The M/M/1 Working Vacation Queueing System with N-policy and Different Arrival Rates

Shengli Lv, *Member, IAENG*, Jingyi Wen, Man Yang

***Abstract*—We study the M/M/1 working vacation queueing system with N-policy and different arrival rates in this paper. The arrival rate of customers in a different period is different. Using the method of matrix geometry solution, We have given the steady-state performance indicators of this system. In addition, we also get the conditional stochastic decomposition structures of the queue length as well as the waiting time. The last part of the article show the numerical analyses and the trend of each performance indicators.**

***Index Terms*—N-policy, different arrival rates, stochastic decomposition, matrix geometric solution, working vacation.**

I. Introduction

In the queueing system, the idle state of the server station can be effectively took advantage of avoiding the waste of resources, so the queueing model with various vacation strategies has an important application in some practical systems, and has become a research hotspot of queueing theory. A detailed introduction can be found in the review papers [1], [2] or monographs [3]. A lot of scholars have made extensive and in-depth study on the classical vacation queueing system [4], [5]. As early as 20 years ago, Servi and Finn [6] introduced a new vacation policy that it was a semi-vacation policy: when a server served customers in a vacation, they served at a lower rate and didn't stop the service. They defined such a vacation policy as working vacation. They got relevant performance indicators. This queueing systems have triggered various studies on discrete-time working vacation queueing. Chen and Jia [7] obtained PGF and the mean of queue length during departure periods. Lv et al. [8] considered the customers' input rates impacted on the queue length in the system. [9] and [10] researched a queueing model with a vacation interruption, working breakdowns and repairs. Yang and Tian [11] built a mathematical model to research the N-policy queueing system. Then they used matrix geometry solution method to obtain the results. Secondly, a conditional stochastic decomposition structure of the queue length and wait time distribution was given. In addition, the queueing model with N-policy was also studied. V. Karthick and V. Suvitha [12] analyzed the repairable Markovian queueing systems which have three service counters with servers on vacation. Under the N-policy, when the system has on customers, the service station stops all work and enters the idle period (if it is a vacation queue, it enters the vacation period). When the number of customers in the system gets to the conversion threshold N, the system resumes normal work and enters the busy period. [13] and [14] considered the retrial queueing system and obtained the steady-state indexes and probability distributions. Ma, Li and Wei [15] studied the social optimization in working vacation queueing model with N-policy. Wang and Zhu [16] analyzed the queue with negative customers and different arrival rates. Studied on other more general models include: Tang et al. [17] analyzed the calculation of the stable length with delay N-policy. [18] and [19] considered the discrete-time case of [17], that is, the calculation problem of the queueing stable length with a delay N-policy. Finally, the research on the queueing model with N-policy can also refer to the literature cited above.

The core content of the theory of vacation queueing system is stochastic decomposition. The performance indicators in the queueing systems can usually be divided into two independent random variables. One of which is the indicator with the same name in the classical system with on vacation, and the other is the additional random variable caused by vacation. So this process is called the stochastic decomposition. The various stochastic decomposition results and the methods that lead to them are a prominent feature of the study of vacation queueing. The stochastic decomposition makes the comparison between the working vacation queueing systems and the classical queueing system clear. It is convenient to analyze the influence of various vacation strategies on the classical queueing model. Since the classical queueing system has been studied in depth. Stochastic decomposition method transforms the analysis of the vacation queueing into the study of additional random variables, thus simplifying the steady-state. Stochastic decomposition is not only important in the theoretical analysis of vacation queueing, but also provides great convenience for practical applications under various backgrounds. Considering that the transition between two states usually requires a certain amount of expenses, the accumulation of a certain number of customers before implementing the transition may produce better economic benefits. When the vacation was over, if the number of customers in the system is equal or great than N, the server will start to service customers. Otherwise, the server will start the other independent and equally distributed vacation. This is a N-policy multiple vacation rule. In daily life, people line up to get on the bus at the station, queue up to see a doctor at the hospital and queue up to handle business at the bank. Such service systems all have one thing in common, that is, the arrival of customers has a strong time-varying feature. The long waiting process not only affects the service

Manuscript received March 24, 2023; revised October 7, 2023.

This work was supported by the National Nature Science Foundation of China (No. 71971189).

Shengli Lv is an associate professor in School of Science, Yanshan University, Qinhuangdao, Hebei 066004, PR China. (e-mail: qhdddlsl@163.com).

Jingyi Wen is a postgraduate student in the School of Science, Yanshan University, Qinhuangdao, Hebei 066004, PR China. (corresponding author to provide e-mail: wenjingyi0609@163.com).

Man Yang is a postgraduate student in School of Science, Yanshan University, Qinhuangdao, Hebei 066004, PR China. (e-mail: y159158@126.com).

efficiency but also easily causes customer dissatisfaction. Therefore, the change of customer arrival rates and the imbalance of service rates have become a research focus of many scholars.

In summary, the queueing systems in the real word will change with the changes of the system state and the arrival rates of customers. Based On the above literatures, we study the queueing system with N-policy, in which a customer arrival rate changes with the queueing system's state. Performance indicators in this system were got by using matrix geometry solution [20] method.

II. Model Description

1) The customer arrival obeys the Poisson process. The customer arrival rate is λ_1 in normal busy period (state 1) as well as λ_2 in working vacation period (state 0).

2) The service time is subject to exponential distribution and abides the exponential distribution of with μ_1 in state 1. During state 0, the server doesn't stop working completely, and serves customers at a rate $\mu_2\,(\mu_2 < \mu_1)$.

3) Once this system has no customers, the server immediately enters a working vacation of random length V. In the end of the working vacation, if the number of customers is j $(j < N)$ in the system, server will again enter a separate, evenly distributed working vacation. Continue to serve the customers after the working vacation; Otherwise, if the number of customers is equal to or great than N (N is given in advance) in the system, the servers are going to immediately stop working vacation, and the service rate will increase from μ_2 to μ_1. It begins a normal busy period, until the service station recovers idle again. The length V of the vacation obeys the exponential distribution of the parameter θ.

