An Effective Algorithm for Solving a System of Lane-Emden Equations Arising from Catalytic Diffusion Reactions

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***Abstract*—In this study, we investigate a system of coupled nonlinear Lane-Emden equations that arise from catalytic diffusion reactions. A highly effective algorithm, which heavily relies on the differential transform method, is proposed to solve this system. The algorithm produces a convergent series solution with components that can be easily computed. The Adomian polynomials corresponding to the given system are utilized for calculating the differential transforms of its nonlinearities with multiple variables. A practical numerical example is provided to validate the effectiveness and accuracy of the present scheme. The numerical results obtained by our developed approach show a significantly lower error rate compared to other existing approaches.**

***Index Terms*—Catalytic diffusion reactions, Lane-Emden equations, Differential transform method, Adomian polynomials.**

I. INTRODUCTION

IN this work, we consider a system of Lane-Emden singular equations with the form

$$\begin{cases} u''(x) + \dfrac{2}{x}u'(x) + f(u(x), v(x)) = 0, \\ v''(x) + \dfrac{2}{x}v'(x) + g(u(x), v(x)) = 0, \end{cases} \tag{1}$$

subject to the Neumann boundary conditions

$$u'(0) = v'(0) = 0, \tag{2}$$

and the Dirichlet boundary conditions

$$u(1) = k_1, v(1) = k_2, \tag{3}$$

where k_1 and k_2 are finite real constants. This issue frequently arises in the modeling of various real-world phenomena, such as chemical reactions, population evolution, pattern formation, and others [1], [2], [3], [4]. Various numerical methods have been proposed for solving the system (1)-(3). To name a few, the Adomian decomposition method was applied by Rach et al. in [5] to solve this issue, resulting in a series solution with easily computable components. In [6], an approximate solution of the system (1)-(3) was derived using a matrix method. In [7], Singh and Wazwaz transformed the given problem into an equivalent Fredholm integral equation to overcome its singular behavior at the origin. They then solved the integral equation using the homotopy analysis method (HAM). In [8], a fast-converging iterative scheme based on the optimized homotopy analysis method (OHAM) was discussed for solving problem (1)-(3). In [9], a new algorithm was developed to obtain numerical results for this problem by constructing Green's function before establishing the recursive scheme for the Adomian series solution. Furthermore, in [10], the Kansa collocation method based on radial basis functions was applied to approximate the solution of problem (1)-(3).

The differential transform method (DTM) was first introduced by Pukhov [11], [12], [13], and has since become a widely utilized semi-numerical-analytic approach for tackling various scientific problems. With this effective technique, nonlinear problems can be solved explicitly and numerically with minimal calculations, without the need for linearization, discretization or perturbation. The use of DTM and its modifications has proven successful in obtaining solutions for a wide range of scientific problems nowadays [14], [15], [16], [17], [18]. The technique is highly effective, yet it encounters certain challenges when tackling diverse equations. The main challenge lies in formulating a direct yet efficient approach to acquire the differential transforms of nonlinearities, such as $f(u(x), v(x))$ and $g(u(x), v(x))$, in system (1). There are several works [16], [19], [20], [21], [22] that pertain to this topic. However, the utilization of the scheme in references [19] and [20] for handling differential equations with two or more nonlinearities will inevitably lead to an increase in computational budget. On the other hand, the effective technique utilizing Adomian polynomials as presented in references [21] and [22] is only applicable to nonlinearities with a single variable instead of multiple variables. Excitingly, Xie et al. [16] have recently established a connection between the Adomian polynomials and differential transforms of multi-variables. The mathematical structures of both the Adomian polynomials and differential transforms for these nonlinearities have been demonstrated to be identical, differing only in their constant components rather than variable ones.

Inspired by the work of [16], our objective in this study is to employ the DTM combined with the Adomian polynomials for solving the mixed boundary value problem (1)-(3). The differential transforms of nonlinearities $f(u(x), v(x))$ and $g(u(x), v(x))$ in equations (1) will be computed utilizing the formula given in [16].

