A Modified Smoothing Newton Method for Solving Weighted Complementarity Problems with a Nonmonotone Line Search

Xiangjing Liu, Jianke Zhang, and Junfeng Chen

Abstract—**We concentrate on the general form of the weighted complementarity problem, which serves as a generalization of the nonlinear complementarity problem and finds extensive applications in various fields, including economics, sciences, engineering, atmospheric chemistry, and multibody dynamics. We introduce a novel Fischer-Burmeister-based one-parameter smoothing complementarity function. The WCP is then reformulated as a smoothing system of equations, and a new smoothing Newton method is devised to solve the problem efficiently on the new one-parameter smoothing complementarity function. To ensure global convergence, we introduce a new line search rule. The new method exhibits both global and local quadratic convergence properties under appropriate conditions, as demonstrated through several numerical experiments that confirm its effectiveness and stability.**

Index Terms—**weighted complementarity problem, nonmonotone line search, smoothing Newton algorithm, convergence analysis.**

I. Introduction

CONSIDER the following weighted complementarity problem (WCP) in this paper: find a triple $(x, s, y) \in R^n \times R^n \times R^m$ that satisfies

$$x \geq 0,\ s \geq 0,\ G(x, s, y) = 0,\ xs = w, \tag{1}$$

where the mapping $G(x, s, y) : R^{2n+m} \rightarrow R^{n+m}$ is nonlinear and the weighted vector $w \in R^n_+$ is known, with xs denoting the element-wise product of x with s.

When $w = 0$, the WCP (1) simplifies to the well-known nonlinear complementarity problem (NCP):

$$x \geq 0,\ g(x) \geq 0,\ \langle x,\ g(x)\rangle = 0. \tag{2}$$

The NCP (2) has been extensively studied by various researchers, see [1–6]. Moreover, when the mapping $G(x, s, y) : R^{2n+m} \rightarrow R^{n+m}$ assumes a linear form, the WCP (1) can be further simplified to the linear weighted complementarity problem (WLCP):

$$x \geq 0,\ s \geq 0,\ Ax + Bs + Cy = t,\ xs = w, \tag{3}$$

where $A,\ B \in R^{(m+n)\times n}$, $C \in R^{(m+n)\times m}$, $t \in R^{m+n}$.

The notation of WCP was initially introduced by Potra [7], who argued that describing certain equilibrium problems in terms of WCP rather than NCP can lead to more effective solutions, particularly in the field of economics. Potra [7] transformed Fisher market equilibrium problem from economics into the WCP and demonstrated that the quadratic programming and weighted centering problem can be reformulated as the monotone WCP. Furthermore, the WCP shows promise for applications in atmospheric chemistry [8, 9] and multibody dynamics [10, 11].

Due to its wide range of applications, the WCP garnered significant attention from researchers aiming to develop efficient algorithms [12–15]. In the context of monotone linear WCP, Potra [7] generalized the approaches of Mcshane [16] and Mizuno et al. [17] to introduce two class of interior-point methods and conduct an analysis on their computational complexity and convergence properties. For a class of monotone WLCP, Asadi et al. [18] developed a new interior-point algorithm and established an iteration bound. Gowda [19] investigated the WLCP within the framework of Euclidean Jordan algebra, while Chi et al. [20, 21] presented infeasible interior-point algorithms for a specific class of WLCPs with favorable computational complexity.

In addition to the interior point algorithm, the smoothing Newton algorithm is another popular method for solving various mathematical programming problems [22–29]. Recently, several researchers have conducted investigations into the feasibility and convergence properties of smoothing Newton algorithms for the WCP. Zhang [30] proposed a smoothing Newton approach for the monotone WCP, while Tang et al. [31] developed a smoothing approach for the WCP over Euclidean Jordan algebra and discussed its convergence properties under certain assumptions.

Motivated by the aforementioned studies, we develop a novel nonmonotone smoothing Newton method for the general form (1). By utilizing a one-parameter smoothing function, we transform the WCP (1) into an equivalent set of smoothing equations and develop a new smoothing Newton algorithm. The feasibility and convergence properties are discussed under appropriate conditions. Our algorithm possesses several advantageous features:

1) We construct a new Fischer-Burmeister-based one-parameter smoothing complementarity function, which exhibits desirable properties of continuous differentiability. By employing this new smoothing function, we effectively convert the WCP (1) into a smoothing set of equations equivalently.
2) Our approach distinguishes itself from the method presented in [32] by incorporating a novel nonmonotone line search technique. This technique can be reduced to a monotone line search by selecting the appropriate parameters.

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This paper is organized as follows. A one-parameter smoothing function is presented along with a discussion of its fundamental features in Section II. In Section III, a new feasible nonmonotone smoothing Newton algorithm specifically designed for WCPs is proposed. The convergence properties of the method are thoroughly analyzed in Section IV. In Section V, we present the results of the numerical experiments performed. In Section VI, we propose a conclusion to summarize the results of this paper.