4) We assumed that all processes are independent of each other. In addition, the queueing system has only one service station and the service discipline is first-come-first-served ($FCFS$).

Let $Q(t)$ be the number of customers in the system at the moment t, and $J(t)$ be the state of service stations at the moment t.

$$J(t) = \begin{cases} 0, \text{the system is in the working vacation} \\ \quad \text{period at time t,} \\ 1, \text{the system is in the normal busy} \\ \quad \text{period at time t.} \end{cases}$$

Then $\{Q(t), J(t)\}$ is a quasi-birth-and-death process (QBD) with the state space $\Omega = \{(0,0)\} \cup \{(k,j),\ k \geq 1,\ j = 0,1\}$.

The state transition diagram for this system is shown in Figure 1.

From the state transition diagram, the infinitesimal generator $\mathcal{Q}$ can be written as

$$\mathcal{Q} = \begin{pmatrix} A_{00} & C_{00} & & & & & \\ B_{10} & A_0 & C & & & & \\ & B & A_0 & C & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & B & A & C & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1}$$

where

$A_{00} = -\lambda_2$, $C_{00} = \begin{pmatrix} \lambda_2 & 0 \end{pmatrix}$, $B_{10} = \begin{pmatrix} \mu_2 \\ \mu_1 \end{pmatrix}$,

$B = \begin{pmatrix} \mu_2 & 0 \\ 0 & \mu_1 \end{pmatrix}$, $C = \begin{pmatrix} \lambda_2 & \\ & \lambda_1 \end{pmatrix}$,

$A_0 = \begin{pmatrix} -(\mu_2+\lambda_2) & 0 \\ 0 & -(\mu_1+\lambda_1) \end{pmatrix}$,

$A = \begin{pmatrix} -(\mu_2+\lambda_2+\theta) & \theta \\ 0 & -(\mu_1+\lambda_1) \end{pmatrix}$.

To analyze this QBD process, the minimal non-negative solution of matrix equation [20]

$$R^2B + RA + C = 0. \tag{2}$$

Theorem 1. If $\rho = \lambda_1{\mu_1}^{-1} < 1$, the matrix equation $R^2B + RA + C = 0$ has the minimal non-negative solution

$$R = \begin{pmatrix} r & \frac{\theta r}{\mu_1(1-r)} \\ 0 & \rho \end{pmatrix}, \tag{3}$$

where r is the root of equation $\mu_2{r_{11}}^2 - (\mu_2+\lambda_2+\theta)\,r_{11} + \lambda_2 = 0$ and $0 < r < 1$.

Proof Assume that R has the upper triangular matrix, let

$$R = \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}.$$

Taking R^2 and R into Eq. (2), the following system of equations can be obtained:

$$\begin{cases} \mu_2{r_{11}}^2 - (\mu_2+\lambda_2+\theta)\,r_{11} + \lambda_2 = 0, \\ \mu_1 r_{12}\,(r_{11}+r_{22}) + \theta r_{11} - (\mu_1+\lambda_1)\,r_{12} = 0, \\ \mu_1{r_{22}}^2 - (\mu_1+\lambda_1)\,r_{22} + \lambda_1 = 0. \end{cases} \tag{4}$$

r_{11}can be derived from the first equation of Eq. (4)

$$r_{11} = \frac{(\mu_2+\lambda_2+\theta) \pm \sqrt{(\mu_2+\lambda_2+\theta)^2 - 4\mu_2\lambda_2}}{2\mu_2}. \tag{5}$$

Evidently, $r_{11} < 1$ when the negative sign works, and $r_{11} > 1$ when the positive sign works. We take the negative sign to get the minimum non-negative solution R,. From the third equation of Eq. (4)

$$\begin{aligned} r_{22} &= \frac{(\mu_1+\lambda_1) - \sqrt{(\mu_1+\lambda_1)^2 - 4\mu_1\lambda_1}}{2\mu_1} \\ &= \frac{(\mu_1+\lambda_1) - |\mu_1-\lambda_1|}{2\mu_1} = \frac{\lambda_1}{\mu_1} = \rho. \end{aligned} \tag{6}$$

We can derive $r_{22} = \rho < 1$. Taking $r_{11} = r$ and $r_{22} = \rho < 1$ into the second equation of Eq. (4), we get the expression for $r_{12} = \frac{\theta r}{\mu_1(1-r)}$, Theorem 1 is proved.

We can obtain that r satisfies the following relation:

$$\lambda_2 + \theta + \mu_2(1-r) = \mu_2 + \frac{\theta}{1-r} = \frac{\lambda_2}{r}. \tag{7}$$

III. Queue Length Distribution

If $\rho = \lambda_1{\mu_1}^{-1} < 1$, let $\prod$ be the steady-state probability vector of $\{Q_v, J\}$, then

$$\prod = (\pi_0, \pi_1, \pi_2, \cdots),$$

and

$$\pi_{kj} = \lim_{t\to\infty} P\{Q_v(t) = k, J(t) = j\},\ (k,j) \in \Omega$$

the steady-state probability is

$$\pi_0 = (\pi_{00}), \pi_k = (\pi_{k0}, \pi_{k1}),\ k \geq 1.$$

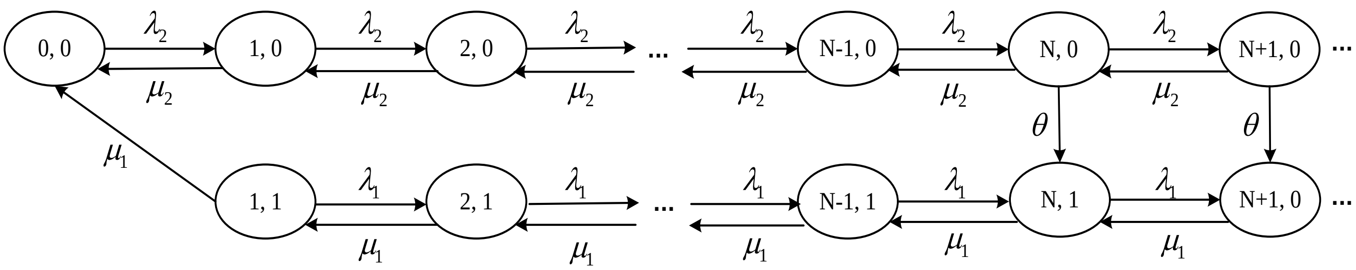

Figure 1. The state transition diagram of the system

Theorem 2. If $\rho = \lambda_1 {\mu_1}^{-1} < 1$, the stationary probability distribution meets the following equations