The rest of the paper is organized as follows: in the next section, we describe the relation between the differential transform of nonlinearities with multiple variables and its corresponding Adomian polynomials. Section 3 provides an algorithm for solving problem (1)-(3), while Section 4 discusses a practical problem to test the effectiveness of our

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TABLE I
THE BASIC OPERATIONS OF THE DTM.

Original function	Transformed function
$af(x) \pm bg(x)$	$aF(k) \pm bG(k)$
$f(x)g(x)$	$\sum_{n=0}^{k} F(n)G(k-n)$
$\mathrm{d}^n f(x)/\mathrm{d}x^n$	$\frac{(k+n)!}{k!} F(k+n)$
x^n	$\delta(k-n) = \begin{cases} 1, \text{ if } k=n, \\ 0, \text{ if } k \neq n. \end{cases}$
$\exp(x)$	$1/k!$
$\sin(ax+b)$	$a^k/k! \sin(k\pi/2 + b)$
$\cos(ax+b)$	$a^k/k! \cos(k\pi/2 + b)$

proposed scheme. Finally, we conclude this paper with a brief summary in Section 5.

II. DIFFERENTIAL TRANSFORM OF NONLINEARITIES WITH MULTIPLE VARIABLES

A. *Definition of differential transform*

The function $f(x)$ is assumed to be differentiable, and its differential transform at $x = 0$ is generally defined as

$$F(k) = \frac{1}{k!}\left(\frac{\mathrm{d}^k f(x)}{\mathrm{d}x^k}\right)\bigg|_{x=0}. \tag{4}$$

Meanwhile, the differential inverse transform of $F(k)$ is described as

$$f(x) = \sum_{k=0}^{\infty} F(k)x^k. \tag{5}$$

The function $f(x)$ is commonly represented as a truncated series in practical applications. It can be approximated as

$$f(x) \approx f_N(x) = \sum_{k=0}^{N} F(k)x^k. \tag{6}$$

For the sake of convenience, we have listed the fundamental operations in Table I, where $f(x)$ and $g(x)$ represent the original functions, while $F(k)$ and $G(k)$ correspond to their respective transformed functions, n is a nonnegative integer, and a, b are two real numbers.

B. *Differential transform of nonlinearities with multiple variables*

The Adomian decomposition method (ADM) has been widely acknowledged as a highly effective tool for solving various scientific problems, both linear and nonlinear. In this approach, the solution to the given problem is typically expressed as a series defined by

$$u = \sum_{m=0}^{\infty} u_m, \quad v = \sum_{m=0}^{\infty} v_m,$$

with the infinite series of polynomials

$$f_i(u,v) = \sum_{m=0}^{\infty} A_{m,i}, \quad i = 1,2.$$

for nonlinearities $f_i(u,v), i = 1,2$. The coefficients $A_{m,i}, i = 1,2$ are determined based on the solution components $u_0, u_1, \ldots; v_0, v_1, \ldots$, and are referred to as Adomian polynomials. The Adomian polynomials of two variables were given in references [23], [24] and [25] via the parametrization:

$$u(\lambda) = \sum_{m=0}^{\infty} u_m \lambda^m, \quad v(\lambda) = \sum_{m=0}^{\infty} v_m \lambda^m,$$

and

$$f_i\big(u(\lambda), v(\lambda)\big) = \sum_{m=0}^{\infty} A_{m,i}\lambda^m, \quad i = 1,2. \tag{7}$$

For the left side of (7), we apply the Taylor expansion at the point $\lambda = 0$ such that it yields

$$A_{n,i} = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} f_i\left(\sum_{m=0}^{\infty} u_m\lambda^m, \sum_{m=0}^{\infty} v_m\lambda^m\right)\bigg|_{\lambda=0}, \quad i = 1,2.$$

Recently, Xie et al. [16] established a connection between the Adomian polynomials and differential transforms of multi-variables. Their findings are summarized as follows:

Lemma 1. *Given a nonlinear function $f(u(x), v(x))$ coupled with the differential transform $F(k)$ and the Adomian polynomials $A_k (k = 0, 1, 2, \ldots)$, it holds that*

$$F(k) = A_k\big(U(0), \ldots, U(k); V(0), \ldots, V(k)\big), \tag{8}$$

where $U(k)$ and $V(k)$ are the transformed functions of $u(x)$ and $v(x)$, respectively.

