4, 0.2\rangle \end{bmatrix},$$
$$C = \begin{bmatrix} \langle 0.4, 0.3, 0.2\rangle & \langle 0.3, 0.2, 0.4\rangle \\ \langle 0.3, 0.2, 0.4\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix} \text{ and}$$
$$K = \begin{bmatrix} \langle 0.0, 0.0, 0.5\rangle & \langle 0.5, 0.5, 0.0\rangle \\ \langle 0.5, 0.5, 0.0\rangle & \langle 0.0, 0.0, 0.5\rangle \end{bmatrix}, \text{ then}$$
$$B + C = \begin{bmatrix} \langle 0.4, 0.4, 0.2\rangle & \langle 0.3, 0.3, 0.4\rangle \\ \langle 0.3, 0.3, 0.4\rangle & \langle 0.4, 0.4, 0.2\rangle \end{bmatrix}$$
$$K(B + C)^T = \begin{bmatrix} \langle 0.3, 0.3, 0.4\rangle & \langle 0.4, 0.4, 0.2\rangle \\ \langle 0.4, 0.4, 0.2\rangle & \langle 0.3, 0.3, 0.4\rangle \end{bmatrix}$$
$$K(B + C)^TK = \begin{bmatrix} \langle 0.4, 0.4, 0.2\rangle & \langle 0.3, 0.3, 0.4\rangle \\ \langle 0.3, 0.3, 0.4\rangle & \langle 0.4, 0.4, 0.2\rangle \end{bmatrix} = B + C.$$

Theorem VI.3. If B and C are K-Centrosymmetric PicFMs then BC is also K-Centrosymmetric PicFM.

Proof: If both B and C are K-Centrosymmetric PicFMs then, $B = KB^TK$ and $C = KC^TK$.
To Prove: BC is also K-Centrosymmetric.
(i.e) To prove: $BC = K(BC)^TK$.
Now, $K(BC)^TK = KC^TB^TK$
$= K[(KCK)(KBK)]K$
$= K^2CK^2BK^2$
$= CB$ Where $K^2 = I$
$= BC$

Example VI.5. Let us consider the PicFMs,

$$B = \begin{bmatrix} \langle 0.3, 0.4, 0.2\rangle & \langle 0.1, 0.3, 0.4\rangle \\ \langle 0.1, 0.3, 0.4\rangle & \langle 0.3, 0.4, 0.2\rangle \end{bmatrix} \text{ and}$$
$$K = \begin{bmatrix} \langle 0.0, 0.0, 0.5\rangle & \langle 0.5, 0.5, 0.0\rangle \\ \langle 0.5, 0.5, 0.0\rangle & \langle 0.0, 0.0, 0.5\rangle \end{bmatrix}, \text{ then}$$
$$BC = \begin{bmatrix} \langle 0.3, 0.3, 0.2\rangle & \langle 0.3, 0.3, 0.4\rangle \\ \langle 0.3, 0.3, 0.4\rangle & \langle 0.3, 0.3, 0.2\rangle \end{bmatrix}$$
$$K(BC)^T = \begin{bmatrix} \langle 0.3, 0.3, 0.4\rangle & \langle 0.3, 0.3, 0.2\rangle \\ \langle 0.3, 0.3, 0.2\rangle & \langle 0.3, 0.3, 0.4\rangle \end{bmatrix}$$
$$K(BC)^TK = \begin{bmatrix} \langle 0.3, 0.3, 0.2\rangle & \langle 0.3, 0.3, 0.4\rangle \\ \langle 0.3, 0.3, 0.4\rangle & \langle 0.3, 0.3, 0.2\rangle \end{bmatrix} = BC.$$

Theorem VI.4. If $B \in \mathscr{P}_2$ be a K-Centrosymmetric PicFM then BB^T is also K-Centrosymmetric PicFM.