II. Preliminaries

First, we propose a new one-parameter smoothing function $\phi_r(\varepsilon,u,v): R^3 \to R$ as follows:

$$\phi_r(\varepsilon,u,v) = (1+\varepsilon)(u+v) - \sqrt{(u+\varepsilon v)^2+(v+\varepsilon u)^2+4(1-\varepsilon)r+2\varepsilon^2}, \quad (4)$$

where $0 \le \varepsilon \le 1$ and $r \ge 0$. The properties of $\phi_r(\varepsilon,u,v)$ can be easily deduced through straightforward reasoning and calculation.

Lemma 1. For any $0 \le \varepsilon \le 1$, $\phi_r(\varepsilon,u,v) = 0$ if and only if $u+\varepsilon v > 0, v+\varepsilon u > 0, (u+\varepsilon v)(v+\varepsilon u) = 2(1-\varepsilon)r+\varepsilon^2$.

Lemma 2. For any $\varepsilon \in (0,1)$, $\phi_r(\varepsilon,u,v)$ is continuously differentiable, with

$$(\phi_r(\varepsilon,u,v))'_\varepsilon = u+v - \frac{v(u+\varepsilon v)+u(v+\varepsilon u)+2(\varepsilon-r)}{\sqrt{(u+\varepsilon v)^2+(v+\varepsilon u)^2+4(1-\varepsilon)r+2\varepsilon^2}}, \quad (5)$$

$$(\phi_r(\varepsilon,u,v))'_u = 1+\varepsilon - \frac{u+\varepsilon v+\varepsilon(v+\varepsilon u)}{\sqrt{(u+\varepsilon v)^2+(v+\varepsilon u)^2+4(1-\varepsilon)r+2\varepsilon^2}}, \quad (6)$$

$$(\phi_r(\varepsilon,u,v))'_v = 1+\varepsilon - \frac{\varepsilon(u+\varepsilon v)+v+\varepsilon u}{\sqrt{(u+\varepsilon v)^2+(v+\varepsilon u)^2+4(1-\varepsilon)r+2\varepsilon^2}}. \quad (7)$$

Moreover,

$$(\phi_r(\varepsilon,u,v))'_u > 0, \quad (8)$$

and

$$(\phi_r(\varepsilon,u,v))'_v > 0. \quad (9)$$

The proof of Lemma 2, which can be obtained through some simple calculations, is omitted here.

For a given $w \in R^n_+$, we define

$$M(\varepsilon,x,s,y) = \begin{pmatrix} \varepsilon \\ G(x,s,y) \\ \phi_w(\varepsilon,x,s) \end{pmatrix}, \quad (10)$$

where

$$\phi_w(\varepsilon,x,s) = \begin{pmatrix} \phi_{w_1}(\varepsilon,x_1,s_1) \\ \vdots \\ \phi_{w_n}(\varepsilon,x_n,s_n) \end{pmatrix}. \quad (11)$$

To simplify the notation, let $z = (\varepsilon,x,s,y)$. The WCP (1) can be converted into the equations $M(z) = 0$ as a direct consequence. By solving $M(z) = 0$, we can obtain a solution to the WCP (1). We first conclude that $M(z)$ is continuously differentiable.

Lemma 3. Define $M(z)$ by (10), then $M(z)$ is continuously differentiable for any $\varepsilon > 0$ with its Jacobian matrix

$$M'(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & G'_x & G'_s & G'_y \\ D_1 & D_2 & D_3 & 0 \end{pmatrix}, \quad (12)$$

where $D_1 = (d_1^1, d_2^1, \ldots, d_n^1)^T$, $D_2 = \mathrm{diag}(d^2)$ and $D_3 = \mathrm{diag}(d^3)$ with $d^2 = (d_1^2, d_2^2, \ldots, d_n^2)^T$ and $d^3 = (d_1^3, d_2^3, \ldots, d_n^3)^T$, where

$$d_i^1 = x_i + s_i - \frac{s_i(x_i+\varepsilon x_i)+x_i(s_i+\varepsilon x_i)+2(\varepsilon-w_i)}{\sqrt{(x_i+\varepsilon s_i)^2+(s_i+\varepsilon x_i)^2+4(1-\varepsilon)w_i+2\varepsilon^2}}, \quad (13)$$

$$d_i^2 = 1+\varepsilon - \frac{x_i+\varepsilon s_i+(s_i+\varepsilon x_i)\varepsilon}{\sqrt{(x_i+\varepsilon s_i)^2+(s_i+\varepsilon x_i)^2+4(1-\varepsilon)w_i+2\varepsilon^2}}, \quad (14)$$

$$d_i^3 = 1+\varepsilon - \frac{\varepsilon(x_i+\varepsilon s_i)+s_i+\varepsilon x_i}{\sqrt{(x_i+\varepsilon s_i)^2+(s_i+\varepsilon x_i)^2+4(1-\varepsilon)w_i+2\varepsilon^2}}. \quad (15)$$

Next, we analyze the nonsingularity of $M'(z)$. To do this, we introduce the following assumption.

Assumption 1. Assume that $\mathrm{Rank}(G'_y) = m$, for any $(\Delta x, \Delta s, \Delta y) \in R^{2n+m}$, if

$$G'_x\Delta x + G'_s\Delta s + G'_y\Delta y = 0,$$

then $\langle \Delta x, \Delta s\rangle \ge 0$.