This result demonstrates that the mathematical structure of Adomian polynomials and differential transforms for nonlinear functions is identical. Consequently, we can derive the differential transform of any nonlinearity by evaluating its corresponding Adomian polynomial using constant values instead of variable components.

III. ALGORITHM TO SOLVE PROBLEM (1)-(3)

In this section, we give a simple and effective algorithm to solve the problem (1)-(3) based on the DTM combined with the Adomian polynomials. We want to find the approximate solutions with the form of

$$u_N(x) = \sum_{k=0}^{N} U(k)x^k, \quad v_N(x) = \sum_{k=0}^{N} V(k)x^k, \tag{9}$$

where the coefficients

$$U(0), U(1), \ldots, U(N) \text{ and } V(0), V(1), \ldots, V(N)$$

are to be determined by using the following steps:

• Firstly, by combining the definition of the differential transform in (4) and the boundary value condition in (2), it reads

$$U(1) = V(1) = 0. \tag{10}$$

Suppose that

$$U(0) = \alpha \text{ and } V(0) = \beta, \tag{11}$$

where α, β are two real parameters to be determined.

• Secondly, we multiply both sides of (1) by the variable x in order to obtain

$$\begin{cases} xu''(x) + 2u'(x) + xf(u(x), v(x)) = 0, \\ xv''(x) + 2v'(x) + xg(u(x), v(x)) = 0. \end{cases} \tag{12}$$

Furthermore, the following recurrence relation can be derived by applying the differential transform (4) to (12) .

$$U(k+1) = -\frac{F(k-1)}{(k+1)(k+2)}, \ \ k = 1, 2, \ldots, N-1, \tag{13}$$

$$V(k+1) = -\frac{G(k-1)}{(k+1)(k+2)}, \ \ k = 1, 2, \ldots, N-1, \tag{14}$$

where $F(k)$ and $G(k)$ are the differential transforms of the nonlinear functions $f(u, v)$ and $g(u, v)$, respectively.

• Thirdly, we use Lemma 1 to compute $F(k)$ and $G(k)$ by means of their corresponding Adomian polynomials denoted by A_k^F and A_k^G, respectively.

$$F(k) = A_k^F, G(k) = A_k^G, \ \ k = 0, 1, 2, \ldots, N. \tag{15}$$

Furthermore, by substituting (15) into (13) and (14), and then combining relations (9)-(11), we can obtain the truncated series solutions of problem (1)-(3), namely,

$$u_N(x) = \alpha - \sum_{k=1}^{N-1} \frac{A_{k-1}^F}{(k+1)(k+2)} x^{k+1}, \tag{16}$$

$$v_N(x) = \beta - \sum_{k=1}^{N-1} \frac{A_{k-1}^G}{(k+1)(k+2)} x^{k+1}. \tag{17}$$

• Finally, by applying the truncated series solutions (16) and (17) to the boundary conditions (3), a system of nonlinear algebraic equation with unknown parameters α and β is obtained:

$$h_1(\alpha, \beta) = 0, h_2(\alpha, \beta) = 0. \tag{18}$$

Solving this system, and substituting the values of α and β into (16) and (17), we get the final result.

IV. A PRACTICAL MODEL FROM CATALYTIC DIFFUSION REACTIONS

To demonstrate the accuracy and efficiency of our proposed scheme, we consider a boundary value problem that arises in catalytic diffusion reactions:

$$\begin{cases} u''(x) + \frac{2}{x}u'(x) - k_{11}u^2(x) - k_{12}u(x)v(x) = 0, \\ v''(x) + \frac{2}{x}v'(x) - k_{21}u^2(x) - k_{22}u(x)v(x) = 0, \end{cases} \tag{19}$$

subject to the Neumann boundary conditions

$$u'(0) = v'(0) = 0, \tag{20}$$

and the Dirichlet boundary conditions

$$u(1) = k_1, v(1) = k_2. \tag{21}$$

Those parameters $k_1, k_2, k_{11}, k_{12}, k_{21}$ and k_{22} can be described for the real chemical reactions. Flocherzi and Sundmacher [4] have shown the existence of a solution to problem (19)-(21).

