A New Technique for Solving a Class of Nonlinear Initial-Boundary Value Problems

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Abstract— **In this paper, a new technique for solving a class of Nonlinear Initial-Boundary Value Problems (NIBVPs) is introduced. The main advantage of this technique is that it utilizes both initial and boundary conditions in the Adomian series solution and consequently, an accurate approximate solution is obtained in both temporal and spatial directions. The sufficient condition that guarantees a unique solution is obtained. Convergence of the series solution is proved and the truncated error is estimated. Some numerical examples are discussed to illustrate the effectiveness of the proposed technique.**

Index Terms—: **adomian series, convergence of the series, new technique, NIBVPs**

I. Introduction

NUMEROUS physics and engineering problems are modeled using nonlinear partial differential equations. Numerous techniques, including the homotopy analysis method [1]-[3], the homotopy perturbation method [3]-[5], the variational iteration method [6], [7], the differential transform method [8], the Adomian Decomposition Method (ADM) [9]-[13] and other methods were developed over time to approximate the solutions to these nonlinear equations. ADM introduces a series solution to a large amount of linear and nonlinear differential equations that quickly converge to exact solutions. Applications of ADM to NIBVPs were studied by many authors [14]-[16], in this paper, we are going to discuss the following NIBVPs,

$$(\mathcal{L}_t + \mathcal{L}_x)\, y(x,t) = N(x,t,y(x,t),\ldots), \quad (1)$$

subjected to suitable initial and boundary conditions. Where, $\mathcal{L}_t$ is the highest partial derivative with respect to t, $\mathcal{L}_x$ is the highest partial derivative with respect to x and N is the general nonlinear term that contains the remainder derivatives and free terms. During the solution using ADM, the unknown function $y(x,t)$ is expanded in an infinite series $y = \sum_{i=0}^{\infty} y_i$, in which y_i is obtained using the different recursive formulas [9], [11], and the nonlinear term N is expanded in an infinite series of polynomials called Adomian polynomials,

$$N = \sum_{n=0}^{\infty} A_n\,. \quad (2)$$

These polynomials can be generated either using the traditional formula:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{n} \lambda^i y_i\right)\right]_{\lambda=0},\ n = 0,1,2,\ldots, \quad (3)$$

or using the accelerated formula [16], [17]:

$$A_n = N(s_n) - \sum_{i=0}^{n-1} A_i\,, \quad (4)$$

where, s_n is the partial sum,

$$s_n = y_0 + y_1 + y_2 + y_3 + \cdots + y_n. \quad (5)$$

Using ADM, equation (1) can be solved either using $\mathcal{L}^{-1}{}_t$ or $\mathcal{L}^{-1}{}_x$ where, $\mathcal{L}^{-1}{}_t$ is the inverse operator of $\mathcal{L}_t$ and $\mathcal{L}^{-1}{}_x$ is the inverse operator of $\mathcal{L}_x$ and these solutions are called partial solutions. Authors in [18] showed that the solutions using $\mathcal{L}^{-1}{}_t$ and $\mathcal{L}^{-1}{}_x$ are equal while authors in [19] showed that the solution using $\mathcal{L}^{-1}{}_x$ is more accurate and efficient. In this paper, a new technique for solving a class of NIBVPs is introduced in section II. The main advantage of this technique is that it utilizes both initial and boundary conditions in the Adomian series solution and consequently, an accurate approximate solution is obtained in both temporal and spatial directions. In section III, the sufficient condition that guarantees a unique solution is obtained, the convergence of the series solution is proved and the truncated error is estimated. In sections IV, and V, some numerical examples are discussed to illustrate the effectiveness of the proposed technique.

II. The Proposed Technique

Equation (1) can be rewritten in the following form:

$$\mathcal{L}_t y = N_1, \quad (6)$$

or

$$\mathcal{L}_x y = N_2, \quad (7)$$

where, N_1 contains the partial derivative with respect to x and N_2 contains the partial derivative with respect to t. Applying $\mathcal{L}^{-1}{}_t$ on both sides of equation (6) and $\mathcal{L}^{-1}{}_x$ on both sides of equation (7) yields,

$$y = \phi_1(x,t) + \mathcal{L}^{-1}{}_t\, N_1\,, \quad (8)$$

and

$$y = \phi_2(x,t) + \mathcal{L}^{-1}{}_x\, N_2. \quad (9)$$

Where, $\phi_1(x,t)$ is the solution of $\mathcal{L}_t y = 0$ satisfying the given Initial Condition (I.C) and $\phi_2(x,t)$ is the solution of $\mathcal{L}_x y = 0$ satisfying the given Boundary Condition (B.C). Adding equation (8), and equation (9) yields,

$$y = \frac{1}{2}\left(\phi_1(x,t) + \phi_2(x,t)\right) + \frac{1}{2}\left(\mathcal{L}^{-1}{}_t\, N_1 + \mathcal{L}^{-1}{}_x\, N_2\right). \quad (10)$$

In equation (8), let $y = \sum_{i=0}^{\infty} \hat{y}_i$ and $N_1 = \sum_{i=0}^{\infty} A_i$ while in equation (9), let $y = \sum_{i=0}^{\infty} \tilde{y}_i$ and $N_2 = \sum_{i=0}^{\infty} B_i$ where A_i and B_i are the Adomian polynomials of N_1 and N_2 respectively.

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Application of ADM on both sides of equation (8) yields the recursive relation:

$$\hat{y}_0 = \phi_1(x,t), \quad (11)$$

$$\hat{y}_{i+1} = \mathcal{L}^{-1}{}_t\, A_i, \quad i \geq 0, \quad (12)$$

while application of ADM on both sides of equation (9) yields the recursive relation;

$$\tilde{y}_0 = \phi_2(x,t), \quad (13)$$

$$\tilde{y}_{i+1} = \mathcal{L}^{-1}{}_x\, B_i, \quad i \geq 0. \quad (14)$$

Finally, the final recursive relation will be,

$$y_0 = \frac{\hat{y}_0 + \tilde{y}_0}{2}, \quad (15)$$

$$y_i = \frac{\hat{y}_i + \tilde{y}_i}{2}, \quad i = 1,2,3,..., \quad (16)$$

and the final solution is,

$$y = \sum_{i=0}^{\infty} y_i\,. \quad (17)$$

III. CONVERGENCE ANALYSIS

Let, $\mathcal{L}^{-1}{}_t = \int_a^t ...$ n − fold $... \int_a^t (.)\, d\tau\, d\tau$ and $\mathcal{L}^{-1}{}_x = \int_b^x ...$ m − fold $... \int_b^x (.)$ ds ds, where, $a \in [0,T]$ and $b \in [0,X]$.

Assume N_1 and N_2 are Lipschitzian with respect to the unknown function y so that;

$|N_1(u) - N_1(v)| \leq L_1|u - v|$ and $|N_2(u) - N_2(v)| \leq L_2|u - v|$. Let, $E = (C[\Omega], \|.\|)$ denotes the Banach space of all continuous functions such that $\Omega = [0,T] \times [0,X]$ with the norm $\|y(x,t)\| = \max_{\Omega} |y(x,t)|$.

A. Uniqueness theorem

Theorem 1. Equation (1) has a unique solution if $0 < \alpha < 1$, where, $\alpha = \frac{1}{2}\left(\frac{L_1 T^n}{n!} + \frac{L_2 X^m}{m!}\right)$. Proof:

Define the mapping H: E→E such that:

$H\, y = \frac{1}{2}\left(\phi_1(x,t) + \phi_2(x,t)\right) + \frac{1}{2}(\mathcal{L}^{-1}{}_t\, N_1 + \mathcal{L}^{-1}{}_x\, N_2)$.