Note that if $G(x,s,y): R^{2n+m} \to R^{n+m}$ is linear, then Assumption 1 simplifies to

$$A\Delta x + B\Delta s + C\Delta y = 0,$$

indicating the monotonicity of $G(x,s,y)$. The feasibility of smoothing methods for WLCPs in this case has been previously discussed in [7, 12, 30].

Theorem 1. If Assumption 1 is satisfied, then for any $\varepsilon > 0$, $M'(z)$ is nonsingular.

Proof. Suppose that there is $\Delta z = (\Delta\varepsilon, \Delta x, \Delta s, \Delta y) \in R^{2n+m+1}$ satisfying

$$M'(z)\Delta z = 0. \quad (16)$$

Next, we only need to demonstrate that $\Delta z = 0$. By substituting (12) into (16), we obtain

$$\begin{aligned} \Delta\varepsilon &= 0, \\ G'_x\Delta x + G'_s\Delta s + G'_y\Delta y &= 0, \\ D_1\Delta\varepsilon + D_2\Delta x + D_3\Delta s &= 0. \end{aligned} \quad (17)$$

According to Lemma 2 and Lemma 3, the diagonal matrices D_2 and D_3 are positive definite. Applying (17), we have

$$\Delta x = -D_2^{-1}D_3\Delta s, \quad (18)$$

and consequently

$$\langle \Delta x, \Delta s\rangle = -\Delta s^T D_3 D_2^{-1}\Delta s \le 0. \quad (19)$$

By virtue of Assumption 1, we have $\langle \Delta x, \Delta s\rangle \ge 0$. This, combined with (19), leads to

$$\langle \Delta x, \Delta s\rangle = -\Delta s^T D_3 D_2^{-1}\Delta s = 0,$$

and $\Delta s = 0$. Furthermore, we have $\Delta x = 0$ by (18). Thus, utilizing the second equation in (17) results in $\Delta y = 0$.

III. A NONMONOTONE SMOOTHING NEWTON METHOD

Based on (10), we set $\theta(z) = \frac{1}{2}\|M(z)\|^2$. Now, we propose the novel nonmonotone smoothing method.

Algorithm 1.

Step 0. Choose τ, $l \in (0,1)$, $\delta \in (0,\sqrt{2})$ and $\varepsilon_0 > 0$ such that $\varepsilon_0 \geq \delta$. $S \geq 0$ is a positive integer and $q_0 = 1$. $\{\xi_k\} \subseteq R_+$ satisfies that $\lim_{k\to\infty} \xi_k = 0$. $(x^0, s^0, y^0) \in R^{2n+m}$ is an arbitrary starting point. Set $z^0 = (\varepsilon_0, x^0, s^0, y^0)$, $\Upsilon_0 = \theta(z^0)$, $W_0 = \theta(z^0)$ and $e = (1,0)^T \in R \times R^{2n+m}$. Let $k = 0$.

Step 1. If $\|M(z^k)\| = 0$, stop. Else, set

$$\gamma_k = \begin{cases} \min\{\delta, \delta\Upsilon_0\}, & k = 0, \\ \min\{\delta, \delta\Upsilon_k, \gamma_{k-1}\}, & k \geq 1. \end{cases} \tag{20}$$

Step 2. Obtain the Newton direction $\Delta z^k = (\Delta\varepsilon_k, \Delta x^k, \Delta s^k, \Delta y^k)$ by solving

$$M'(z^k)\Delta z^k = -M(z^k) + \gamma_k e. \tag{21}$$

Step 3. Let β_k be the maximum among l^0, l, l^2, ... that meets the following inequality:

$$\theta(z^k + \beta_k\Delta z^k) \leq [1 - \tau(2 - \sqrt{2}\delta)\beta_k]\Upsilon_k. \tag{22}$$

Step 4. Let $z^{k+1} = z^k + \beta_k\Delta z^k$, $s(k) = \min(k, S)$ and

$$W_{k+1} = \max_{0\leq i\leq s(k+1)} \{\theta(z^{k+1-i})\}. \tag{23}$$

Step 5. Set

$$\Upsilon_{k+1} = \frac{\xi_k q_k \theta(z^{k+1}) + W_{k+1}}{q_{k+1}}, \tag{24}$$

$$q_{k+1} = \xi_k q_k + 1. \tag{25}$$

and $k = k + 1$. Return to Step 1.

Remark 1.

1) Contrary to [32], Step 5 of Algorithm 1 indicates that Υ_{k+1} is a convex combination some $\theta(z^j)$ with $k + 1 - s(k+1) \leq j \leq k+1$ and $\theta(z^{k+1})$. Considering the definition of q_{k+1}, we can see that (22) can be simplified to the following monotone line search
$$\theta(z^k + \beta_k\Delta z^k) \leq [1 - \tau(2 - \sqrt{2}\delta)\beta_k]\theta(z^k),$$
if $S = 0$.
2) Based on Step 0, we have
$$W_0 = \Upsilon_0 = \theta(z^0).$$
From (23) and (24), it can be observed that $W_k \geq \Upsilon_k \geq \theta(z^k)$ for any $k \geq 0$.
3) It follows from the definition of W_k that
$$\begin{aligned} W_{k+1} &= \max_{0\leq i\leq s(k+1)} \theta(z^{k+1-i}) \\ &\leq \max_{0\leq i\leq s(k)+1} \theta(z^{k+1-i}) \\ &= \max\{W_k, \theta(z^{k+1})\} \\ &= W_k, \end{aligned} \tag{26}$$
where the last equality is derived from (22) and (23). Thus, $\{W_k\}$ is nonincreasing and then convergent.

