Clique Free Number of a Graph

Surekha Ravishankar Bhat, Ravishankar Bhat and Smitha Ganesh Bhat*

***Abstract*—A complete maximal subgraph of a graph H is designated as a clique. A set $S \subseteq V$ is clique free if $\langle S \rangle$, the subgraph induced by the set S does not induce any clique of H. The clique free number $\beta_{vc} = \beta_{vc}(H)$ is the maximum order of a clique free set of H. In this present work, we have deduced few bounds for cilque free number and have substantiated the graphs attaining the same. Also, a Gallai's theorem type result for clique free number is proved and Konig-Egervarey Theorem is extended to clique free sets. An algorithm to find all the maximal clique free sets is derived.**

***Index Terms*—Independent sets, clique transversal sets, clique free sets, clique independent sets.**

I. Introduction

FOR any of the unspecified terminologies refer [1], [25]. By citing a graph H, we connote a connected simple finite graph with q edges and p vertices. A set S is a dominating set of H if and only if, $N(w) \cap D \neq \phi$ for every $w \in V - D$. The domination number $\gamma(H)$ is the minimum number of vertices in a dominating set. Comprehensive survey of domination theory is available in [4], [11]. On the contrary, a set $S \subseteq V$ is said to be independent if none of any two vertices in S are adjacent. The *independence number* $\beta_0 = \beta_0(H)$ (*independent domination number* $i = i(H)$) is the maximum (minimum) order of a maximal independent set of H. A vertex v is a *cutpoint* if $H - v$ is disjoint (unconnected) A maximal subgraph of H with no cutpoint is a *block*. A complete maximal subgraph of H is designated as a *clique*. Let $K(H)$ denote the set of all cliques in H and $|K(H)| = k$. A graph H is referred to as a *block graph* if any and every block of H is a clique. The *minimum clique number* $\vartheta = \vartheta(H)$ (*maximum clique number* $\omega = \omega(H)$) is the order of a minimum (maximum) clique of H. A graph H is t*-clique regular* if $\omega(H) = \vartheta(H) = t$. The properties of clique regular graphs have been studied in [3]. Any triangle free graph without isolates is 2-clique regular. Any *wheel graph* W_n is 3-clique regular. A *wind mill graph* $Wd(n, k)$ is a graph with k copies of complete graph K_n adjacent at a single vertex. In particular, $Wd(3, k) = F_k$ is called the *friendship graph*. Any windmill graph $Wd(n, k)$ is n-clique regular. A *generalized star* denoted $S(n, k)$ is a windmill graph in which each K_n has $n-1$ vertices in common. The corona of two graphs H_1 and H_2 is the graph $H = H_1 \cdot H_2$. It is developed from one copy of H_1 and $|V(H_1)|$ copies of H_2, where the i^{th} vertex of H_1 is conterminous to each and every vertex in i^{th} copy of H_2. Any corona $K_m \cdot K_{m-1}$ is a m-clique regular graph.

A *clique graph* $K_H(H)$ of H is a graph with vertex set as cliques of H and any two vertices in $K_H(H)$ are adjacent if and only if the corresponding cliques in H have a vertex in common. A graph H is called a *clique path* if $K_H(H)$ is a path. Similarly, H is a *clique cycle* if $K_H(H)$ is a cycle. A clique path, two different types of clique cycles and a generalized star are shown in the Fig. 1. Further, H is *clique complete* if any two cliques have atleast one vertex in common. If H is clique complete then $K_H(H)$ is complete. Any generalized star $S(n, k)$ and any windmill graph $Wd(n, k)$ are examples of clique complete graphs.

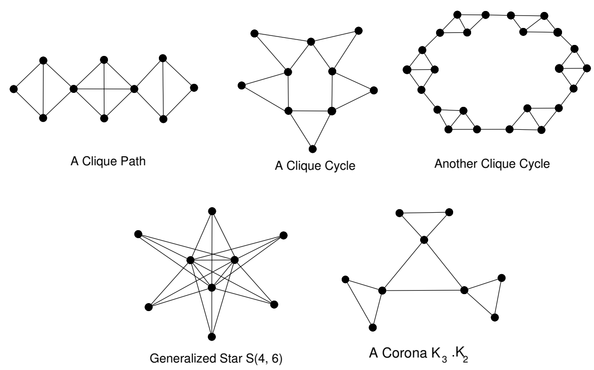

Fig. 1. Examples of a clique path, two different types of clique cycles, a generalized star and a Corona

Furthermore, a comprehensive investigation into the characteristics of cliques in graph structures has been conducted by Surekha et.al [24], Sayinath Udupa N V [22] and Tana et. al [26]. In a parallel line of research, Isabel Cristina Lopes et. al [12] have also explored this intriguing topic.