In the following, we will solve it by taking $k_1 = 1, k_2 = 2, k_{11} = 1, k_{12} = 2/5, k_{21} = 1/2$ and $k_{22} = 1$ with the proposed algorithm given in Section III.

Firstly, as mentioned in (10) and (11), we set

$$U(0) = \alpha, V(0) = \beta, \text{ and } U(1) = V(1) = 0. \tag{22}$$

The Adomian polynomials of nonlinearities $f(u(x), v(x)) = -u^2(x) - 2/5u(x)v(x)$ and $g(u(x), v(x)) = -1/2u^2(x) - u(x)v(x)$ in this problem are computed as

$$\begin{aligned} A_0^F &= -U^2(0) - \tfrac{2}{5}U(0)V(0), \\ A_1^F &= -2U(0)U(1) - \tfrac{2}{5}U(1)V(0) - \tfrac{2}{5}U(0)V(1), \\ A_2^F &= -U^2(1) - 2U(0)U(2) - \tfrac{2}{5}U(2)V(0) - \tfrac{2}{5}U(1)V(1) \\ &\quad -\tfrac{2}{5}U(0)V(2), \\ &\cdots \end{aligned}$$

and

$$\begin{aligned} A_0^G &= -\tfrac{1}{2}U^2(0) - U(0)V(0), \\ A_1^G &= -U(0)U(1) - U(1)V(0) - U(0)V(1), \\ A_2^G &= -\tfrac{1}{2}U^2(1) - U(0)U(2) - U(2)V(0) - U(1)V(1) \\ &\quad -U(0)V(2), \\ &\cdots \end{aligned}$$

respectively. Furthermore, we obtain the differential transforms $U(k)$ in the form of

$$\begin{aligned} &U(2) = \tfrac{1}{6}\alpha^2 + \tfrac{1}{15}\alpha\beta, \\ &U(4) = \tfrac{11}{600}\alpha^3 + \tfrac{1}{75}\alpha^2\beta + \tfrac{1}{750}\alpha\beta^2, \\ &U(6) = \tfrac{1}{560}\alpha^4 + \tfrac{349}{189000}\alpha^3\beta + \tfrac{41}{94500}\alpha^2\beta^2 + \tfrac{1}{78750}\alpha\beta^3, \\ &\cdots \\ &U(k) = 0, \text{ if } k \text{ is odd and } k \geq 3, \end{aligned}$$

and $V(k)$ in the form of

$$\begin{aligned} &V(2) = \tfrac{1}{12}\alpha^2 + \tfrac{1}{6}\alpha\beta, \\ &V(4) = \tfrac{1}{80}\alpha^3 + \tfrac{1}{50}\alpha^2\beta + \tfrac{1}{300}\alpha\beta^2, \\ &V(6) = \tfrac{211}{151200}\alpha^4 + \tfrac{173}{75600}\alpha^3\beta + \tfrac{47}{63000}\alpha^2\beta^2 + \tfrac{1}{31500}\alpha\beta^3, \\ &\cdots \\ &V(k) = 0, \text{ if } k \text{ is odd and } k \geq 3, \end{aligned}$$

by applying relations (13), (14), and (15). Therefore, the truncated series solutions for $N = 6$ can be derived with the help of relations (16) and (17). Specifically,

$$\begin{aligned} u_6(x) = \alpha &+ (\tfrac{1}{6}\alpha^2 + \tfrac{1}{15}\alpha\beta)x^2 \\ &+ (\tfrac{11}{600}\alpha^3 + \tfrac{1}{75}\alpha^2\beta + \tfrac{1}{750}\alpha\beta^2)x^4 \\ &+ (\tfrac{1}{560}\alpha^4 + \tfrac{349}{189000}\alpha^3\beta + \tfrac{41}{94500}\alpha^2\beta^2 \\ &+ \tfrac{1}{78750}\alpha\beta^3)x^6 \end{aligned} \tag{23}$$

and

$$\begin{aligned} v_6(x) = \beta &+ (\tfrac{1}{12}\alpha^2 + \tfrac{1}{6}\alpha\beta)x^2 \\ &+ (\tfrac{1}{80}\alpha^3 + \tfrac{1}{50}\alpha^2\beta + \tfrac{1}{300}\alpha\beta^2)x^4 \\ &+ (\tfrac{211}{151200}\alpha^4 + \tfrac{173}{75600}\alpha^3\beta + \tfrac{47}{63000}\alpha^2\beta^2 \\ &+ \tfrac{1}{31500}\alpha\beta^3)x^6. \end{aligned} \tag{24}$$