By conducting straightforward computations and reasoning, we can derive the following lemma. For brevity, we will only state the conclusion.

Lemma 4. If Assumption 1 is satisfied, then the statements hold that $\varepsilon_k \geq 0$, $\varepsilon_k \geq \gamma_k$ and $\{\varepsilon_k\}$ is nonincreasing for any $k \geq 0$.

Theorem 2. If Assumption 1 is true, then Algorithm 1 is well-defined.

Proof. We can establish the invertibility of $M'(z)$ based on Theorem 1, thereby rendering Step 2 feasible. Then, we demonstrate the viability of Step 3.

From the definition of γ_k, it can be deduced that

$$\gamma_k \leq \delta \cdot \sqrt{\Upsilon_k}. \tag{27}$$

Given that

$$\varepsilon_k \leq \|M(z^k)\| = \sqrt{2\theta(z^k)} \leq \sqrt{2\Upsilon_k},$$

we can infer from (21) and (27) that

$$\begin{aligned} &\theta(z^k + \beta\Delta z^k) \\ &= \theta(z^k) + \beta\theta'(z^k)\Delta z^k + o(\beta) \\ &= \theta(z^k) + \beta M(z^k)^T(\gamma_k e - M(z^k)) + o(\beta) \\ &\leq \theta(z^k) + \sqrt{2}\beta\delta\Upsilon_k - 2\beta\theta(z^k) + o(\beta) \\ &\leq [1 - \beta(2 - \sqrt{2}\delta)]\Upsilon_k + o(\beta), \end{aligned} \tag{28}$$

indicating that there is a constant $\bar{\beta} \in (0,1)$ satisfying

$$\theta(z^k + \beta\Delta z^k) \leq [1 - \tau\beta(2 - \sqrt{2}\delta)]\Upsilon_k$$

holds for any $\beta \in (0, \bar{\beta})$ and $\tau \in (0,1)$ and thus Step 3 is available.

IV. CONVERGENCE PROPERTIES

We begin by demonstrating a significant theorem.

Theorem 3. If Assumption 1 is true, then, $\lim_{k\to\infty} \beta_k\Upsilon_k = 0$.

Proof. For any $k \geq 0$, set the integer $p(k) \in [k - s(k), k]$ such that

$$\begin{aligned} W_k &= \max_{0\leq i\leq s(k)} \theta(z^{k-i}) \\ &= \theta(z^{p(k)}), \end{aligned} \tag{29}$$

Based on (22) and Remark 1 2), we can obtain that

$$\begin{aligned} W_k &= \theta(z^{p(k)}) \\ &= \theta(z^{p(k)-1} + \beta_{p(k)-1}\Delta z^{p(k)-1}) \\ &\leq \Upsilon_{p(k)-1} - (2 - \sqrt{2}\delta)\tau\beta_{p(k)-1}\Upsilon_{p(k)-1} \\ &\leq W_{p(k)-1} - (2 - \sqrt{2}\delta)\tau\beta_{p(k)-1}\Upsilon_{p(k)-1}. \end{aligned} \tag{30}$$

As $\{W_k\}$ is convergent according to Remark 1 3), we have

$$\lim_{k\to\infty} \left(W_k - W_{p(k)-1}\right) = 0. \tag{31}$$

Combining (30) and (31) yields that

$$\lim_{k\to\infty} \beta_{p(k)-1}\Upsilon_{p(k)-1} = 0. \tag{32}$$

We now demonstrate that $\lim_{k\to\infty} \beta_k\Upsilon_k = 0$. By assuming that $\hat{p}(k) = p(k + S + 2)$, we can utilize induction to prove that

$$\lim_{k\to\infty} \beta_{\hat{p}(k)-j}\Upsilon_{\hat{p}(k)-j} = 0, \tag{33}$$

and

$$\lim_{k\to\infty} \theta(z^{\hat{p}(k)-j}) = \lim_{k\to\infty} W_k, \tag{34}$$

for any given $j \geq 1$.

Let $k \geq j-1$, without loss of generality. Since $\{\hat{p}(k)\}$ is a subsequence of $\{p(k)\}$, (33) holds for $j=1$ based on (32).