A. Clique Transversal number

A vertex $v \in V$ and an edge $x \in E$ are said to cover each other if v is incident on the edge x. Minimum number of vertices that cover all the edges of a graph is called the *vertex covering number* $\alpha_0 = \alpha_0(H)$. In 1990, the concept of vertex covering is extended as clique transversal number, defined and studied by Tuza [27], and later by Erdos *et.al* [9] in 1992. A vertex $v \in V$ and a clique $h \in K(H)$ are said to cover each other if v is incident on the clique h. Minimum number of verices that cover all the cliques of H is called *clique transversal number* $\tau_c = \tau_c(H)$. We immediately note that for any triangle free graph, $\alpha_0(H) = \tau_c(H)$.

For any $v \in V$ the *open neighborhood* $N(v) = \{u \in V | u$ is adjacent to $v\}$ and the *closed neighborhood* $N[v] = N(v) \cup \{v\}$. Then degree $d(v) = |N(v)|$. Let $\Delta(H)$ and $\delta(H)$ denote the maximum and minimum degree of H respectively. If $\overline{H}$ denote the complement of H then it is well known that

$$\Delta(H) + \delta(\overline{H}) = \Delta(\overline{H}) + \delta(H) = p - 1. \qquad (*)$$

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Surekha Ravishankar Bhat is Professor at Department of Mathematics, Milagres College, Kallianpur, Udupi, Karnataka, India-574111 (email: surekharbhat@gmail.com)

Ravishankar Bhat is former Professor at Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India-576104 (e-mail: ravishankar.bhats@gmail.com)

Smitha Ganesh Bhat is Assistant Professor at Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India-576104 (corresponding author to provide phone: 9844061970, e-mail: smitha.holla@manipal.edu)

II. CLIQUE FREE NUMBER

We recall the definition of VB independent sets defined in [2]. A set $D \subseteq V$ is said to be VB independent if $\langle D \rangle$ the induced subgraph induced by the set D, does not contain any block of H. The VB independence number $\beta_{vb} = \beta_{vb}(H)$ is the order of a maximum VB independent set of H. On the similar lines we define clique free number of a graph as follows. A set $S \subseteq V$ is said to be *clique free set* if $\langle S \rangle$ the subgraph induced by the set S does not induce any clique of H. The *clique free number* $\beta_{vc} = \beta_{vc}(H)$ is the maximum order of a clique free set of H.

As every independent set is clique free, $\beta_0(H) \leq \beta_{vc}(H)$. Note that if $\langle S \rangle$ is free from edges we get independent sets. If H is any triangle free graph, then every edge is a clique of H and hence we have $\beta_{vc}(H) = \beta_o(H)$. Let $C(H)$ denote the set of cut points of H and $|C(H)| = n$. If H is any block graph, then the set of non-cut points $V - C(H)$ does not induce any clique of H and hence $\beta_{vc} \geq p - n$. For any windmill graph $H = Wd(n, k)$, has $(n-1)k$ non-cut points and only one cut point. Therefore $\beta_{vc}(H) = (n-1)k = |V - C(H)|$.

A property satisfied by set $S \subseteq V$ is called *hereditary* if every subset of S obeys the same property and is called *superhereditary* if every superset of S obeys the same property. We observe that the property of clique freeness is a hereditary property and clique transversal property is superhereditary in the sense that every subset of a clique free set is also clique free and every superset of a clique transversal set is also a clique transversal set of H.

Properties of n-independent sets are studied by the same authors see [23]. We say that a vertex $v \in V$, *n-covers* an edge $x \in E$, if $x \in \langle N[v] \rangle$. A set $S \subseteq V$ is said to be *n-independent* if every edge in $\langle S \rangle$ is n-covered by a vertex in $V - S$. The *n-independence number* $\beta_n = \beta_n(H)$ is the maximum order of n-independent set. It is proved that $\beta_0(H) \leq \beta_n(H)$, We prove that clique free number β_{vc} fits best in between the two.