Secondly, by imposing the truncated series solutions (23) and (24) on the Dirichlet boundary conditions (21), we obtain a nonlinear algebraic equation. Solving it yields the unknown parameters α and β as

$$\alpha = 0.7816027253 \text{ and } \beta = 1.690947017. \tag{25}$$

Finally, by substituting (25) into (23) and (24), we can get the approximate solutions of problem (19)-(21) with a

TABLE II
NUMERICAL VALUES OF $\alpha = u(0)$ AND $\beta = v(0)$.

6	0.7816027253	1.690947017
8	0.7813963345	1.690695454
10	0.7813752563	1.690670401
12	0.7813731705	1.690667967
14	0.7813729689	1.690667735
16	0.7813729497	1.690667713
18	0.7813729479	1.690667711
20	0.7813729478	1.690667711
22	0.7813729478	1.690667711
24	0.7813729478	1.690667711

degree of six as follows:

$$\begin{aligned} u_6(x) =& 0.7816027253 + 0.1899270565x^2 \\ &+ 0.02550703547x^4 + 0.002963182759x^6, \\ v_6(x) =& 1.690947017 + 0.2711833678x^2 \\ &+ 0.03407808768x^4 + 0.003791527200x^6. \end{aligned}$$

Proceeding as before, we have also computed the approximate solutions for $N = 8$ through 24 by a step size of 2. In Table II, we have listed the values of α and β for different N. It can be observed that when sufficiently large N is taken into account, α is almost identical to 0.7813729478 and β is almost identical to 1.690667711. This implies that the unknown solutions $u(x)$ and $v(x)$ of problem (19)-(21) satisfy $u(0) = 0.7813729478$ and $v(0) = 1.690667711$. This performance demonstrates the effectiveness of our present scheme.

Furthermore, as an exact solution for problem (19)-(21) is lacking, we instead assess accuracy by examining the absolute residual error functions and maximal error remainder parameters. The absolute residual error functions are defined as

$$|ER_N^{u(x)}(x)| = |u_N''(x) + \frac{2}{x}u_N'(x) - u_N^2(x) - \frac{2}{5}u_N(x)v_N(x)|$$

and

$$|ER_N^{v(x)}(x)| = |v_N''(x) + \frac{2}{x}v_N'(x) - \frac{1}{2}u_N^2(x) - u_N(x)v_N(x)|,$$

while the maximal error remainder parameters are

$$MER_N^{u(x)} = \max_{0 \le x \le 1} |ER_N^{u(x)}(x)|$$

and

$$MER_N^{v(x)} = \max_{0 \le x \le 1} |ER_N^{v(x)}(x)|.$$

In Figures 1 and 2, we have plotted the absolute error remainder functions $|ER_N^{u(x)}(x)|$ and $|ER_N^{v(x)}(x)|$ for N values ranging from 10 to 20 in increments of 2. Tables III and IV compare the maximal error remainder parameters obtained by the present method with those obtained using other existing approaches, which include Adomian decomposition method (ADM) [5], modified Adomian decomposition method (MADM) [5], and optimized homotopy analysis method (OHAM) [8]. From the two tables, it is evident that our computational results have an advantage over those obtained by the aforementioned methods. In addition,

TABLE III
COMPARISON OF THE MAXIMAL ERROR REMAINDER PARAMETERS $MER_N^{u(x)}$.