$$\begin{cases} \pi_{j0} = K \cdot \left(\frac{\lambda_2}{\lambda_2 - \mu_2} \left(\frac{\theta r}{\lambda_2 (1-r)} - (1-r) \left(\frac{\mu_2}{\lambda_2} \right)^{N-j} \right) \right), \\ j = 0, 1, 2, \cdots, N-1, \\ \pi_{j1} = K \cdot \frac{\theta r}{\mu_1 (1-r)} \cdot \frac{1-\rho^j}{1-\rho}, j = 1, 2, \cdots, N-1, \\ \pi_{ko} = K \cdot r^{k-N+1}, \ k = N, \cdots, \\ \pi_{k1} = K \cdot \frac{\theta r}{\mu_1 (1-r)} \left(\sum_{j=0}^{k-N} r^j \rho^{k-N-j} + \frac{1-\rho^{N-1}}{1-\rho} \rho^{k-N+1} \right), \\ k = N, \cdots, \end{cases} \tag{8}$$

where

$$K = [(N-1) \left(1 - \left(\frac{\mu_2}{\lambda_2} \right)^{-1} \right) \left(1 - \frac{r \mu_2}{\lambda_2} \right) \left(1 - \frac{\mu_2}{\mu_1} \right) \frac{1}{1-\rho} + \frac{1}{1-\rho} \frac{1}{1-r} \left(1 - \frac{\mu_2}{\mu_1} r \right) - (1-r) \left(\left(\frac{\mu_2}{\lambda_2} \right)^2 - \left(\frac{\mu_2}{\lambda_2} \right)^{N+1} \right) \left(1 - \frac{\mu_2}{\lambda_2} \right)^{-2}]^{-1}.$$

Proof In terms of the matrix-geometric solution method [20] (Neuts,1981), we have

$$\begin{aligned} \pi_k &= \pi_{N-1} R^{k-N+1} \\ &= \begin{pmatrix} \pi_{N-1,0} & \pi_{N-1,1} \end{pmatrix} R^{k-N+1}, \ N-1 \leq k, \end{aligned} \tag{9}$$

and $(\pi_{00}, \pi_{10}, \pi_{11}, \cdots, \pi_{\mathrm{N-1,0}}, \pi_{\mathrm{N-1,1}})$ satisfies the set of equations

$$(\pi_{00}, \pi_{10}, \pi_{11}, \cdots, \pi_{\mathrm{N-10}}, \pi_{\mathrm{N-11}}) B[R] = 0, \tag{10}$$

where

$$B[R] = \begin{pmatrix} A_{00} & C_{00} & & & \\ B_{10} & A_0 & C & & \\ & \ddots & \ddots & \ddots & \\ & & B & A_0 & C \\ & & & B & RB + A \end{pmatrix},$$

$$RB + A = \begin{pmatrix} r\mu_2 - (\mu_2 + \lambda_2 + \theta) & \frac{\theta}{1-r} \\ 0 & \mu_1 \rho - (\mu_1 + \lambda_1) \end{pmatrix}.$$

Taking $B[R]$ into the above relation, we obtain the set of equations

$$\begin{cases} -\lambda_2 \pi_{00} + \mu_2 \pi_{10} + \mu_1 \pi_{11} = 0, \\ -(\lambda_1 + \mu_1) \pi_{11} + \mu_1 \pi_{21} = 0, \\ \lambda_2 \pi_{j-i,0} - (\lambda_2 + \mu_2) \pi_{j,0} + \mu_2 \pi_{j+1,0} = 0, \\ \lambda_2 \pi_{N-2,0} + [r\mu_2 - (\lambda_2 + \mu_2 + \theta)] \pi_{N-1,0} = 0, \\ \lambda_1 \pi_{j-i,1} - (\lambda_1 + \mu_1) \pi_{j,1} + \mu_1 \pi_{j+1,1} = 0, \\ \lambda_1 \pi_{N-2,1} + \frac{\theta}{1-r} \pi_{N-1,0} + [\mu_1 \rho - (\lambda_1 + \mu_1)] \pi_{N-1,1} = 0. \end{cases} \tag{11}$$

Taking $\pi_{N-1} = K$, from the fifth equation of Eq. (11)

$$\mu_1 (\pi_{j+1,1} - \pi_{j,1}) = \lambda_1 (\pi_{j,1} - \pi_{j-1,1}),$$

$$\pi_{j+1,1} - \pi_{j,1} = \frac{\lambda_1}{\mu_1} (\pi_{j,1} - \pi_{j-1,1}) = \rho (\pi_{j,1} - \pi_{j-1,1}).$$

From the second equation of Eq. (11), $\mu_1 (\pi_{21} - \pi_{11}) = \lambda_1 \pi_{11}$, is that

$$\pi_{21} - \pi_{11} = \frac{\lambda_1}{\mu_1} \pi_{11} = \rho \pi_{11}.$$

We get

$$\pi_{j+1,1} - \pi_{j,1} = \rho^j \pi_{11},$$

$$\pi_{j,1} = \sum_{i=0}^{j-1} \rho^i \pi_{11} = \frac{1-\rho^j}{1-\rho} \pi_{11}.$$

$\pi_{j,1}$and $\mu_1 \pi_{11} = \frac{\theta r}{1-r} \pi_{\mathrm{N-1,0}}$ are added in turn, we have

$$\pi_{j,1} = \frac{\theta r}{\mu_1 (1-r)} \cdot \frac{1-\rho^j}{1-\rho} \pi_{\mathrm{N-1,0}}, \ j \leq N-1.$$