Let, y and $y^* \in$ E then,

$\|y - y^*\| = \frac{1}{2} \max_{\Omega} |\mathcal{L}^{-1}{}_t\, N_1(y) + \mathcal{L}^{-1}{}_x\, N_2(y) - \mathcal{L}^{-1}{}_t\, N_1(y^*) - \mathcal{L}^{-1}{}_x\, N_2(y^*)|$,

$\|y - y^*\| \leq \frac{1}{2} \int_a^t ...$ n − fold $... \int_a^t \max_{\Omega} |N_1(y) - N_1(y^*)|\, d\tau\, d\tau + \frac{1}{2} \int_b^x ...$ m − fold $... \int_b^x \max_{\Omega} |N_1(y) - N_1(y^*)|\, ds\, ds$.

$\leq \|y - y^*\| \left(\frac{L_1}{2} \int_a^t ...\ n - fold\ ... \int_a^t (.)\, d\tau\, d\tau + \frac{L_2}{2} \int_b^x ...\ m - fold\ ... \int_b^x (.)\, ds\, ds\right)$,

$\leq \|y - y^*\| \left(\frac{L_1}{2}\left(\frac{(t-a)^n}{n!}\right) + \frac{L_2}{2}\left(\frac{(x-b)^m}{m!}\right)\right)$,

$\leq \frac{1}{2} \|y - y^*\| \left(\frac{L_1 T^n}{n!} + \frac{L_2 X^m}{m!}\right)$.

$\leq \alpha \|y - y^*\|$.

B. Convergence theorem

Theorem 2. The series solution (17) of equation (1) converges if $0 < \alpha < 1$, and $|y_1(x,t)| < \infty$ on Ω.Proof**:**

Let, S_n and S_m be arbitrary partial sums with n > m. We are going to prove that $\{S_n\}$ is a Cauchy sequence in E. From Theorem 1 we have,

$\|S_{m+1} - S_m\| \leq \alpha \|S_m - S_{m-1}\| \leq \alpha^2 \|S_{m-1} - S_{m-2}\| ... \leq \alpha^m \|S_1 - S_0\|$.

Using the triangle inequality, we have,

$\|S_n - S_m\| \leq \|S_{m+1} - S_m\| + \|S_{m+2} - S_{m+1}\| + \cdots \|S_n - S_{n-1}\| \leq [\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-1}]\|S_1 - S_0\| \leq \alpha^m[1 + \alpha + \alpha^2 + \cdots + \alpha^{n-m-1}]\|S_1 - S_0\| \leq \alpha^m\left(\frac{1-\alpha^{n-m}}{1-\alpha}\right)\|y_1(x,t)\|$.

Since $0 < \alpha < 1$ so, $1 - \alpha^{n-m} \leq 1$ then,

$$\|S_n - S_m\| \leq \left(\frac{\alpha^m}{1-\alpha}\right)\|y_1(x,t)\|, \quad (18)$$

but $\max_{\Omega} |y(x,t)| < \infty$ then $\|S_n - S_m\| \to 0$ as $m \to \infty$, from which we conclude that $\{S_n\}$ is a Cauchy sequence in E and consequently series $\sum_{i=0}^{\infty} y_i(x,t)$ converges to the unique solution $y(x,t)$.

C. Error estimate

Theorem 3. The truncated error of the series solution (17) to equation (1) is estimated to be:

$$\max_{\Omega} \left|y(x,t) - \sum_{i=0}^{m} y_i(x,t)\right| \leq \left(\frac{\alpha^m}{1-\alpha}\right) \max_{\Omega} |y_1(x,t)|.$$

Proof: From Theorem 2 inequality (18) we have,

$$\|S_n - S_m\| \leq \left(\frac{\alpha^m}{1-\alpha}\right)\|y_1(x,t)\|.$$

As $n \to \infty$ then $S_n \to y(x,t)$, so we have

$$\|y(x,t) - S_m\| \leq \left(\frac{\alpha^m}{1-\alpha}\right)\|y_1(x,t)\|.$$

Finally, the truncated error in the region Ω is given by the relation:

$$\max_{\Omega} \left|y(x,t) - \sum_{i=0}^{m} y_i(x,t)\right| \leq \left(\frac{\alpha^m}{1-\alpha}\right) \max_{\Omega} |y_1(x,t)|. \quad (19)$$

IV. NUMERICAL EXAMPLES

In this section, the proposed technique is applied to solve some NIBVPs with different kinds of nonlinearity terms to show the comparison between the ADM in x, t-directions solutions, and the proposed technique to the exact solution.

Example 1

Consider the Non-Linear Partial Differential Equation (NLPDE),

$$y_{tt} - y_{xx} + y^2 = 12\, t^2\, x^2(x^2 - t^2) + t^8\, x^8. \quad (20)$$

Subject to the conditions:

$$y(x,0) = y_t(x,0) = 0,$$

$$y(1,t) = t^4 \ and\ y_x(1,t) = 4\, t^4. \quad (21)$$

The exact solution for this example is $y(x,t) = x^4\, t^4$**.** We applied ADM in the x-direction, the t-direction, and the proposed ADM technique (the average solution) to solve equation (20). Obtained results are explained through Tables I, II, and Figs. 1-8.

Numerical implementation of ADM

By solving the above example using ADM in the t-direction, we obtain the following solution,

$$\mathrm{Y}(x,t) = -\frac{2}{5}t^6x^2 + t^4x^4 + \frac{t^{10}x^8}{90} - \frac{1}{2}t^4x^4 - \frac{t^{22}x^{16}}{14968800} + \frac{1}{10}t^{10}\left(-\frac{2x^2}{45} + \frac{4x^3}{75} - \frac{79x^8}{450}\right) + \frac{1}{8}t^8\left(-\frac{2}{35} + \frac{2x^4}{7} - \frac{12x^5}{35} + \frac{2x^{10}}{35}\right) + \frac{1}{6}t^6\left(-\frac{1}{5} + \frac{12x}{25} + \frac{264x^2}{125} - \frac{2x^6}{25} + \frac{12x^7}{125} - \frac{x^{12}}{125}\right) + \frac{1}{16}t^{16}\left(-\frac{x^2}{750} + \frac{x^3}{675} - \frac{x^{12}}{1125}\right) + \frac{1}{14}t^{14}\left(\frac{3x^4}{650} - \frac{x^5}{117} + \frac{x^8}{1170} - \frac{x^9}{975} + \frac{x^{14}}{975}\right) + \frac{1}{12}t^{12}\left(-\frac{1}{110} + \frac{52x}{2475} - \frac{2x^2}{165} + \frac{311x^6}{4950} + \frac{x^7}{495} - \frac{x^{10}}{990} + \frac{x^{11}}{825} - \frac{x^{16}}{4950}\right) + \frac{1}{20}t^{20}\left(\frac{x^8}{34200} - \frac{x^9}{30780} + \right.$$

$$\frac{x^{18}}{307800}\Big) + \frac{1}{18}t^{18}\Big(-\frac{1}{6800} + \frac{x}{3060} - \frac{x^2}{5508} + \frac{x^{10}}{10200} + \frac{x^{11}}{27540} - \frac{x^{20}}{550800}\Big) + \cdots. \tag{22}$$

The average solution is given by,

$$\mathrm{Z}(x,t) = \frac{1}{2}\Big(-\frac{2}{5}t^6x^2 + 2t^4x^4 + \frac{t^{10}x^8}{90} - \frac{2}{5}t^2(5 - 6x + x^6) - \frac{1}{90}t^8(9 - 10x + x^{10})\Big) + \frac{1}{1885619736000}(-62985t^{22}x^{16} + 377123947200t^2(5 - 6x + x^6) + 1001t^{20}(1615 - 1710x + 1377x^8 - 1530x^9 + 248x^{18}) + 29930472t^8(4878 - 5950x + x^4(2175 - 2106x + 778x^6)) + 5755860t^{14}(247 - 350x + x^4(327 - 282x + 2x^4(5 - 6x + 12x^6))) - 364t^{18}(27702 + x(-55440 + 26125x + x^9(1881 - 19760x + 682x^{10}))) - 6734356200(-1 + x)^4(35 + x(20 + x(10 + x(4 + x)))) + 1813968t^6(-231825 + x(721980 + x(-544698 + 269500x + x^4(-41580 + 38016x - 3493x^6)))) + 75582t^{12}(54675 + x(-49616 - 12600x + x^5(40362 + 35100x - 7x^4(370 - 396x + 129x^6)))) + 34535160t^4(-39325 + x(46950 + x(13650 + x(-10920 - 24024x + x^5(195 - 182x + 6x^6))))) + 969t^{16}(-14805 + x(-534690 + x(1135134 + x(-810810 + 250250x + x^9(-66339 - 3850x + 65x^{10}))))) + 342t^{10}(-120329825 + x(77588550 + x(125585460 + x(-91483392 + 30630600x + x^5(-67102893 + 70x(-12155 + x^3(3315 - 3366x + 286x^6)))))))) + \cdots. \tag{23}$$