As $M(z^{\hat{p}(k)-1})$ is invertible according to Theorem 1 for $\varepsilon_{\hat{p}(k)-1} > 0$, there exists some $\zeta > 0$ such that

$$\|[M(z^{\hat{p}(k)-1})]^{-1}\| \leq \zeta. \tag{35}$$

By utilizing (21) and (27), we get

$$\begin{aligned} &\|\Delta z^{\hat{p}(k)-1}\| \\ &= \|[M(z^{\hat{p}(k)-1})]^{-1}[\gamma_{\hat{p}(k)-1}e - M(z^{\hat{p}(k)-1})]\| \\ &\leq \zeta \cdot \left(\|\gamma_{\hat{p}(k)-1}e\| + \|M(z^{\hat{p}(k)-1})\|\right) \\ &= \zeta(\delta + \sqrt{2}) \cdot \sqrt{\Upsilon_{\hat{p}(k)-1}}, \end{aligned} \tag{36}$$

and subsequently,

$$\begin{aligned} \|z^{\hat{p}(k)} - z^{\hat{p}(k)-1}\| &= \|\beta_{\hat{p}(k)-1}\Delta z^{\hat{p}(k)-1}\| \\ &\leq \zeta(\delta + \sqrt{2}) \cdot \sqrt{\beta^2_{\hat{p}(k)-1}\Upsilon_{\hat{p}(k)-1}}. \end{aligned} \tag{37}$$

Notice that (33) is valid for $j = 1$. Furthermore, (37) demonstrates that

$$\lim_{k\to\infty} \|z^{\hat{p}(k)} - z^{\hat{p}(k)-1}\| = 0, \tag{38}$$

since $\beta_{\hat{p}(k)-1} \in (0,1)$. We know that (34) holds for $j=1$ due to the continuity of $\theta(z^k)$.

Now, we assume that both (33) and (34) hold for some $j > 1$ and examine the case of $j+1$. By employing (22), we obtain

$$\begin{aligned} \theta(z^{\hat{p}(k)-j}) &\leq [1 - \tau(2-\sqrt{2}\delta)\beta_{\hat{p}(k)-(j+1)}]\Upsilon_{\hat{p}(k)-(j+1)} \\ &\leq C_{\hat{p}(k)-(j+1)} - \tau(2-\sqrt{2}\delta)\beta_{\hat{p}(k)-(j+1)}\Upsilon_{\hat{p}(k)-(j+1)}, \end{aligned}$$

which, when combined with $\lim_{k\to\infty} \theta(z^{\hat{p}(k)-j}) = \lim_{k\to\infty} W_k$ and $\lim_{k\to\infty} \hat{p}(k) - (j+1) = \infty$, yields

$$\lim_{k\to\infty} \beta_{\hat{p}(k)-(j+1)}\Upsilon_{\hat{p}(k)-(j+1)} = 0.$$

Furthermore, we have

$$\lim_{k\to\infty} \|z^{\hat{p}(k)-j} - z^{\hat{p}(k)-(j+1)}\| = 0,$$

by employing an inequality comparable to (38). Consequently, we deduce

$$\lim_{k\to\infty} \theta(z^{\hat{p}(k)-(j+1)}) = \lim_{k\to\infty} \theta(z^{\hat{p}(k)-j}) = \lim_{k\to\infty} W_k,$$

by utilizing (34) and the continuity of $\theta(z^k)$. Therefore, for any $j \geq 1$, both (33) and (34) hold.

Combining (33) and (36) yields

$$\lim_{k\to\infty} \|z^{k+1} - z^{\hat{p}(k)}\| = 0,$$

since

$$z^{k+1} = z^{\hat{p}(k)} + \sum_{j=1}^{\hat{p}(k)-k-1} \beta_{\hat{p}(k)-j}\Delta z^{\hat{p}(k)-j}.$$

Additionally, by the continuity of $\theta(z^k)$, we determine that

$$\begin{aligned} \lim_{k\to\infty} \theta(z^{k+1}) &= \lim_{k\to\infty} \theta(z^{\hat{p}(k)}) \\ &= \lim_{k\to\infty} \theta(z^{p(k)}) = \lim_{k\to\infty} W_k, \end{aligned} \tag{39}$$

as $\{W_k\}$, which corresponds to $\{\theta(z^{p(k)})\}$, is convergent.

On the other hand, by utilizing (22), we can deduce that

$$\begin{aligned} \theta(z^{k+1}) &\leq \Upsilon_k - \tau(2-\sqrt{2}\delta)\beta_k\Upsilon_k \\ &\leq W_k - \tau(2-\sqrt{2}\delta)\beta_k\Upsilon_k, \end{aligned}$$

which combining with (39) results in $\lim_{k\to\infty} \beta_k\Upsilon_k = 0$.

Theorem 4. If Assumption 1 is satisfied, then any accumulation point of $\{z^k\}$ is a solution to the WCP (1).