From the third equation of Eq. (11)

$$\lambda_2 (\pi_{j-1,0} - \pi_{j,0}) = \mu_2 (\pi_{j,0} - \pi_{j+1,0}),$$

$$\pi_{j-1,0} - \pi_{j,0} = \frac{\mu_2}{\lambda_2} (\pi_{j,0} - \pi_{j+1,0}) = \left(\frac{\mu_2}{\lambda_2} \right)^{N-j} \pi_{N-1,0}.$$

From the fourth equation of Eq. (11)

$$\pi_{N-2,0} - \pi_{N-1,0} = \frac{\mu_2 (1-r) + \theta}{\lambda_2},$$

$$\pi_{N-j,0} - \pi_{N-j-1,0} = (1-r) \left(\frac{\mu_2}{\lambda_2} \right)^{j-1} \pi_{N-1,0}.$$

Thus,

$$\pi_{N-j,0} = \frac{\lambda_2}{\lambda_2 - \mu_2} [r \left(1 - \frac{\mu_2}{\lambda_2} \right) + (1-r) \left(1 - \left(\frac{\mu_2}{\lambda_2} \right)^j \right)] \pi_{N-1,0}, \ j \leq N,$$

$$\pi_{N-j,0} = \frac{\lambda_2}{\lambda_2 - \mu_2} [\frac{\theta r}{\lambda_2 (1-r)} - (1-r) \left(1 - \left(\frac{\mu_2}{\lambda_2} \right)^j \right)] \pi_{N-1,0}, \ j \leq N.$$

Notice

$$R^{k-N+1} = \begin{pmatrix} r^{k-N+1} & \frac{\theta r}{\mu_1 (1-r)} \cdot \frac{1-\rho^j}{1-\rho} \pi_{N-1,0} \\ 0 & \rho^{k-N+1} \end{pmatrix}.$$

Substituting $(\pi_{N-1,0}, \pi_{N-1,1})$ and R^{k-N+1} into Eq. (9), we can obtain Eq. (11). Through the normalization condition, we obtain the constant factor K.

IV. The Conditional Stochastic Decomposition Structure

Theorem 3. If $\rho = \lambda_1 {\mu_1}^{-1} < 1$, the conditional random variable ${Q_v}^{(N)}$ can be decomposed into ${Q_v}^{(N)} = Q_0 + Q_d$, where Q_0 is the conditional queue length of the classical M/M/1 queue without vacation, and $Q_{\rm d}$ is the additional queue length caused by working vacation [11]. The distribution function is

$$P\left\{Q_d = k\right\} = \begin{cases} \frac{1}{\xi}\left(\frac{1}{1-\rho} + \frac{\left(1-\rho^{N-1}\right)\rho}{(1-\rho)^2}\right), \ k=0, \\ \frac{1}{\xi}\cdot\frac{1}{1-\rho}\cdot r^k, \ k \geq 1, \end{cases} \tag{12}$$

where

$$\xi = \frac{1}{1-r}\cdot\frac{1}{1-\rho} + \frac{\rho\left(1-\rho^{N-1}\right)}{(1-\rho)^2}.$$

Proof The probability of the system in state 1 and the number of customers not less than N,

$$\begin{aligned} &P\left\{Q_v \geq N, J=1\right\} \\ &= \sum_{k=N}^{\infty} \pi_{k1} \\ &= K\cdot\frac{\theta r}{\mu_1(1-r)}\left[\frac{1}{1-r}\cdot\frac{1}{1-\rho} + \frac{\rho\left(1-\rho^{N-1}\right)}{(1-\rho)^2}\right] \\ &= K\cdot\frac{\theta r}{\mu_1(1-r)}\cdot\xi. \end{aligned}$$

Thus, we have

$$\begin{aligned} &P\left\{{Q_v}^{(N)} = k\right\} \\ &= P\left\{Q_v - N = k | Q_v \geq N, J=1\right\} \\ &= \frac{1}{\xi}\left(\sum_{j=0}^{k} r^j\rho^{k-j} + \frac{1-\rho^{N-1}}{1-\rho}\cdot\rho^{k+1}\right), \ k = 0,1,\cdots. \end{aligned}$$

The PGF of ${Q_v}^{(N)}$ is

$$\begin{aligned} &{Q_v}^{(N)}(z) \\ &= \sum_{k=0}^{\infty} z^k P\left\{{Q_v}^{(N)} = k\right\} \\ &= \frac{1-\rho}{1-\rho z}\cdot\frac{1}{\xi}\cdot\left(\frac{1}{1-\rho}\cdot\frac{1}{1-r}\cdot\frac{1-r}{1-rz} + \frac{\rho\left(1-\rho^{N-1}\right)}{(1-\rho)^2}\right) \\ &= Q_0(z)\cdot Q_d(z), \end{aligned}$$

where $Q_0(z) = \frac{1-\rho}{1-\rho z}$, then the PGF of Q_d is

$$Q_d(z) = \frac{1}{\xi}\cdot\left(\frac{1}{1-\rho}\cdot\frac{1}{1-r}\cdot\frac{1-r}{1-rz} + \frac{\rho\left(1-\rho^{N-1}\right)}{(1-\rho)^2}\right).$$

Expanding it to a power function of z can have the distribution of $Q_{\rm d}$.