The solution using ADM in the x-direction is given by,

$$U(x,t) = t^4x^4 - \frac{2}{5}t^2(5 - 6x + x^6) - \frac{1}{90}t^8(9 - 10x + x^{10}) + \Big(\frac{6}{7} - \frac{12t^2}{5} + \frac{313t^4}{182} + \frac{72t^6}{55} - \frac{17t^8}{90} + \frac{4421t^{10}}{157080} - \frac{58t^{12}}{10125} - \frac{t^{14}}{468} - \frac{457t^{16}}{831600} + \frac{t^{18}}{307800} - \frac{t^{20}}{550800}\Big)(-1 + x) + \Big(-2 + t^4 - \frac{14t^6}{5} + \frac{t^{10}}{10} + \frac{t^{16}}{400}\Big)\Big(-\frac{1}{2} + \frac{x^2}{2}\Big) + \Big(\frac{6}{5} - \frac{6t^4}{5} + \frac{14t^6}{9} - \frac{26t^{10}}{225} - \frac{t^{16}}{360}\Big)\Big(-\frac{1}{3} + \frac{x^3}{3}\Big) + \Big(-\frac{38t^4}{25} + \frac{2t^8}{15} + \frac{2t^{10}}{45} + \frac{t^{14}}{150} + \frac{t^{16}}{972}\Big)\Big(-\frac{1}{4} + \frac{x^4}{4}\Big) + \Big(-\frac{3t^8}{25} - \frac{t^{14}}{180}\Big)\Big(-\frac{1}{5} + \frac{x^5}{5}\Big) + \Big(\frac{12t^2}{5} - \frac{2t^6}{5} - \frac{3t^{12}}{250}\Big)\Big(-\frac{1}{6} + \frac{x^6}{6}\Big) + \Big(\frac{2t^6}{5} + \frac{t^{12}}{54}\Big)\Big(-\frac{1}{7} + \frac{x^7}{7}\Big) + \Big(-\frac{2}{35} + \frac{2t^4}{35} - \frac{19t^{10}}{350}\Big)\Big(-\frac{1}{8} + \frac{x^8}{8}\Big) + \Big(-\frac{3t^4}{50} - \frac{t^{10}}{360}\Big)\Big(-\frac{1}{9} + \frac{x^9}{9}\Big) + \Big(\frac{79t^8}{450} - \frac{t^{12}}{810} - \frac{t^{18}}{16200}\Big)\Big(-\frac{1}{10} + \frac{x^{10}}{10}\Big) + \Big(\frac{t^{12}}{750} + \frac{t^{18}}{16200}\Big)\Big(-\frac{1}{11} + \frac{x^{11}}{11}\Big) + \Big(-\frac{32t^6}{495} + \frac{t^{10}}{990} - \frac{t^{16}}{6600}\Big)\Big(-\frac{1}{12} + \frac{x^{12}}{12}\Big) + \Big(-\frac{t^{10}}{900} - \frac{t^{16}}{19440}\Big)\Big(-\frac{1}{13} + \frac{x^{13}}{13}\Big) + \Big(\frac{t^4}{325} + \frac{t^{14}}{975}\Big)\Big(-\frac{1}{14} + \frac{x^{14}}{14}\Big) - \frac{t^{12}\Big(-\frac{1}{16} + \frac{x^{16}}{16}\Big)}{1125} + \Big(\frac{t^{10}}{7650} + \frac{t^{20}}{550800}\Big)\Big(-\frac{1}{18} + \frac{x^{18}}{18}\Big) - \frac{t^{18}\Big(-\frac{1}{20} + \frac{x^{20}}{20}\Big)}{307800} +, \frac{t^{16}\Big(-\frac{1}{22} + \frac{x^{22}}{22}\Big)}{680400} + \cdots. \tag{24}$$

Example 2

Consider the NLPDE,

$$y_x - y_t + y^2 = e^t\, x\,(2 - x + e^t\, x^3). \tag{25}$$

Subject to the conditions:

$$y(x,0) = x^2, \;\; y(1,t) = e^t. \tag{26}$$

The exact solution for this example is $y(x,t) = x^2\, e^t$. Applying ADM in the x-direction, the t-direction, and the proposed ADM technique (the average solution) to the above problem. Obtained results are explained through Tables III, IV, and Figs. 9-16.

Numerical implementation of ADM:

By solving the above example using ADM in the t-direction, we obtain the following solution,

$$\mathrm{Y}(x,t) = x\big(2 + e^t(-2 + x - x^3\mathrm{Sinh}[t])\big) + \frac{1}{14400}(14324 + 480x(-64 + 57x) + 9e^{4t}(2 + (5 - 2x)x^4)^2 - 80e^{3t}(-1 + x)(1 + (-5 + x)x)\big(-2 + x^4(-5 + 2x)\big) + 900t\big(16 + x^2(16 + 16x + 8x^3 + x^6)\big) - 1200e^t(12 + x(-28 + x(30 + x(-24 + x(3 + x(6 + (-6 + x)x)))))) - x^3(6960 + x(9020 + x(-10712 + x(5480 + x(160 + x(-835 + 36x(5 + x))))))) + 20e^{2t}(10 + x(-192 + x(480 + x(-1100 + x(642 + x(-300 + 82x + 9x^3(-5 + 2x)))))))) + \cdots. \tag{27}$$

The average solution is given by,

$$\mathrm{Z}(x,t) = \frac{1}{2}\Big(\frac{1}{15}e^t\big(5 - 5(-3 + x)x^2 + 3e^t(-1 + x^5)\big) + x\big(2 + e^t(-2 + x - x^3\mathrm{Sinh}(t))\big)\Big) + \frac{1}{2}(\frac{1}{14400}(14324 + 480x(-64 + 57x) + 9e^{4t}(2 + (5 - 2x)x^4)^2 - 80e^{3t}(-1 + x)(1 + (-5 + x)x)(-2 + x^4(-5 + 2x)) + 900t(16 + x^2(16 + 16x + 8x^3 + x^6)) - 1200e^t(12 + x(-28 + x(30 + x(-24 + x(3 + x(6 + (-6 + x)x)))))) - x^3(6960 + x(9020 + x(-10712 + x(5480 + x(160 + x(-835 + 36x(5 + x))))))) + 20e^{2t}(10 + x(-192 + x(480 + x(-1100 + x(642 + x(-300 + 82x + 9x^3(-5 + 2x)))))))) + \frac{1}{1663200}(-11550(-61 + 48x^3 + 12x^6 + x^9) - 42e^{4t}(-773 + 396x + x^5(396 - 132x + x^4(275 + 18x(-11 + 2x)))) + 495e^t(-1 + x)^2(-103 + x(354 + x(-1429 + x(148 + 5x(-19 + x(-34 + 7x)))))) + 385e^{3t}(-1 + x)^2(53 + x(250 + x(15 + x(68 + x(85 + x(174 + x(-121 + 16x))))))) - 132e^{2t}(-1257 + x(2870 + x(-3360 + x(5600 + x(-6475 + x(4494 + x(-2170 + 410x + 7x^3(-25 + 9x)))))))))) + \cdots. \tag{28}$$