Proposition II.1. *For any graph H,*

$$\beta_0(H) \leq \beta_{vc}(H) \leq \beta_n(H) \quad (1)$$

.

Proof: Since every independent set is a clique free set, and every clique free set is n-independent set the result follows. ■

Example 2.1

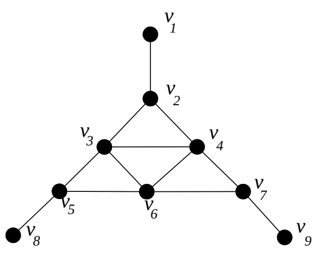

Fig. 2. A graph H with β_{vc}=5

For the graph H in Fig. 2, $S_1 = \{v_1, v_8, v_9, v_6\}$ is a β_0-set, $S_2 = \{v_1, v_8, v_9, v_6, v_3\}$ is a β_{vc}-set and $S_3 = \{v_1, v_8, v_9, v_6, v_3, v_4\}$ is a β_n-set of H, Thus $\beta_0(H) = 4 < \beta_{vc}(H) = 5 < \beta_n(H) = 6$ and hence for the graph H strict inequality in equation (1) holds. Equality in equation (1) holds for any triangle free graph J as, $\beta_0(J) = \beta_{vc}(J) = \beta_n(J)$.

A. *An algorithm to find maximal clique free sets*

An algorithm to find all maximal independent sets of a graph is developed (see [18]) using boolean arithmetic. We extend the same algorithm with few modifications to find all maximal clique free sets in any graph. Let us treat each vertex in the graph as a boolean variable. Let $a + b$ denote the logical (or boolean) sum which indicates the process of including vertex a or b or both. Let ab denote the logical multiplication of operation of including both vertices a and b, and let the boolean complement a' denote that vertex a is not included. We make use of the following identities of boolean Algebra.

(i) $a + a = a$ and $aa = a$ (*Idempotent Laws*)
(ii) $a + (ab) = a$ and $a(a + b) = a$ (*Absorbtion Identity*)

For a given graph H we must find a maximal subset of vertices that doesnot include all the vertices of any clique in H. Let us express a clique $k = (v_1, v_2, \ldots, v_t)$ as a boolean product, $(v_1 v_2 ....v_t)$. Let us sum all products in H to get a boolean expression

$$\phi = \sum (v_1 v_2 ....v_t) \qquad \text{for all } (v_1, v_2, \ldots, v_t) \text{ in } H$$

Let us further take the boolean complement ϕ' of this expression, and express it as a sum of boolean products:

$$\phi' = f_1 + f_2 + \cdots + f_n$$

A vertex set is a maximal clique free set if and only if $\phi = 0$ (logically false) which is possible if and only if $\phi' = 1$(true), which is possible if and only if at least one $f_i = 1$, which is possible if and only if each vertex appearing in f_i (in complement form) is excluded from the vertex set of H. Thus, each f_i will yeild a maximal clique free set and every maximal clique free set will be produced by this method. This procedure can be best explained by an example.

Example 2.2

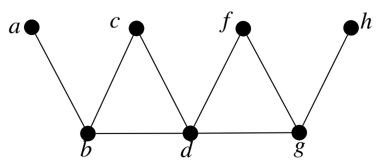

Fig. 3. A clique path H_1 with 7 maximal clique free sets

Consider the clique path graph H_1 shown in the Fig. 3. Let us sum all the cliques in H_1 to get

$$\phi = ab + bcd + dfg + gh.$$

Then the boolean complement ϕ' is given by

$$\begin{aligned}\phi' &= (a'+b')(b'+c'+d')(d'+f'+g')(g'+h') \\ &= (a'b'+a'c'+a'd'+b'+b'c'+b'd') \\ &\qquad (d'+f'+g')(g'+h') \\ &= (b'+a'c'+a'd')(d'+f'+g')(g'+h') \\ &= (b'd'+b'f'+b'g'+a'c'd'+a'c'f'+a'c'g' \\ &\qquad +a'd'+a'd'f'+a'd'g')(g'+h') \\ &= (a'd'+b'g'+b'd'+b'f'+a'c'f'+a'c'g') \\ &\qquad (g'+h') \\ &= a'c'g'+a'c'f'h'+b'g'+a'd'g'+a'd'h' \\ &\qquad +b'd'h'+b'f'h'.\end{aligned}$$

Now if we exclude from the vertex set of H_1, vertices appearing in any one of these seven terms, we get a maximal clique free set. Thus, the seven maximal clique free sets of the graph H_1 are

$$\{bdfh\}, \{bdg\}, \{acdfh\}, \{bcfh\}, \{bcfg\}, \{acfg\}, \{acdg\}$$

Once all the maximal clique free sets of H have been obtained, we find the order of the one with the largest number of vertices to get the clique free number. Therefore $\beta_{vc}(H_1) = 5$.