The mean additional queue length of system

$$E(Q_d) = {Q_d}'(z)|_{z=1} = \frac{1}{\xi}\left[\frac{1}{1-\rho}\frac{r}{(1-r)^2}\right].$$

Then, we get

$$\begin{aligned} &E\left({Q_v}^{(N)}\right) \\ &= E(Q_0) + E(Q_d) \\ &= \frac{\rho}{1-\rho} + \frac{1}{\xi}\left[\frac{1}{1-\rho}\frac{r}{(1-r)^2}\right]. \end{aligned} \tag{13}$$

Theorem 4. If $\rho = \lambda_1 {\mu_1}^{-1} < 1$, the conditional random variable ${W_v}^{(N)}$ can be decomposed into ${W_v}^{(N)} = W_0 + W_{\rm N-1} + W_d$. Where W_0 is the conditional waiting time of the classical M/M/1 queue without vacation, and W_{N-1} obeys the Erlang distribution of parameters μ_1 and $N-1$. And W_d is the additional waiting time caused by working vacation [11]. The distribution function

$$W_d(x) = 1 - \frac{1}{\xi}\frac{1}{1-\rho}\frac{r}{1-r}e^{-\mu_1(1-r)x}, \ x \geq 0, \tag{14}$$

where

$$\xi = \frac{1}{1-r}\cdot\frac{1}{1-\rho} + \frac{\rho\left(1-\rho^{N-1}\right)}{(1-\rho)^2}.$$

Proof If a customer arrive under state $(j,1)$, the LST of ${W_j}^{(N)}$ is

$${W_j}^{(N)*}(s) = \left(\frac{\mu_1}{s+\mu_1}\right)^j, \ j \geq N.$$

The LST of ${W_v}^{(N)}$

$$\begin{aligned} &{W_v}^{(N)*}(s) \\ &= \sum_{k=0}^{\infty}\left(\frac{\mu_1}{s+\mu_1}\right)^{N+k} P\left\{{Q_v}^{(N)} = k\right\} \\ &= \frac{1}{\xi}\sum_{k=0}^{\infty}\left(\frac{\mu_1}{s+\mu_1}\right)^{N+k}\left(\sum_{j=0}^{k} r^j\rho^{k-j} + \frac{1-\rho^{N-1}}{1-\rho}\cdot\rho^{k+1}\right) \\ &= {W_0}^*(s)\,{W_{N-1}}^*(s)\,{W_d}^*(s), \end{aligned}$$

where ${W_0}^*(s) = \frac{\mu_1(1-\rho)}{s+\mu_1(1-\rho)}, {W_{N-1}}^*(s) = \left(\frac{\mu_1}{s+\mu_1}\right)^{N-1}$, and ${W_d}^*(s) = \frac{1}{\xi}\left[\frac{1}{1-\rho}\cdot\frac{r}{1-r}\cdot\frac{\mu_1(1-r)}{s+\mu_1(1-r)} + \frac{1-\rho^N}{(1-\rho)^2}\right]$.

Then, we get the PGF of W_d caused by the working vacation

$${W_d}^*(s) = \frac{1}{\xi}\left[\frac{1}{1-\rho}\cdot\frac{r}{1-r}\cdot\frac{\mu_1(1-r)}{s+\mu_1(1-r)} + \frac{1-\rho^N}{(1-\rho)^2}\right].$$

V. The Busy Period Analysis and the Other Performance Indicators

A. The Busy Period Analysis

We used B to represent the period that the server continues to serve customers with μ_1. C is denoted as full working vacation that the period between two sequential normal busy periods. It consists of several vacations V. A busy cycle is expressed by C, where $C = B + V_g$. Since the distribution of V is memoryless, if the number of customers during the vacation isn't less than N, the normal busy period may start at any time, so

$$P\left\{Q_b = k\right\} = P\left\{Q_v = k | j=0, Q_v \geq N\right\}, \ k \geq N,$$

$$P\left\{j=0, Q_v \geq N\right\} = K\cdot\frac{r}{1-r}.$$

Thus, the distribution, PGF and mean function of Q_b are respectively

$$P\left\{Q_b = k\right\} = (1-r)\,r^{k-N}, \ k \geq N, \tag{15}$$

$$Q_b(z) = \frac{(1-r)\,z^N}{1-rz} \tag{16}$$

and

$$E(Q_b) = N + \frac{r}{1-r}. \tag{17}$$

B_0 represents the classical busy period of M/M/1 queue without vacation, then

$$B_0{}^*(s) = B_b{}^*\left(s + \lambda_1\left(1 - B_0{}^*(s)\right)\right),$$

$$B_0{}^*(s) = \frac{\mu_1}{\mu_1 + s + \lambda_1\left(1 - B_0{}^*(s)\right)},$$

$$E(B_0) = \frac{1}{\mu_1(1-\rho)}.$$

When $Q_b = k$ and $k \geq N$, then a normal busy period begins, so

$$B^*(s) = Q_b\left(B_0{}^*(s)\right) = \frac{(1-r)\left(B_0{}^*(s)\right)^N}{1 - rB_0{}^*(s)},$$

$$E(B) = E(Q_b)\,E(B_0) = \left(N + \frac{r}{1-r}\right)\frac{1}{\mu_1(1-\rho)}.$$

Using the joint probability distribution and the limit theorem of the alternating renewal process, we get

$$\begin{aligned} &P\{J=1\} = \tfrac{E(B)}{E(C)} \\ &= \sum_{j=1}^{N-1} K\tfrac{1-\rho^j}{1-\rho}\tfrac{\theta r}{\mu_1(1-r)} + \\ &\sum_{k=N}^{\infty} K\tfrac{\theta r}{\mu_1(1-r)}\left(\sum_{j=0}^{k-N} r^j\rho^{k-N-j} + \tfrac{1-\rho^{N-1}}{1-\rho}\rho^{k-N+1}\right) \\ &= K\tfrac{1}{1-\rho}\tfrac{\theta r}{\mu_1(1-r)}\left[(N-1) + \tfrac{1}{1-r}\right]. \end{aligned} \tag{18}$$

$$\begin{aligned} &P\{J=0\} = \tfrac{E(V_g)}{E(C)} \\ &= \sum_{j=0}^{N-1} K\left(\tfrac{\lambda_2}{\lambda_2-\mu_2}\left(\tfrac{\theta r}{\lambda_2(1-r)} - (1-r)\left(\tfrac{\mu_2}{\lambda_2}\right)^{N-j}\right)\right) \\ &\quad + \sum_{k=N}^{\infty} K\cdot r^{k-N+1} \\ &= K[(N-1)\left(1-\tfrac{\mu_2}{\lambda_2}\right)^{-1}\left(1-\tfrac{r\mu_2}{\lambda_2}\right) \\ &\quad + (1-r)\tfrac{\mu_2}{\lambda_2}\left(1-\left(\tfrac{\mu_2}{\lambda_2}\right)^N\right)\left(1-\tfrac{\mu_2}{\lambda_2}\right)^{-2}]. \end{aligned} \tag{19}$$