The solution using ADM in the x-direction is given by,

$$\mathrm{U}(x,t) = \frac{1}{15}e^t\big(5 - 5(-3 + x)x^2 + 3e^t(-1 + x^5)\big) + \frac{1}{1663200}(-11550(-61 + 48x^3 + 12x^6 + x^9) - 42e^{4t}(-773 + 396x + x^5(396 - 132x + x^4(275 + 18x(-11 + 2x)))) + 495e^t(-1 + x)^2(-103 + x(354 + x(-1429 + x(148 + 5x(-19 + x(-34 + 7x)))))) + 385e^{3t}(-1 + x)^2(53 + x(250 + x(15 + x(68 + x(85 +$$

$x(174 + x(-121 + 16x)))))) - 132e^{2t}(-1257 + x(2870 + x(-3360 + x(5600 + x(-6475 + x(4494 + x(-2170 + 410x + 7x^3(-25 + 9x)))))))) + \cdots.$ (29)

Example 3

Consider the NLPDE,

$$y_t - y_{xx} + y\, y_x = \frac{1}{2}\pi\, x \cos\left(\frac{\pi\, t}{2}\right) + x \sin^2\left(\frac{\pi\, t}{2}\right). \quad (30)$$

Subject to the conditions:

$$y(x,0) = 0 \text{ and } y(1,t) = y_x(1,t) = \sin\left(\frac{\pi\, t}{2}\right). \quad (31)$$

The exact solution for this example is $y(x,t) = x\ \sin\left(\frac{\pi\, t}{2}\right)$. Applying ADM in the x-direction, the t-direction, and the proposed ADM technique (the average solution) to the above problem. Obtained results are explained through Tables V, VI, and Figs. 17-24.

Numerical implementation of ADM:

By solving the problem using ADM in the t-direction solution, we obtain the following solution,

$$\mathrm{Y}(x,t) = \frac{1}{2}x\left(t + 2\mathrm{Sin}\left(\frac{\pi t}{2}\right) - \frac{\mathrm{Sin}(\pi t)}{\pi}\right) + \frac{1}{2304\pi^3}(2\pi(-3\pi^4 t(-1+x)^3(1+x)(2+x) + 128\pi(1-3x+2x^3) + 6(1-3x-6tx+2x^3) - 3\pi^2(-2(8+t(3+2t)) + (48+t(297+4t(3+2t)))x + 6tx^2 - 4(8+t(3+2t))x^3 + 3tx^5)) + 48\pi^2(-7+3(7+16t)x - 14x^3)\mathrm{Cos}\left(\frac{\pi t}{2}\right) - 96\pi\big(3tx + \pi^2(1-3x+2x^3)\big)\mathrm{Cos}(\pi t) + 80\pi^2(1-3x+2x^3)\mathrm{Cos}[\tfrac{3\pi t}{2}] - 12\pi(1-3x+2x^3)\mathrm{Cos}(2\pi t) + 24\pi(-144x + \pi^2(2+4t(1-3x+2x^3) - x(51+x(2-4x+x^3))))\mathrm{Sin}\left(\frac{\pi t}{2}\right) - 6(-48x + \pi^4(-1+x)^3(1+x)(2+x) + 4\pi^2(2+t(2-6x+4x^3) - x(75+x(2-4x+x^3))))\mathrm{Sin}(\pi t) + 8\pi(-48x + \pi^2(-1+x)^3(1+x)(2+x))\mathrm{Sin}\left(\frac{3\pi t}{2}\right) - 3(-12x + \pi^2(-1+x)^3(1+x)(2+x))\mathrm{Sin}(2\pi t)) + \cdots. \quad (32)$$

The average solution is given by,

$$\mathrm{Z}(x,t) = \frac{1}{2}\left(\frac{1}{12}(-1+x)^2(2+x)\left(-1 - \pi\mathrm{Cos}\left(\frac{\pi t}{2}\right) + \mathrm{Cos}(\pi t)\right) + x\mathrm{Sin}\left(\frac{\pi t}{2}\right) + \frac{1}{2}x\left(t + 2\mathrm{Sin}\left(\frac{\pi t}{2}\right) - \frac{\mathrm{Sin}(\pi t)}{\pi}\right)\right) + \frac{1}{161280\pi^3}(\pi(1260(1 - 2(1+t)x + x^3) + 8960\pi(1-3x+2x^3) + 3\pi^2(7807 + 476t + 840t^2 - 70(190 + t(297 + 8t(3+t)))x - 70(3+10t)x^2 + 35(163 + 8t(4+3t))x^3 + 35x^4 - 14(3+19t)x^5 + 5x^7) + \pi^4(-1+x)^3(-210t(1+x)(2+x) + (-1+x)^2(33+5x(5+x)))) - 2\pi^2(840(11 - 3(9+16t)x + 16x^3) - \pi^2(-1+x)^2(6687 - 84t(-1+x)(1+x(3+x)) + x(3434 + x(-29 + x(-27 + 5x(2+x))))))\mathrm{Cos}(\tfrac{\pi t}{2}) + \pi(-10080tx + \pi^4(-1+x)^5(33+5x(5+x)) + 4\pi^2(-5847 + 42t(-1+x)^3(1+x(3+x)) + x(9940 + x(210 - x(4305 + x(35 - 42x + 5x^3))))))\mathrm{Cos}(\pi t) + 2\pi^2(280(17 - 33x + 16x^3) - \pi^2(-1+x)^5(33+5x(5+x)))\mathrm{Cos}(\tfrac{3\pi t}{2}) + \pi(-1260(1-2x+x^3) + \pi^2(-1+x)^5(33+5x(5+x)))\mathrm{Cos}(2\pi t) + 84\pi(-1440x + \pi^4(-1+x)^4(4+x) + \pi^2(31 + 40t(5-9x+4x^3) - x(510 + x(75 - 95x + 21x^3))))\mathrm{Sin}(\tfrac{\pi t}{2}) - 42(-240x + \pi^4(-1+x)^3(2+17x(3+x)) + 4\pi^2(11 + 30t(1-2x+x^3) - x(375 + x(15 - 25x + 6x^3))))\mathrm{Sin}(\pi t) + 140\pi\left(-96x + \pi^2(-1+x)^3\big(7 + 5x(3+x)\big)\right)\mathrm{Sin}\left(\frac{3\pi t}{2}\right) - 21\big(-60x + \pi^2(-1+x)^3(2+3x)(7+3x)\big)\mathrm{Sin}(2\pi t)) + \cdots. \quad (33)$$

The solution using ADM in the x-direction is given by,

$$\mathrm{U}(x,t) = \frac{1}{12}(-1+x)^2(2+x)\left(-1 - \pi\mathrm{Cos}\left(\frac{\pi t}{2}\right) + \mathrm{Cos}(\pi t)\right) + x\mathrm{Sin}\left(\frac{\pi t}{2}\right) + \frac{1}{80640\pi^2}(-1+x)^2(420(2+x) + \pi^4(-1+x)^3\big(33+5x(5+x)\big) + 3\pi^2(6687 + 280t^2(2+x) - 56t(-1+x)\big(1+x(3+x)\big) + x(3434 + x(-29 + x(-27 + 5x(2+x))))) - 2\pi(1680(2+x) + \pi^2(-6687 + 84t(-1+x)(1+x(3+x)) + x(-3434 + x(29 + x(27 - 5x(2+x))))))\mathrm{Cos}(\tfrac{\pi t}{2}) + \pi^2(\pi^2(-1+x)^3(33 + 5x(5+x)) + 4(-5007 + 42t(-1+x)(1 + x(3+x)) + x(-2594 + x(29 + x(27 - 5x(2+x))))))\mathrm{Cos}(\pi t) + 2\pi(1680(2+x) - \pi^2(-1+x)^3(33+5x(5+x)))\mathrm{Cos}(\tfrac{3\pi t}{2}) - 840\mathrm{Cos}(2\pi t) + (-420x + \pi^2(-1+x)^3(33 + 5x(5+x)))\mathrm{Cos}(2\pi t) + 84\pi^2(11 + 22x + \pi^2(-1+x)^2(4+x) + (2+x)(80t - 11x^2))\mathrm{Sin}(\tfrac{\pi t}{2}) - 168\pi(1 + 2x - (2+x)(-10t + x^2) + \pi^2(-1+x)(-2+3x(3+x)))\mathrm{Sin}(\pi t) + 420\pi^2(-1+x)(1+x(3+x))\mathrm{Sin}(\tfrac{3\pi t}{2}) - 84\pi(-1+x)(1+x(3+x))\mathrm{Sin}(2\pi t)) + \cdots. \quad (34)$$