Proof. Considering Remark 1, we can conclude that

$$0 \leq \theta(z^{k+1}) \leq \Upsilon_{k+1} \leq W_k \leq W_1 < \infty,$$

indicating that, $\{\theta(z^k)\}$ and $\{\Upsilon_k\}$ are bounded, and $\{W_k\}$ is convergent. Assume that $\{z^k\} \supseteq \{z^k\}_{k\in\mathbb{N}}$ converges to $z^* = (\varepsilon_*, x^*, s^*, y^*)$. Consequently, we obtain

$$\lim_{\mathbb{N}\ni k\to\infty} \|M(z^k)\| = \|H(z^*)\|, \quad \lim_{\mathbb{N}\ni k\to\infty} \Upsilon_k = \Upsilon^*.$$

If $\Upsilon^* = 0$, then

$$\begin{aligned} \lim_{\mathbb{N}\ni k\to\infty} \|M(z^k)\| &= \lim_{\mathbb{N}\ni k\to\infty} \sqrt{2\theta(z^k)} \\ &\leq \lim_{\mathbb{N}\ni k\to\infty} \sqrt{2\Upsilon_k} = \sqrt{2\Upsilon^*}, \end{aligned}$$

implying that $\lim_{\mathbb{N}\ni k\to\infty} \|M(z^k)\| = 0$. Now, we suppose that $\Upsilon^* > 0$.

According to Theorem 3, it is evident that $\lim_{\mathbb{N}\ni k\to\infty} \beta_k = 0$. Let $\hat{\beta} = \dfrac{\beta_k}{l}$, for sufficiently large k, $\hat{\beta}$ does not meet (22), i.e.,

$$\theta(z^k + \hat{\beta}\Delta z^k) > [1 - \tau(2-\sqrt{2}\delta)\hat{\beta}]\Upsilon_k. \tag{40}$$

Meanwhile, we deduce from (28) that

$$\theta(z^k + \hat{\beta}\Delta z^k) \leq [1 - \hat{\beta}(2-\sqrt{2}\delta)]\Upsilon(z^k) + o(\hat{\beta}). \tag{41}$$

Combining (40) with (41) yields

$$(2-\sqrt{2}\delta)(1-\tau)\Upsilon(z^k) < \frac{o(\hat{\beta})}{\hat{\beta}}, \tag{42}$$

for any sufficiently large k.

Let $k \to \infty$ on both sides of (42), we have

$$(2-\sqrt{2}\delta)(1-\tau)\Upsilon^* \leq 0,$$

which contradicts the conditions of $\tau \in (0,1)$, $\delta \in (0,\sqrt{2})$ and $\Upsilon^* > 0$.

Finally, the local convergence property is investigated. Using a method similar to the one in Theorem 8 of [33], whose proof is omitted, we can reach the following conclusion.

Theorem 5. Suppose that Assumption 1 is satisfied, all $D \in \partial M(z^*)$ are nonsingular and $G'(x,s,y)$ is Lipschitz continuous around x^*, then $\{z^k\}$ converges to z^* locally quadratically.

V. Numerical Experiments

To showcase the efficacy and efficiency of Algorithm 1, we employ it to deal with a WLCP and a nonlinear WCP. The implementation of all algorithms is carried out in Matlab R2018b, running on a computer equipped with a 2.30GHz CPU and 16.00GB RAM. The stopping criterion is defined as $\|M(z^k)\| \leq 10^{-6}$, and the parameters are set as

$$\tau = 0.05,\ l = 0.7,\ \delta = 0.0000001 \text{ and } \varepsilon_0 = 0.001.$$

In the following tables, TM(ATM) represents the (average) running time of the algorithm in seconds, Iter(AIter) stands for the (average) number of iterations and GAP(AGAP) indicates the (average) value of $\|M(z^k)\|$ in the final iteration.

First, we consider the WLCP (3) with

$$A=\begin{pmatrix}M\\N\end{pmatrix},B=\begin{pmatrix}0\\-I\end{pmatrix},C=\begin{pmatrix}0\\-M^T\end{pmatrix},t=\begin{pmatrix}Mr\\f\end{pmatrix},$$

where $N\in R^{n\times n}$, $M=randn(m,n)$, $f=rand(n,1)$ and $r=-rand(n,1)$. $w\in R^n$ is obtained by $w=uv$ where $v=Nu-f$ with $u=rand(n,1)$.