B. Independent set Transeversal number and Independence free number

The above discussion leads to define the following. Minimum number of vertices that cover all the maximal independent sets of H is called *independent set transversal number* $\tau_i = \tau_i(H)$. On the otherhand a set $S \subseteq V$ is *independent set free* if no maximal independent set is contained in S. The *independence free number* $\beta_i = \beta_i(H)$ is the maximum order of an independent set free set.

Proposition II.2. *For any graph H,*

$$\tau_i(H) = \tau_c(\overline{H})$$
$$\beta_i(H) = \beta_{vc}(\overline{H})$$

Proof: The result follows from the fact that independent sets and cliques exchange their properties on complementation. ■

We now proceed to find a lower bound for clique free number.

Proposition II.3. *For any graph H,*

$$\Delta \le \beta_{vc}$$

Equality holds iff $N(v)$ is a maximum clique free set of H for every $v \in V_\Delta$ where $V_\Delta = \{u \in V | deg(v) = \Delta\}$

Proof: If v is a vertex of maximum degree Δ, then $\langle N(v) \rangle$ is clique free. Therefore $S = N(v)$ is a clique free set. Thus $\beta_{vc} \ge |S| = |N(v)| = \Delta$

If $N(v)$ is a maximum clique free set of H for every $v \in V_\Delta$ then $\beta_{vc}(H) = |N(v)| = \Delta$.

Conversely, suppose $\beta_{vc}(H) = \Delta$. If possible $N(v)$ is not a maximum clique free set of H for some $v \in V_\Delta$, then there exists at least one vertex $u \in V$ such that $N(v) \cup \{u\}$ is a clique free set of H. Hence $\beta_{vc} \ge N(v) \cup \{u\} = \Delta + 1 > \Delta$ a contradiction. ■

Applying Proposition II.2 in the above proposition and and using the result $(*)$, we have the following

Corollary II.3.1. *For any graph H,*

$$p - \delta - 1 \le \beta_i$$

III. Gallai's Theorem Type Results

In 1959, the graph parameters vertex covering number and independence number are related by the well known, now classical Gallai's Theorem [10], $\alpha_0(H) + \beta_0(H) = p$. Since then, the study of Gallai's type results took momentum and sveral authors got interested and gave similar results, for example see [5], [15], [20], [21]. We now prove another result similar to Gallai's Theorem. We need following lemma.

Lemma III.1. *Let H be a (p, q) graph. Then a set $S \subseteq V$ is a clique transversal set of H if and only if $V - S$ is a clique free set of H.*

Proof: Let S be a clique transversal set of H. Since every clique is covered by some vertex in S, atleast one vertex of every clique is in S. Hence $\langle V - S \rangle$ cannot induce any clique of H. Thus $V - S$ is clique free set.

Conversely, let S be a clique free set of H. Suppose that $V - S$ is not a clique transversal-set. Then there exists atleast one clique $h \in K(H)$ which is not covered by any vertex in $V - S$. This implies all the vertices of h are in S. Then $\langle S \rangle$ contains the clique h - a contradiction. ■

Theorem III.2. *For any graph H with p vertices,*

$$\tau_c(H) + \beta_{vc}(H) = p \quad (2)$$
$$\tau_i(H) + \beta_i(H) = p \quad (3)$$

Proof: Let S be a τ_c-set of H. Then from Lemma III.1, $V - S$ is a clique free set of H. Hence $\beta_{vc} \ge |V - S| = p - \tau_c$. Therefore $\tau_c + \beta_{vc} \ge p$...... .....(i)