V. DISCUSSION

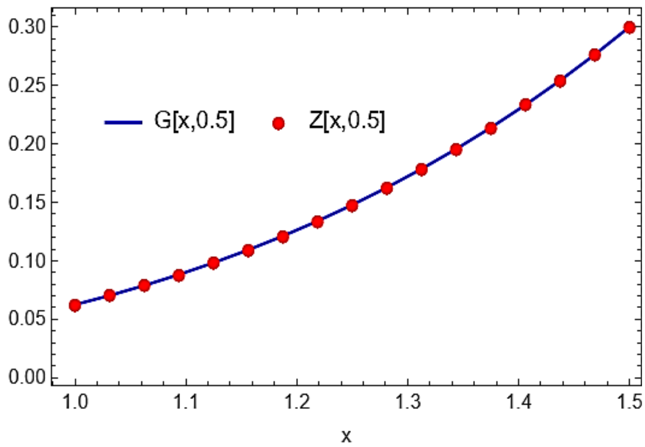

Fig. 1. Graphical interpretation of the exact solution $G(x,t)$ and the average solution $Z(x,t)$ at $t = 0.5$, for example 1.

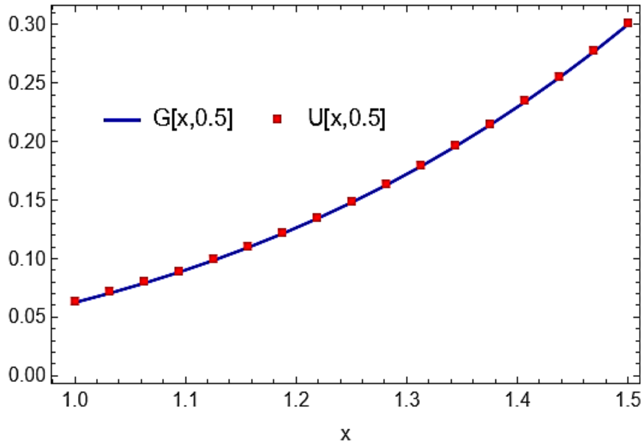

Fig. 2. Graphical interpretation of the exact solution $G(x,t)$ and the x-direction solution $U(x,t)$ at $t = 0.5$, for example 1.

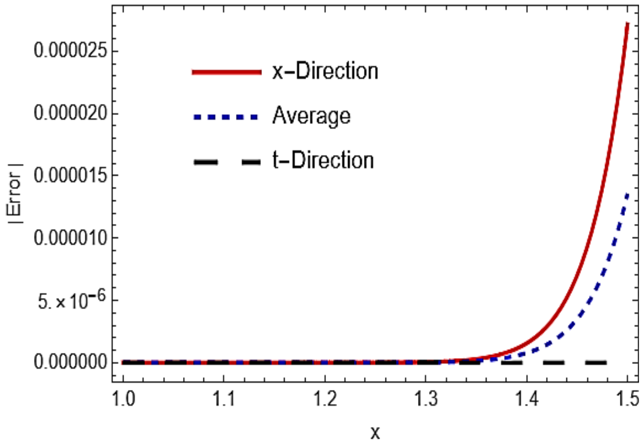

Fig. 5. The AE for the t-direction solution $Y(x,t)$, the average solution $Z(x,t)$, and the x-direction solution $U(x,t)$ at $t = 0.5$, for example 1**.**

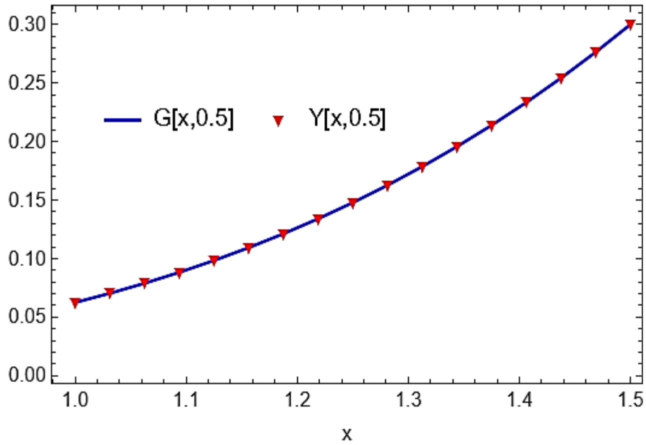

Fig. 3. Graphical interpretation of the exact solution $G(x,t)$ and the t-direction solution $Y(x,t)$ at $t = 0.5$, for example 1.

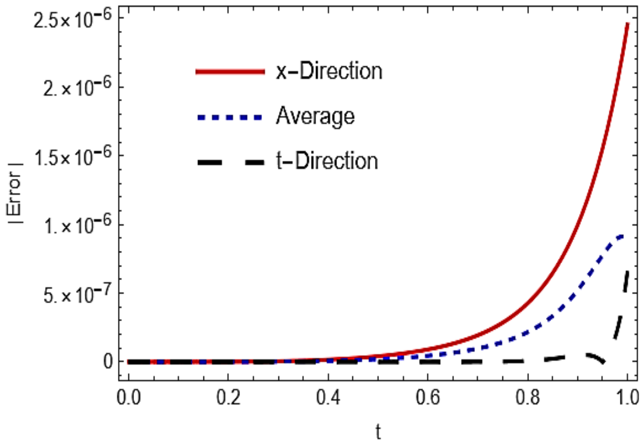

Fig. 6. The AE for the t-direction solution $Y(x,t)$, average solution $Z(x,t)$, and the x-direction solution $U(x,t)$ at $x = 1.3$, for example 1.

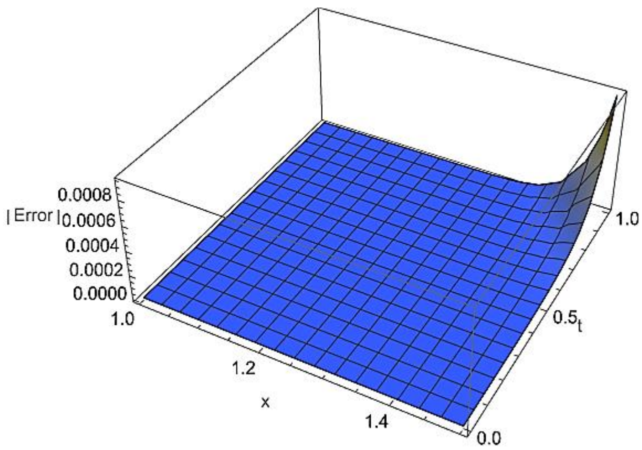

Fig. 4. 3D graph of the Absolute Error (AE) for the average solution, for example 1.

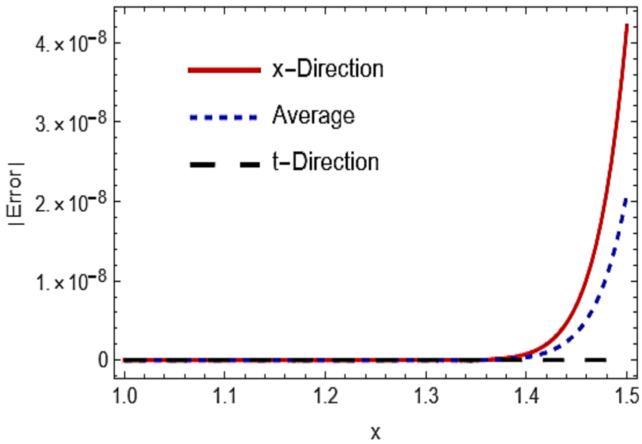

Fig. 7. The AE for the t-direction solution $Y(x,t)$, the average solution $Z(x,t)$, and the x-direction solution $U(x,t)$ at $t = 0.1$, for example 1.