TABLE I
NUMERICAL COMPARISON RESULTS FOR SOLVING THE WLCP WITH N_1

		Algorithm 1			[30], $\mathrm{SN_{zhang}}$		
m	n	Iter	TM	GAP	Iter	TM	GAP
		6	1.6005	1.7014×10^{-7}	6	2.7134	2.6577×10^{-9}
		6	1.4960	1.8576×10^{-7}	6	2.7820	1.3676×10^{-9}
500	1000	6	1.5263	2.2546×10^{-7}	6	2.6051	5.5531×10^{-9}
		7	1.7162	9.1566×10^{-10}	7	2.9324	5.5531×10^{-9}
		7	1.7594	1.4407×10^{-10}	8	3.3145	6.2509×10^{-9}
		6	9.4619	1.4377×10^{-8}	8	15.8012	2.9857×10^{-7}
		7	10.8039	1.9064×10^{-7}	8	13.2422	4.8900×10^{-9}
1000	2000	7	10.7726	1.3502×10^{-9}	8	18.0144	2.8000×10^{-7}
		7	10.8088	4.4566×10^{-9}	8	18.9309	3.5054×10^{-10}
		7	10.9075	2.4927×10^{-7}	8	19.1643	1.1193×10^{-9}
		6	29.0347	2.7727×10^{-7}	8	77.7497	1.7961×10^{-9}
		7	32.7497	2.0578×10^{-9}	8	79.1600	8.8037×10^{-8}
1500	3000	7	35.9581	1.0358×10^{-8}	8	75.8348	1.6339×10^{-8}
		7	36.1701	1.6646×10^{-9}	8	76.5640	4.5752×10^{-8}
		7	32.5845	2.3333×10^{-9}	8	76.5480	5.6413×10^{-10}
		7	79.6557	3.9503×10^{-7}	8	145.6442	1.3405×10^{-7}
		7	85.6767	1.7166×10^{-8}	8	171.5185	3.9481×10^{-8}
2000	4000	7	92.6848	5.7068×10^{-7}	8	170.4471	1.5378×10^{-8}
		7	72.8459	8.7711×10^{-9}	8	170.5121	4.1869×10^{-8}
		7	72.6069	3.1610×10^{-9}	8	180.7858	7.4412×10^{-8}
		7	137.2600	8.3807×10^{-8}	8	298.3107	2.3009×10^{-8}
		7	136.2246	2.7135×10^{-7}	8	320.0004	4.4900×10^{-8}
2500	5000	7	160.5432	2.2443×10^{-7}	8	312.1355	4.1091×10^{-8}
		7	158.5533	2.8880×10^{-9}	8	312.1363	1.6451×10^{-8}
		7	161.5871	9.1816×10^{-9}	8	311.0846	8.9347×10^{-8}

For this problem, we evaluate the performance of Algorithm 1 under different values of N, specifically denoted by N_1 and N_2, respectively. N_1 is produced by setting $N_1=BB^T/\|BB^T\|$ where B is uniformly generated from $[0,1]$. On the other hand, $N_2=diag(rand(n,1))$. In order to ensure robustness, we conduct 5 trials for each instance. For each trial, the initial points x^0, s^0 and y^0 are chosen as $(1,0,\dots,0)^T$ with the appropriate dimension. Additionally, we implement the method proposed in [30] and refer to it as $\mathrm{SN_{zhang}}$ to demonstrate the performance of Algorithm 1. The test results, presented in Tables I and II, demonstrate that Algorithm 1 performs fewer iterations than $\mathrm{SN_{zhang}}$. Moreover, the running time of Algorithm 1 is much less than that of $\mathrm{SN_{zhang}}$, particularly for higher-dimensional problems.

TABLE II
NUMERICAL COMPARISON RESULTS FOR SOLVING THE WLCP WITH N_2

		Algorithm 1			[30], $\mathrm{SN_{zhang}}$		
m	n	Iter	TM	GAP	Iter	TM	GAP
		7	2.3473	1.1794×10^{-8}	6	2.2938	1.7135×10^{-9}
		7	1.6890	8.7344×10^{-10}	6	2.2381	7.3915×10^{-7}
500	1000	7	1.5116	1.3626×10^{-9}	7	3.2154	1.6683×10^{-10}
		7	1.4724	2.2892×10^{-7}	7	2.8470	2.8737×10^{-7}
		7	1.5066	1.4819×10^{-8}	7	2.2989	2.3545×10^{-9}
		6	7.4253	7.0920×10^{-7}	6	17.5515	5.1926×10^{-12}
		7	9.3816	5.5287×10^{-7}	7	20.7418	2.1289×10^{-8}
1000	2000	7	9.2737	1.3493×10^{-9}	7	22.2149	1.6028×10^{-9}
		7	8.6559	1.7792×10^{-8}	8	23.5621	4.0224×10^{-11}
		7	8.5201	1.3211×10^{-9}	8	24.8038	3.5177×10^{-11}
		7	24.0990	5.4242×10^{-9}	7	66.0803	1.9924×10^{-8}
		7	27.2737	5.2470×10^{-9}	7	71.3545	1.3117×10^{-9}
1500	3000	7	27.0466	1.8006×10^{-9}	8	88.4625	1.0130×10^{-11}
		7	27.2405	9.6434×10^{-8}	8	74.6301	6.6432×10^{-10}
		7	27.2752	2.0286×10^{-7}	8	66.0803	1.9924×10^{-8}
		7	54.0027	1.2151×10^{-8}	7	178.5136	2.6987×10^{-10}
		7	53.5682	2.9794×10^{-9}	7	171.5552	5.5322×10^{-11}
2000	4000	7	53.8382	4.5529×10^{-8}	8	204.7930	2.5617×10^{-10}
		7	61.0597	6.3556×10^{-8}	8	205.2599	5.1473×10^{-8}
		7	52.7841	2.1679×10^{-7}	9	204.0851	2.1492×10^{-8}
		7	99.0762	6.9838×10^{-8}	7	346.8380	2.5480×10^{-11}
		7	120.8204	8.3114×10^{-8}	7	343.7178	1.7028×10^{-9}
2500	5000	7	128.4566	1.3047×10^{-8}	8	399.0091	1.0139×10^{-8}
		7	122.1990	1.0983×10^{-7}	8	382.8897	7.9749×10^{-9}
		7	123.5217	5.1374×10^{-8}	9	457.4124	2.6399×10^{-7}

Then, we consider the WCP (1) with

$$G(x,s,y)=\begin{pmatrix}Tx+A^Ty-s+d\\A(x-b)\end{pmatrix},$$

where $T=diag(rand(n,1))$, $A=randn(m,n)$, $b=rand(n,1)$, $d=rand(n,1)$ and $w=rand(n,1)$.