On the otherhand let D be a β_{vc}-set of H. Then again from Lemma III.1, we have $V - D$ is clique transversal set of H. Hence $\tau_c \le |V - D| = p - \beta_{vc}$. Therefore $\tau_c + \beta_{vc} \le p$ ....(ii)

Then the result (2) follows from (i) and (ii). On complementing the equation (2) and applying Proposition II.2, the result (3) follows. ■

A. cc-independent sets

A set of cliques are said to be *clique-clique independent (cc-independent)* if no two cliques have a vertex in common. Similarly, a set of maximal independent sets are said to be *ii-independent* if they are pairwise disjoint. The *cc-independence number* $\beta_{cc} = \beta_{cc}(H)$ (*ii-independence number* $\beta_{ii} = \beta_{ii}(H)$) is the maximum order of a cc-independent (ii-independent) set of H. It is immediate that $\beta_{cc}(H) = \beta_{ii}(\overline{H})$. For any triagle free graph $\beta_{cc}(H) = \beta_1(H)$ and $\tau_c(H) = \alpha_0(H)$. It is proved that $\alpha_0(H) \ge \beta_1(H)$ (see [17]). Similarly, we now show that $\beta_{cc}(H)$ and $\tau_c(H)$ are comparable.

Proposition III.3. *For any graph H,*

$$\tau_c(H) \ge \beta_{cc}(H)$$
$$\tau_i(H) \ge \beta_{ii}(H)$$

Proof: Let $\{k_1, k_2, \ldots, k_t\}$ be the β_{cc}-set of H. Choose $v_i \in k_i, i = 1, 2, ...t$ such that the $d(v_i)$ is as big as possible. Let $S = \{v_1, v_2, \ldots, v_t\}$ be the set of vertices so obtained. If S covers all the cliques of H then S is τ_c-set of H and $\beta_{cc}(H) = \tau_c(H)$. If not, we consider another set S_1 of minimum number of vertices that cover all those cliques not coverd by the set S. Then $S \cup S_1$ forms a τ_c set of H. Therefore $\tau_c(H) = |S \cup S_1| \geq |S| = \beta_{cc}(H)$. The second result follows on complementation. ■

B. *Extended Konig-Egervary Theorem*

A min-max relation is a theorem stating equality between the answers to a minimization problem and a maximization problem over a class of instances. The Konig- Egervary Theorem [16], [8] is such a relation for matching and vertex covering which states that if H is a bipartite graph then $\beta_1(H) = \alpha_0(H)$. We extend this theorem to the newly defined parameters clique free number and clique transeversal number. We need the following definitions before we proceed.

The operation of clique removal from a block graph is defined as follows. Let H be a block graph and h be any clique of H. Then $H - \{h\}$ is a graph obtained by removing all the unicliqual vertices and edges incident on the clique h. A clique h is a pendant clique if h is incident on only one cut point. Let H_1 be the graph obtained by removing all pendant cliques of H. Then a cut point v in H is said to be an end-cut point if v is not a cut point in H_1. Note that the above operations are defined only on block graph and cannot be defined for any general graph.

Similar to Konig-Egervery Theorem we have the following.

Proposition III.4. *If H is either a bipartite graph or a block graph, then*

$$\tau_c(H) = \beta_{cc}(H)$$

$$\tau_i(H) = \beta_{ii}(H)$$

Proof: If H is a bipartite graph then it is triangle free. Hence $\tau_c(H) = \alpha_0(H) = \beta_1(H) = \beta_{cc}(H)$ and the result follows. Therefore, we assume H is any block graph. If H has no cut points then $H = \cup_{i=1}^{k} H_i$ where each H_i is a complete graph. In this case $\tau_c(H) = k = \beta_{cc}(H)$ holds. So, we need to consider any connected graph H with n cut points. We prove the result by induction on number of cut points n.