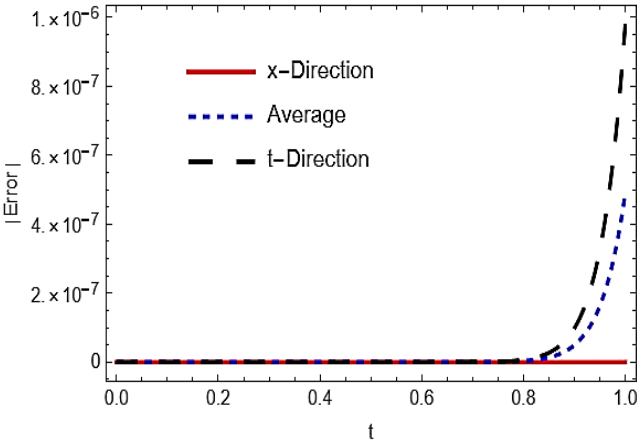

Fig. 8. The AE for the t-direction solution $Y(x,t)$, the average solution $Z(x,t)$, and the x-direction solution $U(x,t)$ at $x = 1.1$, for example 1.

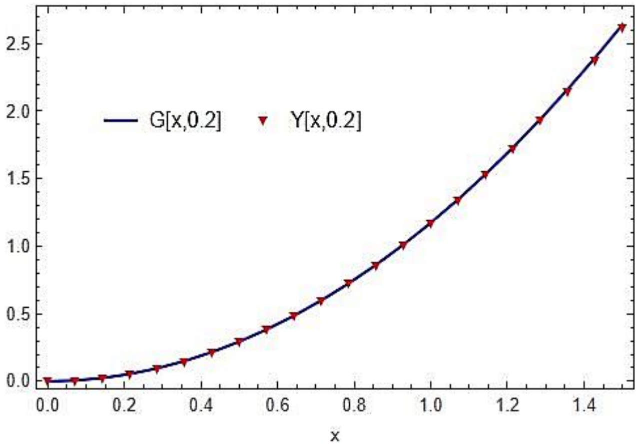

Fig. 11. Graphical representation of the exact solution $G(x,t)$ and the t-direction solution $Y(x,t)$ at $t = 0.2$, for example 2.

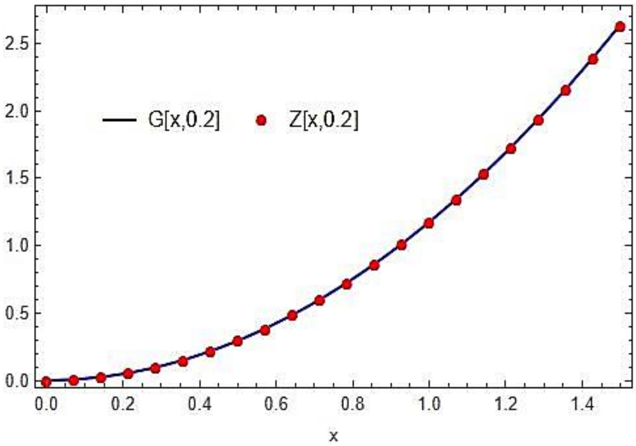

Fig. 9. Graphical representation of the exact solution $G(x,t)$ and the average solution $Z(x,t)$ at $t = 0.2$, for example 2.

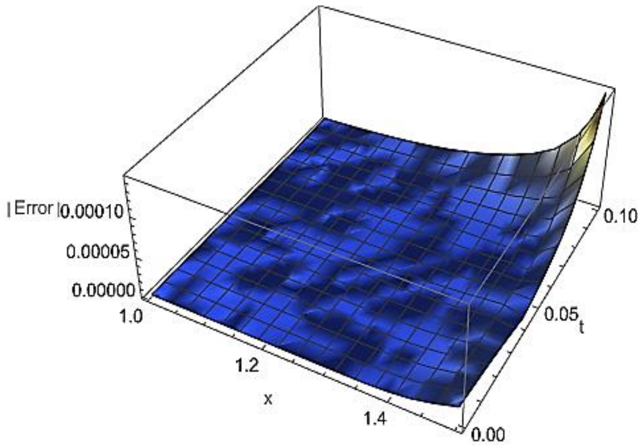

Fig. 12. 3D graph of the AE for the average solution, for example 2.

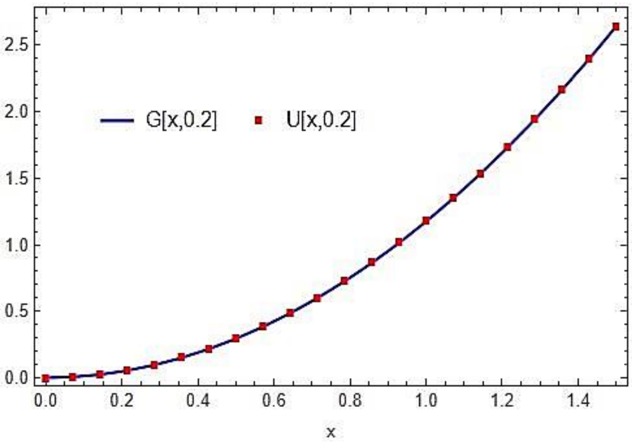

Fig. 10. Graphical representation of the exact solution $G(x,t)$ and the x-direction solution $U(x,t)$ at $t = 0.2$, for example 2.

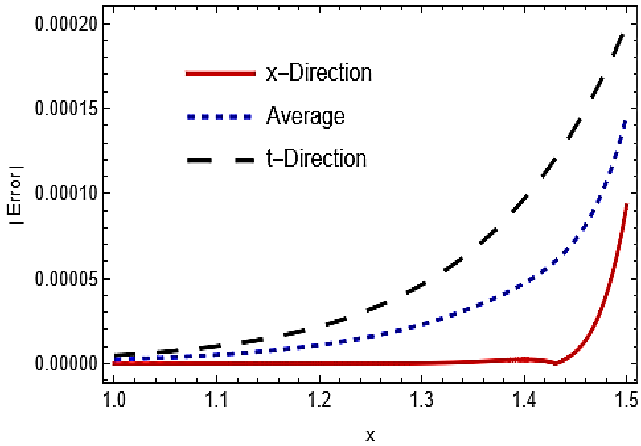

Fig. 13. The AE for the t-direction solution $Y(x,t)$, the average solution $Z(x,t)$, and the x-direction solution $U(x,t)$ at $t = 0.1$, for example 2.

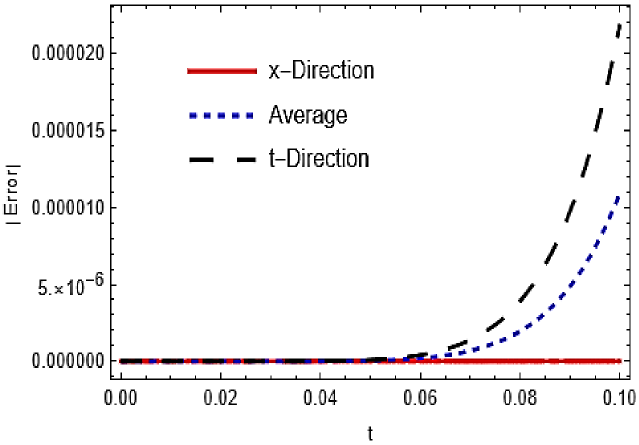

Fig. 14. The AE for the t-direction solution $Y(x,t)$, the average solution $Z(x,t)$, and the x-direction solution $U(x,t)$ at $x = 1.2$, for **e**xample 2.