Substituting $E(B)$ into Eq. (19), we have $E(C) = \frac{1-r}{K\theta r}$. From the relationship between B, V_g and C we get

$$\begin{aligned} &E(V_g) = E(C) - E(B) = \tfrac{1-r}{K\theta r} - \left(N + \tfrac{r}{1-r}\right)\tfrac{1}{\mu_1(1-\rho)} \\ &= [(N-1)\left(1-\tfrac{\mu_2}{\lambda_2}\right)^{-1}\left(1-\tfrac{r\mu_2}{\lambda_2}\right) \\ &\quad + (1-r)\tfrac{\mu_2}{\lambda_2}\left(1-\left(\tfrac{\mu_2}{\lambda_2}\right)^N\right)\left(1-\tfrac{\mu_2}{\lambda_2}\right)^{-2}]\cdot\tfrac{1-r}{\theta r}. \end{aligned}$$

$E(V_g)$ is dependent on λ_2, θ and μ_2 but independent on μ_1. This is in line with the reality that the length of full vacation has no relationship to service rates during regular busy periods.

B. System Performance Indicators

1) The probability that the system is in the normal busy period

$$P\{J=1\} = K\tfrac{1}{1-\rho}\tfrac{\theta r}{\mu_2(1-r)}\left[(N-1) + \tfrac{1}{1-r}\right].$$

2) The probability that the system is in the working vacation period

$$\begin{aligned} &P\{J=0\} \\ &= K[(N-1)\left(1-\tfrac{\mu_2}{\lambda_2}\right)^{-1}\left(1-\tfrac{r\mu_2}{\lambda_2}\right) \\ &\quad + (1-r)\tfrac{\mu_2}{\lambda_2}\left(1-\left(\tfrac{\mu_2}{\lambda_2}\right)^N\right)\left(1-\tfrac{\mu_2}{\lambda_2}\right)^{-2}]. \end{aligned}$$

3) The mean queue length

$$\begin{aligned} E(L) &= \sum_{j=0}^{N-1} j\cdot\pi_{j,0} + \sum_{j=1}^{N-1} j\cdot\pi_{j,1} + \sum_{k=N}^{\infty} k\cdot\pi_{k,0} + \sum_{k=N}^{\infty} k\cdot\pi_{k,1} \\ &= K\tfrac{\theta r}{\mu_1(1-r)}\tfrac{1}{1-\rho}\cdot\tfrac{1}{2}\left[\tfrac{\left(1-2\rho-\rho^2\right)N(N-1)+2\rho^N(N+\rho-2N\rho)}{(1-\rho)^2}\right] \\ &+ K\tfrac{\theta r}{\mu_1(1-r)}\left[\tfrac{N}{(1-r)(1-\rho)} + \tfrac{r+\rho(1-2r)}{(1-r)^2(1-\rho)^2} - \tfrac{(N-N\rho+\rho)\left(\rho-\rho^N\right)}{(1-\rho)^3}\right] \\ &+ Kr\left(\tfrac{N}{1-r} + \tfrac{r}{(1-r)^2}\right) + K\tfrac{\lambda_2}{\lambda_2-\mu_2}[\tfrac{\theta r}{\lambda_2(1-r)}\tfrac{N(N-1)}{2} - \\ &(1-r)\tfrac{\mu_2\left[\lambda_2(N-1)-N\mu_2+\lambda_2\left(\frac{\mu_2}{\lambda_2}\right)^N\right]}{(\lambda_2-\mu_2)^2}]. \end{aligned}$$

VI. Numerical Experiments

For different queueing models, the change of the parameters will directly affect the performance indicators of this system. This section will use specific numerical examples to prove the reasonability and actual operability of the model.

Assuming that $\mu_1 = 0.8, \theta = 0.5, \lambda_2 = 0.2, N = 4$, λ_1 takes values of 0.3, 0.4 and 0.5. The range of variation of μ_2 is $0.1 \leq \mu_2 \leq 0.5$. We account for the trend of the $E\left(Q_v{}^{(N)}\right)$ with λ_1 and μ_2 in figure 2. When λ_1 is fixed, $E\left(Q_v{}^{(N)}\right)$ gradually decreases with the increase of μ_2. When μ_2 is fixed, $E\left(Q_v{}^{(N)}\right)$ smoothly increase with the increase of the λ_1. Obviously, the influence of λ_1 on $E\left(Q_v{}^{(N)}\right)$ is significant.

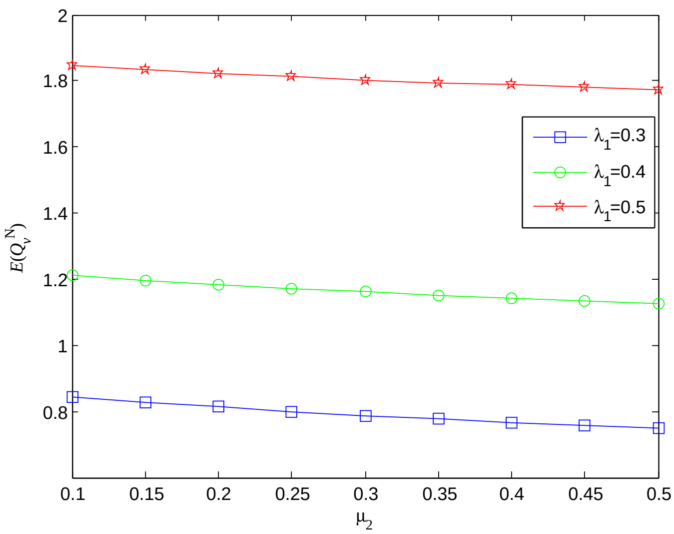

Figure 2. The mean queue length $E\left(Q_v{}^{(N)}\right)$ versus μ_2.