Suppose H is a block graph with $n = 1$ cut point and m cliques. Then H must be a clique complete graph. Consequently, $\tau_c(H) = 1 = \beta_{cc}(H)$. Let H be a block graph with m cliques and $n = 2$ cut points say, c_1, c_2. Then there exists at least two pendant cliques k_1 incident on c_1 and k_2 incident on c_2. Now clearly, $\{c_1, c_2\}$ is a τ_c-set of H and $\{k_1, k_2\}$ is a $\beta_{cc}-$set of H and $\tau_c(H) = 2 = \beta_{cc}(H)$. Thus, the result hold for primary values of $n = 1, 2$. We assume the result is true for all the graphs with less than n cut points. Suppose H be a block graph with n cut points. Consider any end-cut point v and J be the graph obtained by removing all the cliques incident on v. Then J is a graph with less than n cut points and by induction hypothysis, we have $\tau_c(J) = \beta_{cc}(J) = t$. Let D_1 be the τ_c-set of J and L_1 be the $\beta_{cc}-$set of J. Since v is an end-cut point, there exists at least one pendant clique say h incident on v. Then $D = D_1 \cup \{v\}$ is a τ_c-set of H. Again, the set $L = L_1 \cup \{h\}$ is a $\beta_{cc}-$set of H. Thus $\tau_c(H) = |D| = |D_1 \cup \{v\}| = t+1 = |L_1 \cup \{h\}| = |L| = \beta_{cc}(H)$. Hence by induction principles the result is true for all n.

Again, the second result follows on complementation. ■

A factor of a graph H is a spanning subgraph of H. A k-factor of H is a k-regular spanning subgraph of H. A clique covering is a factor of H. A clique covering in which every clique is of order ϑ, is a $(\vartheta - 1)$-regular spanning subgraph of H and hence is a $(\vartheta - 1)$-factor of H. In particular a 1-factor is a perfect matching. In the next result we obtain an upper bound for cc-independence number in terms of minimum clique number ϑ. In what follows, by $V(J)$ we mean the vertex set of the graph J.

Proposition III.5. *For any graph H,*

$$\beta_{cc} \leq \frac{p}{\vartheta} \tag{4}$$

Further, $\beta_{cc} = \frac{p}{\vartheta}$ *if and only if* H *has a* $(\vartheta - 1)$*-factor.*

Proof: Let $\beta_{cc} = t$ and $L = \{k_1, k_2, \ldots, k_t\} \subseteq K(H)$ be the β_{cc}-set of H. Let $|k_i|$ denote the order of the clique k_i. Then $|k_i| \geq \vartheta(H)$, $1 \leq i \leq t$. Since each k_i, $1 \leq i \leq t$ is clique independent, any two cliques in L are mutually disjoint. As there can be some vertices not incident on any of the cliques $k_i \in L$ we have $V(k_1 \cup k_2 \cup \cdots \cup k_t) \subseteq V$. Therefore, $t\vartheta(H) \leq |k_1| + |k_2| + \cdots + |k_t| = |V(k_1 \cup k_2 \cup \cdots \cup k_t)| \leq |V(H)| = p$ proving the desired inequality.

Suppose that H has a $(\vartheta - 1)$-factor. Then H has a clique covering in which every clique is of order ϑ. Let $L = \{k_1, k_2, \ldots, k_t\}$ be such a partition of H. This partition forms the maximum clique independent set of H. Then it is immediate that $t\vartheta(H) = p$. Conversely, let $\beta_{cc} = \frac{p}{\vartheta}$. Then we show that the β_{cc}-set is the required $(\vartheta - 1)$-factor of H. Suppose that β_{cc}-set doesnot form such a partition. Then there are two possibilities.

Case (i). β_{cc}-set doesnot cover all the vertices of H. Let $L = \{k_1, k_2, \ldots, k_t\}$ be a β_{cc}-set of H. Then $V(k_1 \cup k_2 \cup \cdots \cup k_t) \subset V$. Hence $\beta_{cc}(H)\vartheta(H) < p$, - a contradiction.

Case (ii). β_{cc}-set forms a partition but each is not of order $\vartheta(H)$. In this case, there exists atleast one k_i such that $\vartheta(H) < |k_i|$ for some $1 \leq i \leq t$. Then again, we get that $\beta_{cc}(H)\vartheta(H) < p$, - a contradiction. This completes the proof. ■

The bound is sharp for any complete graph K_n, for any even cycle C_{2n} and for any corona $K_m \cdot K_n$ with $m \geq n$.

The following well known result for matching number is straight forward from the above proposition.

Corollary III.5.1. *For any graph H,*

$$\beta_1 \leq \frac{p}{2}$$

Further, $\beta_1 = \frac{p}{2}$ *if and only if* H *has a* 1*-factor.*

Proposition III.6. *For any graph H,*

$$\beta_{cc}(H) \leq \frac{\beta_{vc}}{\vartheta - 1} \tag{5}$$

Further, the bound is sharp.