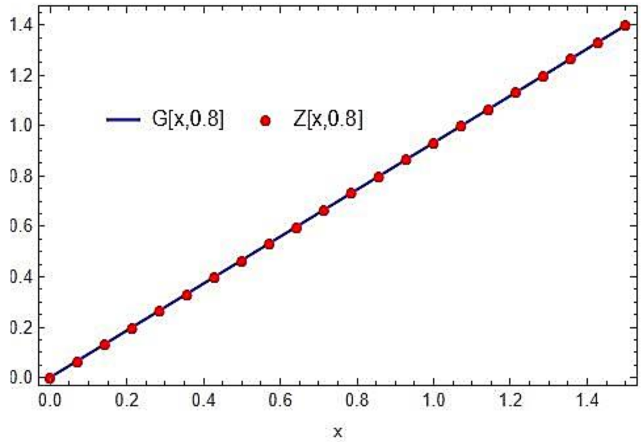

Fig. 17. Graphical representation of the exact solution $G(x,t)$ and the average solution $Z(x,t)$ at $t = 0.8$, for example 3.

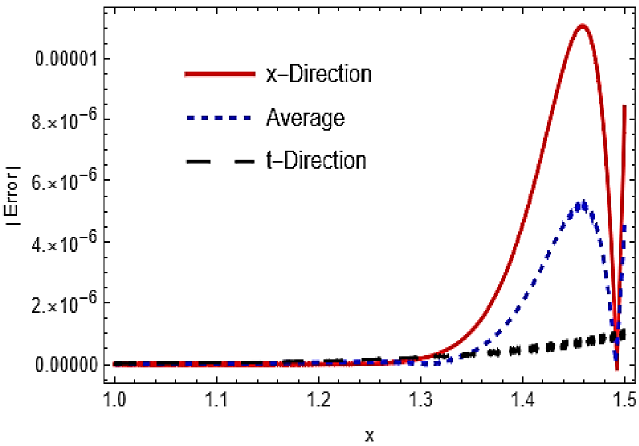

Fig. 15. The AE for the t-direction solution $Y(x,t)$, the average solution $Z(x,t)$, and the x-direction solution $U(x,t)$ at $t = 0.05$, for example 2**.**

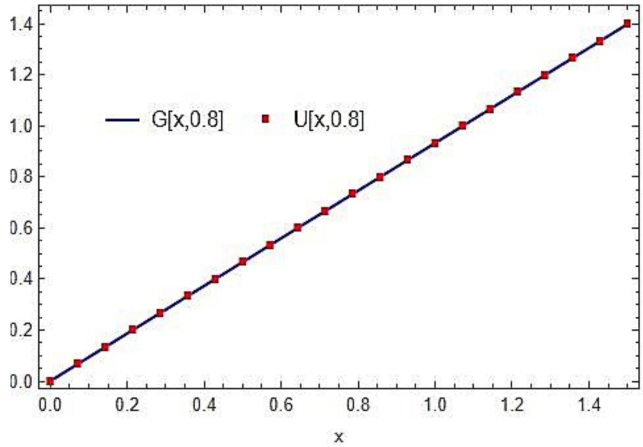

Fig. 18. Graphical representation of the exact solution $G(x,t)$ and the x-direction solution $U(x,t)$ at $t = 0.8$, for example 3.

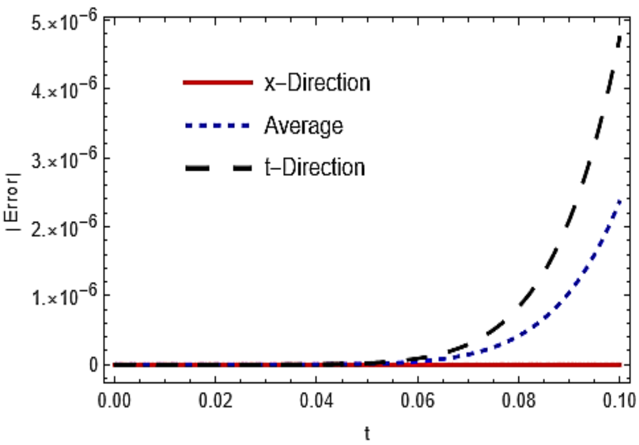

Fig. 16. The AE for the t-direction solution $Y(x,t)$, the average solution $Z(x,t)$, and the x-direction solution $U(x,t)$ at $x = 1$, for example 2.

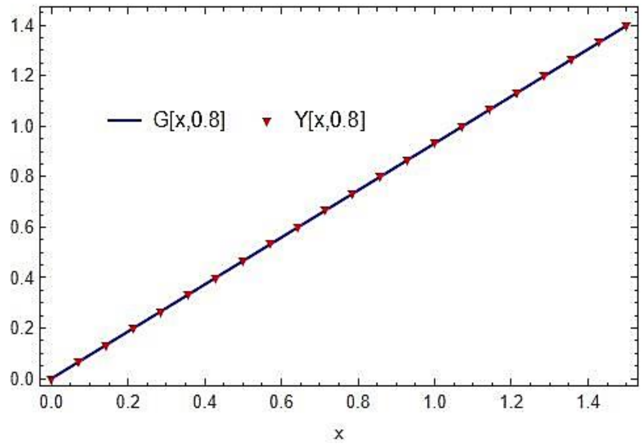

Fig. 19. Graphical representation of the exact solution $G(x,t)$ and the t-direction solution Y (x, t) at $t = 0.8$, for example, 3.

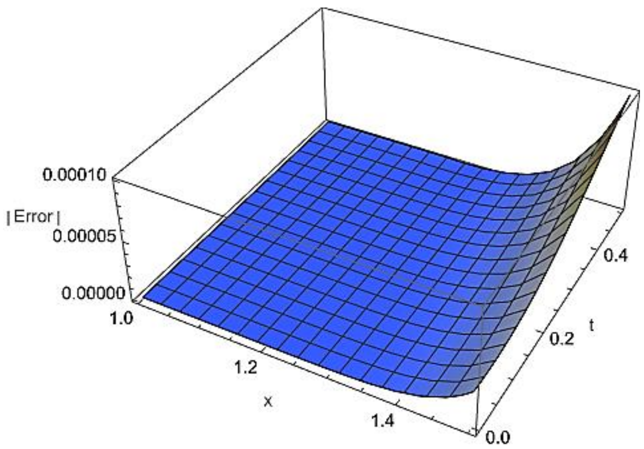

Fig. 20. 3D graph of the AE for the average solution, for example 3.

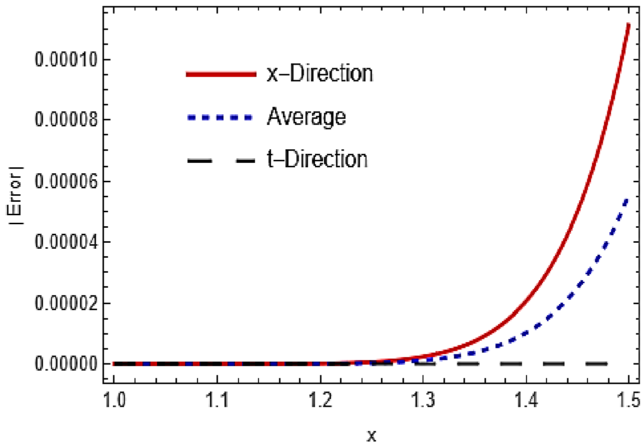

Fig. 21. The AE for the t-direction solution $Y(x,t)$, the average solution $Z(x,t)$, and the x-direction solution $U(x,t)$ at $t = 0.3$, for example 3.

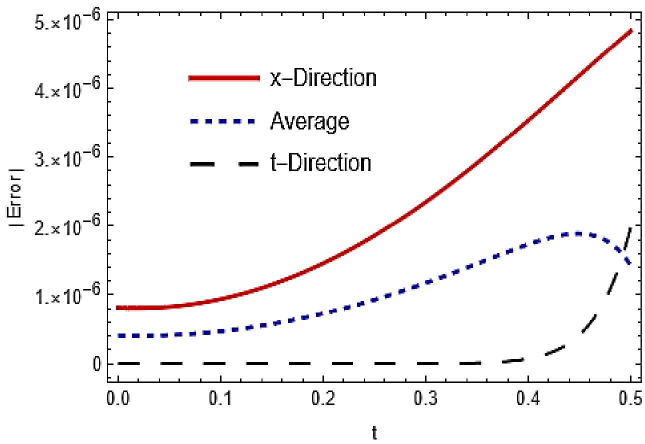

Fig. 22. The AE for the t-direction solution $Y(x,t)$, the average solution $Z(x,t)$, and the x-direction solution $U(x,t)$ at $x = 1.3$, for example 3.