Assuming that $\mu_1 = 0.8, \lambda_1 = 0.5, \lambda_2 = 0.2, N = 4$, μ_2 takes values of 0.3, 0.4 and 0.5 and the range of variation of θ is $0.3 \leq \theta \leq 0.7$. Figure 3 illustrates the effects of μ_2 and θ on $E\left(Q_v{}^{(N)}\right)$. When μ_2 is fixed, $E\left(Q_v{}^{(N)}\right)$ gradually decreases with the increase of θ. When θ is fixed, $E\left(Q_v{}^{(N)}\right)$ decreases with the increase of μ_2. When the working vacation policy is added to a queueing system to save resources, it is important to determine an appropriate service rate on the system.

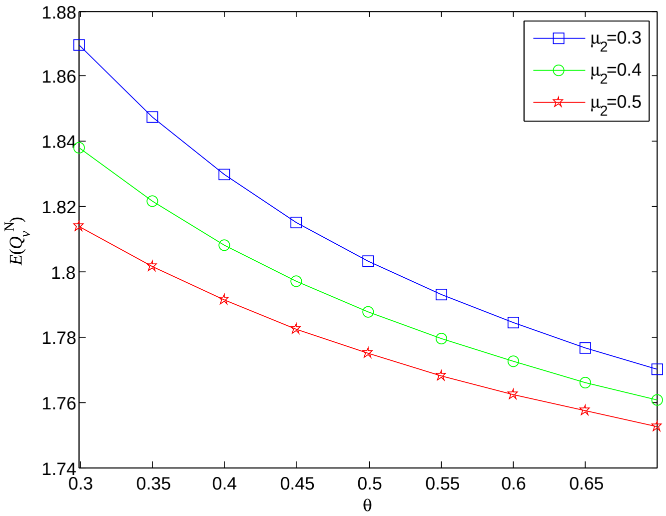

Figure 3. The mean queue length $E\left(Q_v{}^{(N)}\right)$ versus θ.

Assuming that $\mu_2 = 0.5, \lambda_1 = 0.6, \lambda_2 = 0.3, N = 4$, the range of variation of μ_1 is $0.8 \leq \mu_1 \leq 1.3$ and the range of variation of θ is $0.3 \leq \theta \leq 0.7$. Figure 4 illustrates the effects of μ_1 and θ on $E\left(Q_v{}^{(N)}\right)$. This figuer reflects $E\left(Q_v{}^{(N)}\right)$ decreases with the increase of θ and μ_1.

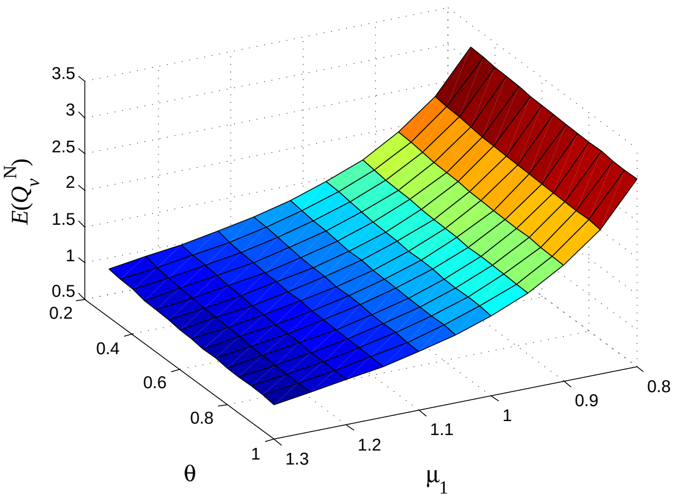

Figure 4. The mean queue length $E\left(Q_v{}^{(N)}\right)$ for different values of θ and μ_1.

Assuming that $\mu_1 = 0.8, \lambda_1 = 0.5, \lambda_2 = 0.2, N = 4$, the range of variation of μ_2 is $0.3 \leq \mu_2 \leq 0.5$ and the range of variation of θ is $0.3 \leq \theta \leq 0.7$. We show the effects of μ_2 and θ on $E\left(Q_v{}^{(N)}\right)$ in figure 5. The result indicates $E\left(Q_v{}^{(N)}\right)$ decreases with the increase of parameter θ and μ_2.

Assuming that $\mu_1 = 0.8, \lambda_1 = 0.5, \lambda_2 = 0.2, N = 4$, the range of variation of μ_2 is $0.3 \leq \mu_2 \leq 0.5$ and the range of variation of θ is $0.3 \leq \theta \leq 0.7$. The figure 6 illustrates the effects of μ_2 and θ on the mean of busy cycle $E(C)$. When μ_2 is fixed, the mean of busy cycle $E(C)$ in the system decreases with the increase of θ. When the θ is fixed, $E(C)$

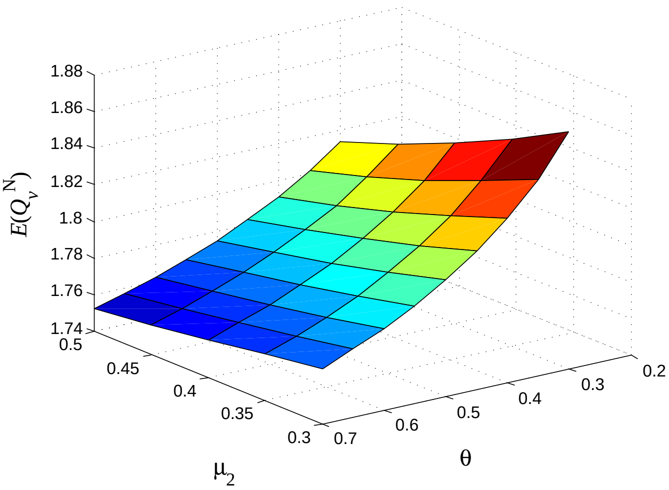

Figure 5. The mean queue length $E\left(Q_v{}^{(N)}\right)$ for different values of θ and μ_2.

grows as the service rate μ_2 grows.