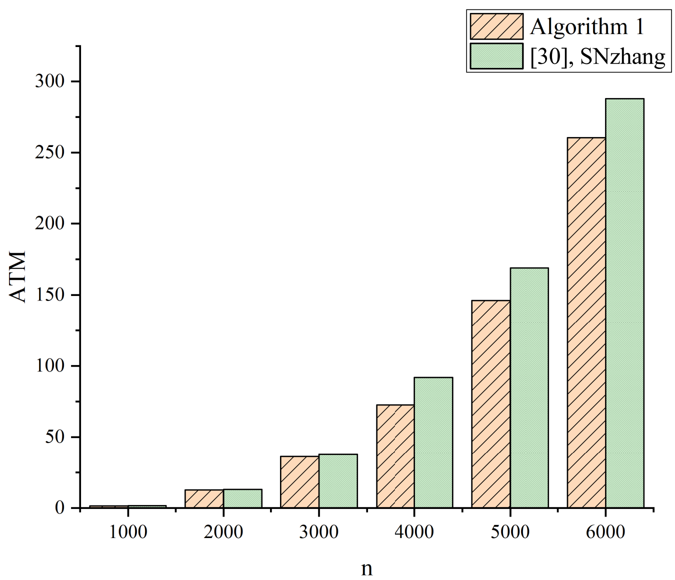

Fig. 1. The average running time of Algorithm 1 and $\mathrm{SN_{zhang}}$ based on 10 trials.

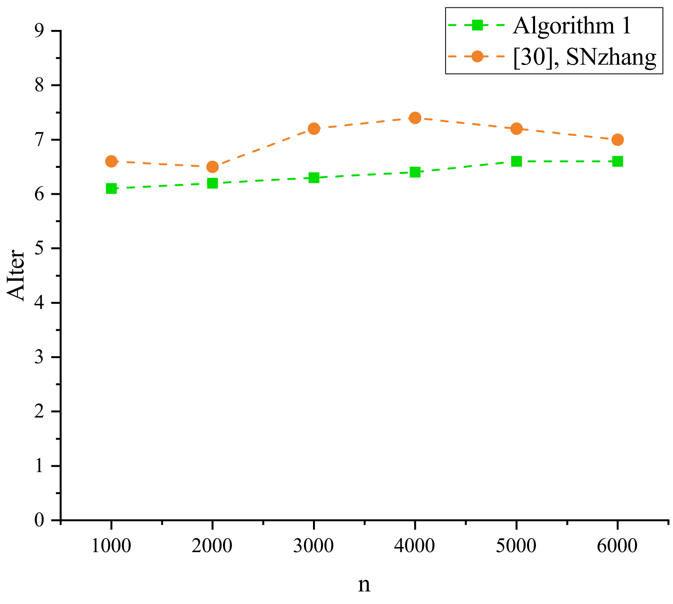

Fig. 2. The average number of iterations of Algorithm 1 and $\mathrm{SN_{zhang}}$ based on 10 trials.

We conducted 10 tests for each size. The starting points x^0, s^0 and y^0 are vectors whose elements are generated randomly in $[0, 1]$. The numerical experimental results are shown in Table III, Figures 1 and 2. We can observe that Algorithm 1 is more efficient than $\mathrm{SN_{zhang}}$. As the dimensionality increases, Algorithm 1 requires less time compared to $\mathrm{SN_{zhang}}$.

TABLE III
NUMERICAL RESULTS FOR A WCP

		Algorithm 1			[30], $\mathrm{SN_{zhang}}$		
m	n	AIter	ATM	AGAP	AIter	ATM	AGAP
500	1000	6.1	1.4948	2.4291×10^{-7}	6.6	1.6792	1.2633×10^{-7}
1000	2000	6.2	12.6265	3.6072×10^{-7}	6.5	13.0823	5.8239×10^{-8}
1500	3000	6.3	36.1982	3.3831×10^{-7}	7.2	37.5804	2.4647×10^{-8}
2000	4000	6.4	72.7058	1.0191×10^{-7}	7.4	92.0072	2.0717×10^{-7}
2500	5000	6.6	146.1622	3.2427×10^{-7}	7.2	168.9312	1.1941×10^{-7}
3000	6000	6.6	260.4692	3.3894×10^{-7}	7.0	287.9610	6.9728×10^{-8}

VI. CONCLUSIONS

For the WCP, a new smoothing Newton method is suggested. By utilizing a one-parameter smoothing function, the WCP is converted into an equivalent system of smoothing equations. The solution to the WCP is obtained by solving the smoothing equation. Through appropriate parameter selection, the nonmonotone line search can be simplified to a monotone line search. The viability and convergence properties of the proposed algorithm are thoroughly discussed under suitable conditions. Furthermore, numerical results are provided to demonstrate the efficiency of the suggested algorithm.

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