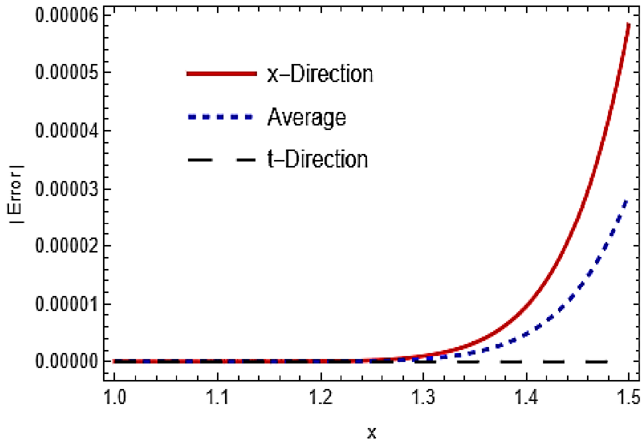

Fig. 23. The AE for the t-direction solution $Y(x,t)$, the average solution $Z(x,t)$, and the x-direction solution $U(x,t)$ at $t = 0.1$, for example 3.

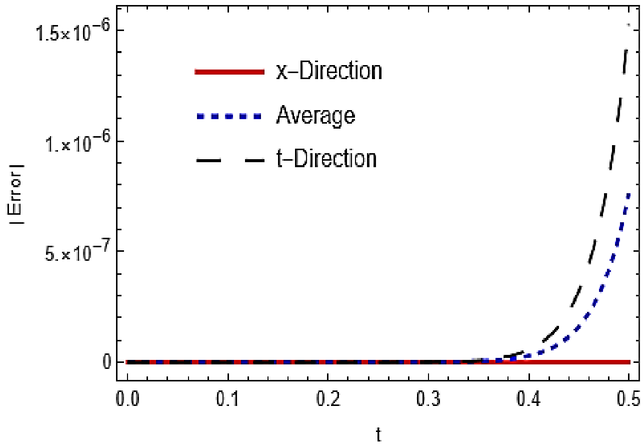

Fig. 24. The AE for the t-direction solution $Y(x,t)$, the average solution $Z(x,t)$, and the x-direction solution $U(x,t)$ at $x = 1$, for example 3.

Table I: Comparison between AE for the t-direction solution $Y(x,t)$, the average solution $Z(x,t)$, and the x-direction solution $U(x,t)$ at $t = 0.5$, for example 1.

c	AE by T-direction solution	AE by Average solution	AE by X-direction solution
1	7.198×10^{-14}	3.599×10^{-14}	0
1.1	1.039×10^{-13}	7.693×10^{-14}	5.020×10^{-14}
1.2	1.438×10^{-13}	1.278×10^{-10}	2.555×10^{-10}
1.3	1.910×10^{-13}	2.019×10^{-8}	4.039×10^{-8}
1.4	2.427×10^{-13}	7.733×10^{-7}	1.546×10^{-6}
1.5	2.932×10^{-13}	1.356×10^{-5}	2.713×10^{-5}

Table II: Comparison between AE for the t-direction solution $Y(x,t)$, the average solution $Z(x,t)$, and the x-direction solution $U(x,t)$ at $x = 1.3$, for example 1.

t	AE by T-direction solution	AE by Average solution	AE by X-direction solution
0	0	1.09×10^{-12}	2.185×10^{-12}
0.1	0	7.41×10^{-12}	1.482×10^{-11}
0.2	0	4.15×10^{-10}	8.314×10^{-10}
0.3	3.469×10^{-18}	2.395×10^{-9}	4.791×10^{-9}
0.4	9.436×10^{-16}	7.937×10^{-9}	1.587×10^{-8}
0.5	1.910×10^{-13}	2.019×10^{-8}	4.039×10^{-8}
0.6	1.359×10^{-11}	4.512×10^{-8}	9.023×10^{-8}
0.7	4.495×10^{-10}	9.707×10^{-8}	1.937×10^{-7}
0.8	7.543×10^{-9}	2.170×10^{-7}	4.265×10^{-7}
0.9	4.761×10^{-8}	5.209×10^{-7}	9.942×10^{-7}
1	6.609×10^{-7}	8.955×10^{-7}	2.452×10^{-6}

Table III: Comparison between AE for the t-direction solution $Y(x,t)$, the average solution $Z(x,t)$, and the x-direction solution $U(x,t)$ at $t = 0.1$, for example 2.

x	AE by T-direction solution	AE by Average solution	AE by X-direction solution
1	4.765×10^{-6}	2.382×10^{-6}	1.651×10^{-12}
1.1	1.024×10^{-5}	5.123×10^{-6}	3.452×10^{-12}
1.2	2.189×10^{-5}	1.094×10^{-5}	1.655×10^{-9}
1.3	4.632×10^{-5}	2.305×10^{-5}	2.072×10^{-7}
1.4	9.679×10^{-5}	4.722×10^{-5}	2.354×10^{-6}
1.5	1.993×10^{-4}	1.461×10^{-4}	9.291×10^{-5}

Table IV: Comparison between AE for the t-direction solution $Y(x,t)$, the average solution $Z(x,t)$, and the x-direction solution $U(x,t)$ at $x = 1.2$, for example 2.

t	AE by T-direction solution	AE by Average solution	AE by X-direction solution
0	5.514×10^{-10}	1.03×10^{-10}	3.13×10^{-10}
0.02	2.655×10^{-10}	4.15×10^{-10}	5.39×10^{-10}
0.04	1.831×10^{-8}	7.916×10^{-9}	7.94×10^{-10}
0.06	4.153×10^{-7}	2.076×10^{-7}	1.07×10^{-9}
0.08	3.913×10^{-6}	1.956×10^{-6}	1.36×10^{-9}
0.1	2.189×10^{-5}	1.094×10^{-5}	1.65×10^{-9}

Table V: Comparison between AE for the t-direction solution $Y(x,t)$, the average solution $Z(x,t)$, and the x-direction solution $U(x,t)$ at $t = 0.3$, for example 3.

c	AE by T-direction solution	AE by Average solution	AE by X-direction solution
1	7.561×10^{-10}	3.78×10^{-10}	0
1.1	8.317×10^{-10}	3.578×10^{-11}	7.602×10^{-10}
1.2	9.073×10^{-10}	5.718×10^{-8}	1.152×10^{-7}
1.3	9.83×10^{-10}	1.172×10^{-6}	2.346×10^{-6}
1.4	1.058×10^{-9}	1.023×10^{-5}	2.046×10^{-5}
1.5	1.134×10^{-9}	5.550×10^{-5}	1.110×10^{-4}

Table IV: Comparison between AE for the t-direction solution $Y(x,t)$, the average solution $Z(x,t)$, and the x-direction solution $U(x,t)$ at $x = 1.3$, for example 3.

t	AE by T-direction solution	AE by Average solution	AE by X-direction solution
0.1	6.938×10^{-16}	4.69×10^{-7}	9.386×10^{-7}
0.2	2.268×10^{-12}	7.28×10^{-7}	1.456×10^{-6}
0.3	9.83×10^{-10}	1.172×10^{-6}	2.346×10^{-6}
0.4	7.201×10^{-8}	1.73×10^{-6}	3.533×10^{-6}
0.5	1.985×10^{-6}	1.424×10^{-6}	4.834×10^{-6}

VI. CONCLUSIONS

In this paper, we propose a semi-analytical technique to solve a class of NIBVPs. This technique utilizes both initial and boundary conditions in the Adomian series solution so that the recurrence relation of the proposed technique's approximate solution agrees with the formulation of this type of problem. In the three previous examples, the proposed technique improved the solution in the x-direction when x diverges about 1 and the solution in the t-direction when t diverges about 0 so that; an accurate approximate solution is obtained in both temporal and spatial directions.

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