Proof: Let B be a K-Centrosymmetric PicFM then, $B = KB^TK$.
To Prove: BB^T is also K-Centrosymmetric.
(i.e) To prove: $BB^T = K(BB^T)^TK$.
Now, $K(BB^T)^TK = K(B^T)^TB^TK$
$= K[BB^T]K$
$= BB^TK.K$
$= BB^TK^2$
$= BB^T$ Where $K^2 = I$

Example VI.6. Let us consider the PicFMs,
$B = \begin{bmatrix} \langle 0.3, 0.4, 0.2\rangle & \langle 0.1, 0.3, 0.4\rangle \\ \langle 0.1, 0.3, 0.4\rangle & \langle 0.3, 0.4, 0.2\rangle \end{bmatrix}$ and
$K = \begin{bmatrix} \langle 0.0, 0.0, 0.5\rangle & \langle 0.5, 0.5, 0.0\rangle \\ \langle 0.5, 0.5, 0.0\rangle & \langle 0.0, 0.0, 0.5\rangle \end{bmatrix}$, then

$$BB^T = \begin{bmatrix} \langle 0.3, 0.4, 0.2\rangle & \langle 0.1, 0.3, 0.4\rangle \\ \langle 0.1, 0.3, 0.4\rangle & \langle 0.3, 0.4, 0.2\rangle \end{bmatrix}$$

$$(BB^T)^T K = \begin{bmatrix} \langle 0.1, 0.3, 0.4\rangle & \langle 0.3, 0.4, 0.2\rangle \\ \langle 0.3, 0.4, 0.2\rangle & \langle 0.1, 0.3, 0.4\rangle \end{bmatrix}$$

$$K(BB^T)^T K = \begin{bmatrix} \langle 0.3, 0.4, 0.2\rangle & \langle 0.1, 0.3, 0.4\rangle \\ \langle 0.1, 0.3, 0.4\rangle & \langle 0.3, 0.4, 0.2\rangle \end{bmatrix} = BB^T.$$

Theorem VI.5. If $B \in \mathscr{P}_2$ be a K-Centrosymmetric PicFM then $B + B^T$ is also K-Centrosymmetric PicFM.

Proof: Let B be a K-Centrosymmetric PicFM then, $B = KB^T K$.
To Prove: $B + B^T$ is also K-Centrosymmetric.
(i.e) To prove: $B + B^T = K(B + B^T)^T K$.
Now, $K(B + B^T)^T K = K(B^T + (B^T)^T)K$
$= K[B^T + B]K$
$= [B^T + B]K.K$
$= [B^T + B]K^2$
$= [B^T + B]$ Where $K^2 = I$
$= [B + B^T]$

Example VI.7. Let us consider the PicFMs,
$B = \begin{bmatrix} \langle 0.3, 0.4, 0.2\rangle & \langle 0.1, 0.3, 0.4\rangle \\ \langle 0.1, 0.3, 0.4\rangle & \langle 0.3, 0.4, 0.2\rangle \end{bmatrix}$ and
$K = \begin{bmatrix} \langle 0.0, 0.0, 0.5\rangle & \langle 0.5, 0.5, 0.0\rangle \\ \langle 0.5, 0.5, 0.0\rangle & \langle 0.0, 0.0, 0.5\rangle \end{bmatrix}$, then

$$B + B^T = \begin{bmatrix} \langle 0.3, 0.4, 0.2\rangle & \langle 0.1, 0.3, 0.4\rangle \\ \langle 0.1, 0.3, 0.4\rangle & \langle 0.3, 0.4, 0.2\rangle \end{bmatrix}$$

$$K(B + B^T)^T = \begin{bmatrix} \langle 0.1, 0.3, 0.4\rangle & \langle 0.3, 0.4, 0.2\rangle \\ \langle 0.3, 0.4, 0.2\rangle & \langle 0.1, 0.3, 0.4\rangle \end{bmatrix}$$

$$K(B + B^T)^T K = \begin{bmatrix} \langle 0.3, 0.4, 0.2\rangle & \langle 0.1, 0.3, 0.4\rangle \\ \langle 0.1, 0.3, 0.4\rangle & \langle 0.3, 0.4, 0.2\rangle \end{bmatrix} = B + B^T.$$

VII. Conclusion

This paper has elucidated a sequence of theorems grounded in the characteristics of symmetric, pseudo-similar and semi-similar PicFMs. The scrutiny of the pseudo-similar relationship has proven especially insightful in its ability to maintain the idempotency and regularity within PicFMs. The characterization of symmetric PicFMs and their group inverse has also been exhaustively discussed. Finally, the introduction of 2x2 Centrosymmetric and K-Centrosymmetric PicFMs, along with their properties and relevant examples, has been incorporated.

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