Proof: Let $\beta_{cc} = t$ and $\{k_1, k_2, ..., k_t\}$ be the β_{cc}-set. Let V_i be the vertex set of k_i, $1 \leq i \leq t$. Then $|V_i| \leq \vartheta$ for every $1 \leq i \leq t$. Therefore, union of $(\vartheta - 1)$ vertices taken from each V_i form a β_{vc}-set of H. Thus $(\vartheta - 1)t \leq \beta_{vc}$. ■

For the m-clique regular graph the corona $H = K_m \cdot K_{m-1}$, we have $\beta_{cc}(H) = m = \dfrac{m(m-1)}{m-1} = \dfrac{\beta_{vc}}{\vartheta - 1}$ and the bound is sharp in equation (5). Also, any cycle C_n and any generalized star $H = S(n, k)$ attain the bound in equation (5)

Proposition III.7. *For any graph H,*

$$\beta_{cc} \leq \beta_{vc}(H) \leq p - \beta_{cc}(H)$$

Proof: From Proposition III.6, $\beta_{cc} \leq \dfrac{\beta_{vc}}{\vartheta - 1} \leq \beta_{vc}$....... ......(A)

Then from Proposition III.3 we have $\tau_c \geq \beta_{cc}$. Then using Theorem III.2, we get $p - \beta_{vc} \geq \beta_{cc}$. This yeilds $\beta_{vc}(H) \leq p - \beta_{cc}(H)$. ....(B)

Combining (A) and (B) we get the desired result. ■

C. Nordhaus-Gaddum Type inequality

Relationship between graphical parameters of great interest and important. Nordhaus and Gaddum [19] began the study of sum and product of the chromatic numbers $\chi(H)$ and $\chi(\overline{H})$. Later similar results for different parameters are studied in [7], [13]. We provide such a relationship for clique transversal numbers $\tau_c(H)$ and $\tau_c(\overline{H})$.

Proposition III.8. *For any graph H,*

$$\tau_c(H) + \tau_c(\overline{H}) \leq p + 1 \quad (6)$$

$$p - 1 \leq \beta_{vc}(H) + \beta_{vc}(\overline{H}) \quad (7)$$

Further the bound is sharp.

Proof: From Theorem III.2 we have $p - \tau_c(H) = \beta_{vc}(H) \geq \Delta$. This implies $\tau_c(H) \leq p - \Delta$. Hence $\tau_c(H) + \tau_c(\overline{H}) \leq p - \Delta + p - \overline{\Delta} = p - \Delta + p - (p - \delta - 1)$ using result $(*)$
$= p + 1 - (\Delta - \delta) \leq p + 1$ which yeilds the result (6)

To establish the result (7): We have $p + 1 \geq \tau_c(H) + \tau_c(\overline{H}) = p - \beta_{vc}(H) + p - \beta_{vc}(\overline{H})$ using Theorem III.2. Hence $\beta_{vc}(H) + \beta_{vc}(\overline{H}) \geq 2p - (p+1) = p - 1$ as desired.

For any complete graph, K_n, $\tau_c(K_n) + \tau_c(\overline{K_n}) = 1 + n$. Similarly, $\beta_{vc}(K_n) + \beta_{vc}(\overline{K_n}) = n - 1 + 0 = n - 1$. Thus the bound is sharp in (6) and (7) for any complete graph. ■

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Surekha Ravishankar Bhat is Professor in the department of Mathematics, Milagres College, Kallianpur, Udupi. She obtained Ph.D from Manipal university in 2013. Her research fields are Inverse domination in graphs, Cliques and block domination parameters in graphs.

Ravishankar Bhat is former Professor in the department of Mathematics, Manipal institute of Technology, Manipal. He obtained Ph.D from National Institute of Technology, Surathkal in the year 2007. He guided two Ph.D students. His research fields are Domination in graphs, Energy of a graphs, Strong independent sets, strong covering sets, Cliques and Neighbourhood number.

Smitha Ganesh Bhat is Assistant Professor in the department of Mathematics, Manipal institute of Technology, Manipal. She obtained Ph.D from Manipal university in 2019. She obtained M.Sc in Mathematics from Mangalore University in the year 2007. Her research fields are Domination in graphs, Cliques and Neighbourhood number.