N	ADM [5]	MADM [5]	OHAM [8]	Present method
6	2.3094e-1	6.4341e-1	1.2697e-2	2.5843e-2
8	1.2671e-1	4.7917e-2	5.6250e-3	4.0190e-3
12	4.1605e-2	1.0941e-2	1.6953e-4	7.2909e-5
18	9.1302e-2	1.7990e-3	2.5134e-6	1.2008e-7
24	2.2402e-3	2.7100e-4	3.9764e-8	3.2820e-10

logarithmic plots of the maximal error remainder parameters $MER_N^{u(x)}$ and $MER_N^{v(x)}$ for $N = 6$ to 24 with a step size of 2 are presented in Figures 3 and 4. The points on both graphs lie on a straight line, indicating an approximately exponential rate of convergence.

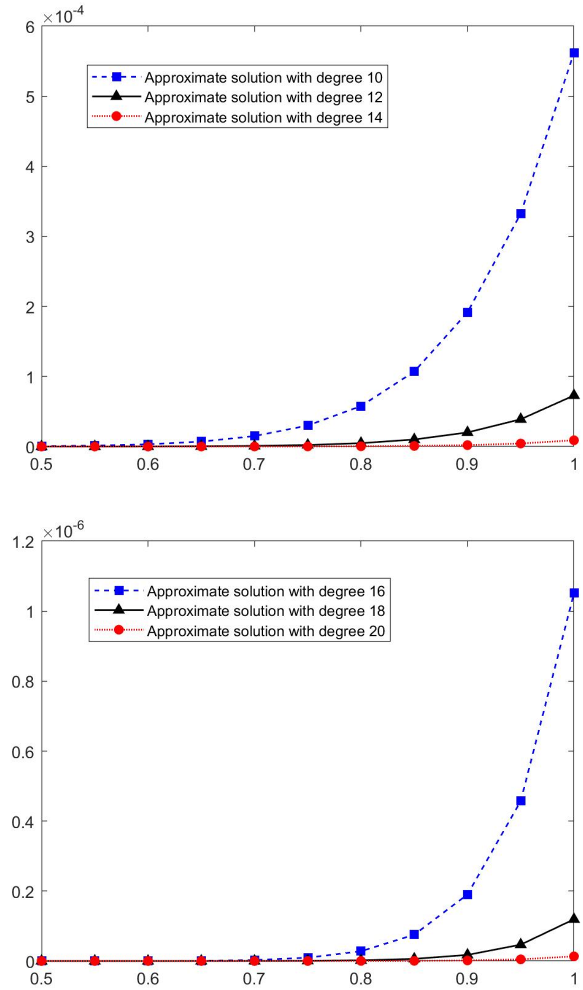
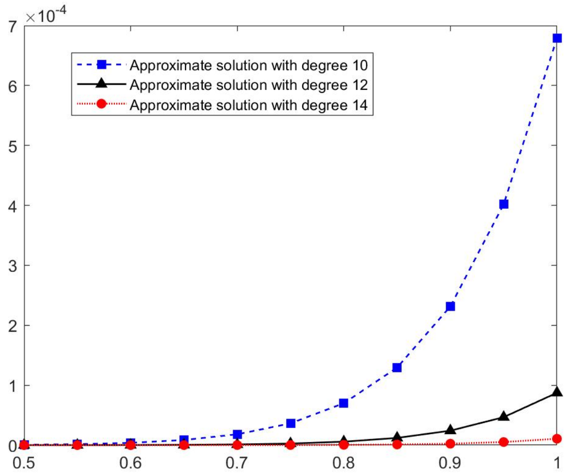


Fig. 1. The curves of absolute residual error functions $|ER_N^{u(x)}(x)|$ for $N = 10, 12, 14$ (up) and $N = 16, 18, 20$ (down).