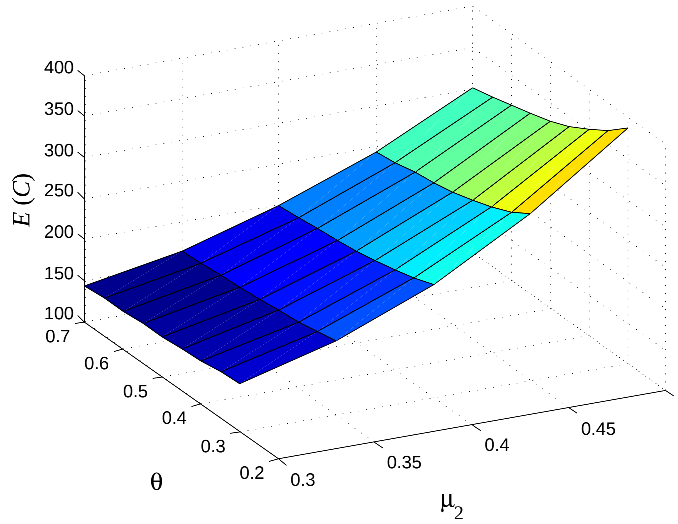

Figure 6. The mean busy cycle $E(C)$ for different values of θ and μ_2.

VII. Conclusion

This paper investigated the M/M/1 working vacation queueing system with N-policy and different arrival rates. We first obtained the stationary distribution via the matrix geometry solution method. Through contrastive analysis, the conditional stochastic decomposition structure of the queue length and waiting time are obtained. We compute various performance indicators of the system and analyzed the busy period. The impacts of system parameters on the performance indexes are obtained by numerical experiments. From the above results, it can be concluded that it is essential to consider service rates when during the working vacation to enhance efficiency. We can carry on the research in the future to extend the system with general distribution of service time or PH distribution.

References

[1] B. Doshi, "Queueing Systems with Vacations-a Survey," *Queueing Systems*, vol. 1, pp. 29-66, 1989.

[2] N. S. Tian, J. H. Li and Z. G. Zhang, "Matrix Analytic Method and Working Vacation Queues-a Survey," *International Journal of Information and Management Sciences*, vol. 20, pp. 603-633, 2009.
[3] N. S. Tian and Z. G. Zhang, "Vacation Queueing Models-Theory and Applications," *New York: Springer 2006*, pp. 13.
[4] N. S. Tian, "Stochastic Service System with Vacation," *Beijing: Peking University Press 2001*, pp. 36.
[5] N. S. Tian and D. Q. Yue, "Quasi-birth-death process and Matrix geometric solution," *Beijing: Science Press 2002*, pp. 22-30.
[6] L. D. Servi and S. G. Finn, "M/M/1 Queues with Working Vacations (M/M/1/WV)," *Performance Evaluation*, vol. 50, no. 1, 2002.
[7] Y. H. Chen and S. F. Jia, "The Discrete Time Queue with Negative Customers and Multiple Working Vacations," *International Journal of Engineering Research*, vol. 2, no. 6, pp. 393-398, 2013.
[8] S. L. Lv, Y. Y. Wu, J. B. Li, et al. , "The M/M/1 Repairable Queueing System with Variable Input Rates and Failure Rates," *IAENG International Journal of Applied Mathematics*, vol. 52, no. 4, pp. 784-790, 2022.
[9] M. Zhang and Z. T. Hou, "Performance Analysis of M/G/1 Queue with Working Vacations and Vacation Interruption," *Journal of Computational and Applied Mathematics*, vol. 234, no. 10, pp. 2977-2985, 2010.
[10] D. Y. Yang and Y. C Cho, "Analysis of the N-policy GI/M/1/K Queueing Systems with Working Breakdowns and Repairs," *Comput. J.*, vol. 62, no. 1, pp. 130-143, 2019.
[11] S. L. Yang and N. S. Tian, "The M/M/1 Queue with Working Vacation and N-policy," *Operations Research and Management Science*, vol. 16, no. 4, pp. 50, 2007.
[12] V. Karthick and V. Suvitha, "An Analysis of Three Servers Markovian Multiple Vacation Queueing System with Servers Breakdown," *IAENG International Journal of Applied Mathematics*, vol. 53, no. 2, pp. 670-677, 2023.
[13] J. Li and T. Li, "An M/G/1 G-queue with Server Breakdown, Working Vacations and Bernoulli Vacation Interruption," *IAENG International Journal of Applied Mathematics*, vol. 50, no. 2, pp. 421-428, 2020.
[14] J. T. Li and T. Li, "An M/M/1 Retrial Queue with Working Vacation, Orbit Search and Balking," *Engineering Letters*, vol. 27, no. 1, pp. 97-102, 2019.
[15] Q. Q. Ma, J. H. Li and Q. Y. Wei, "Social Optimization in M/M/1 queue with working vacation and N -policy," *RAIRO - Operations Research*, vol. 52, no. 2, pp. 439-452, 2018.
[16] L. Wang and Y. L. Zhu, "Geo/Geo/1 Queue with Working Vacations with N-policy, Negative Customers and Different Arrival Rates," *Journal of Northwest Normal University (Natural Science)*, vol. 53, no. 04, pp. 17-22, 2017.
[17] Y. H. Tang, Y. L. Zhu and W. Q. Wu, "Analysis of M/G/1 Repairable Queueing System with N-policy and a Replaceable Repair Facility," *System Engineering Theory and Practice*, vol. 34, pp. 746-755, 2014.
[18] Y. Y. Wei, Y. H. Tang and J. X. Gu, "Queue Length Distribution and Numerical Calculation for Geo/G/1 Queueing System with Delayed N-policy," *System Engineering Theory and Practice*, vol. 31, pp. 2151-2160 130-134, 2011.
[19] H. Y. Wei and Y. Y. Wei, "Delayed N-strategy Geo/G/1 Queue System with Start-up Time and Its Application in ATM Networks," *Journal of Hexi University*, vol. 35, no. 02, pp. 6-16, 2019.
[20] M. Neuts, "Matrix-geometric Solutions in Stochastic Models," *Johns Hopkins Univ Press*, pp. 1-6 41-82, 1981.