V. CONCLUSION

The improved DTM, which combines the DTM with Adomian polynomials to handle nonlinear functions, has been successfully employed in this study to solve the system of Lane-Emden equations derived from catalytic diffusion reactions. The proposed method effectively addresses the



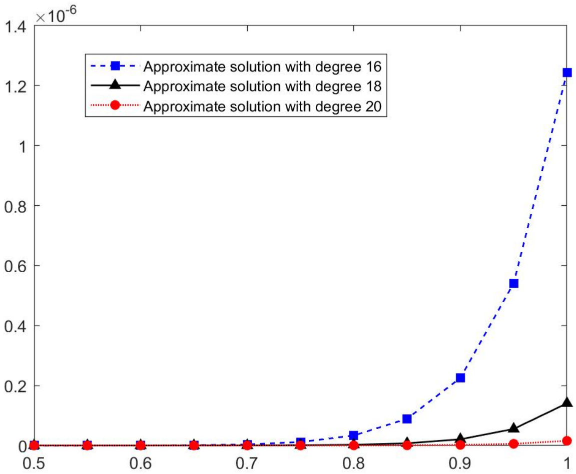


Fig. 2. The curves of absolute residual error functions $|\mathrm{ER}_N^{v(x)}(x)|$ for $N = 10, 12, 14$ (up) and $N = 16, 18, 20$ (down).

TABLE IV
COMPARISON OF THE MAXIMAL ERROR REMAINDER PARAMETERS $MER_N^{v(x)}$.

N	ADM [5]	MADM [5]	OHAM [8]	Present method
6	2.9383e-1	8.5109e-1	1.2606e-2	3.2094e-2
8	1.5989e-1	6.0905e-2	7.7651e-3	4.9121e-3
12	5.2026e-2	1.3647e-2	1.1949e-4	8.7319e-5
18	1.1343e-2	2.0949e-3	1.9772e-6	1.4070e-7
24	2.7736e-3	3.1128e-4	2.6049e-8	4.1787e-10

computational challenges associated with existing methods, as all calculations can be performed through straightforward manipulations without the need for linearization, discretization, or perturbation. Numerical results show that our proposed scheme works well with a satisfyingly low error compared to other existing approaches. The precision of the acquired solution can be enhanced by integrating additional terms. Furthermore, the Adomian polynomials have overcome the challenge of classical DTM in dealing with nonlinear terms involving multiple variables. The proposed technique for evaluating the differential transform of such functions involves only basic arithmetic calculations and the computation of Adomian polynomials, which is anticipated

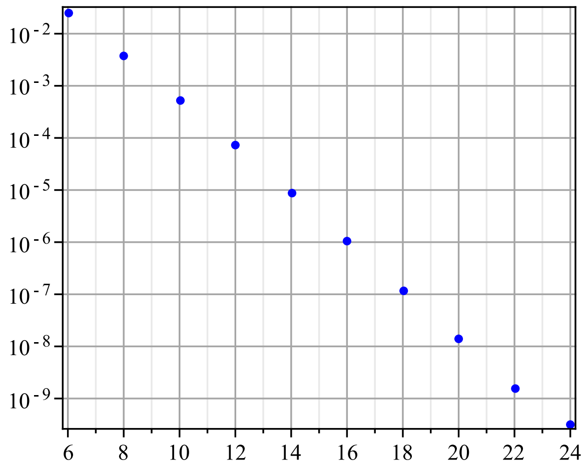


Fig. 3. The logarithmic plot for the maximal error remainder parameters $MER_N^{u(x)}$ for $N = 6$ through 24 by step 2.

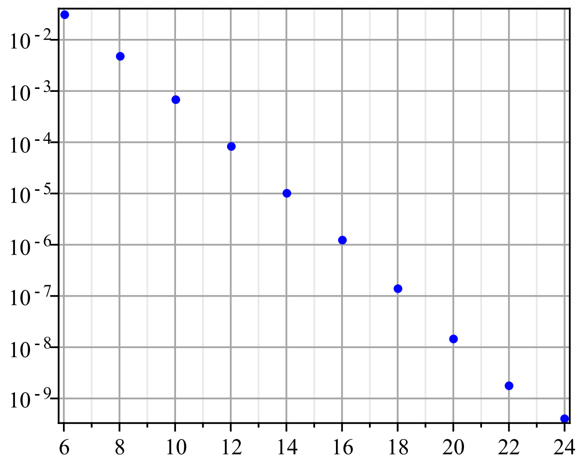


Fig. 4. The logarithmic plot for the maximal error remainder parameters $MER_N^{v(x)}$ for $N = 6$ through 24 by step 2.

to expand the applications of DTM.

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