# Global Asymptotic Stability and Asymptotically Periodic Oscillation in Fractional-Order Fuzzy Cohen-Grossberg Neural Networks with Delays

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***Abstract*—This paper focuses on the $S$-asymptotically $\varpi$-periodic oscillation for a type of fractional-order fuzzy Cohen-Grossberg neural networks (CGNNs) by employing some properties of Mittag-Leffler mappings and fixed point theorems. Further, the global asymptotic stability of CGNNs is received. For CGNNs, our works in this paper not only enrich its theoretical achievements, but also expand its application scope.**

***Index Terms*—Cohen-Grossberg, Mittag-Leffler function, global stability, asymptotic periodicity.**

## I. Introduction

COHEN and Grossberg in 1983 [1] produced Cohen-Grossberg neural networks (CGNNs), which are of interest to numerous academics by virtue of its prospective applications. These manipulations are reliant on the networks' dynamics. Therefore, learning the above dynamics is the prerequisite required for the programming of the operation to neural networks. As well known, an optimization problem is strictly related to their equilibriums, so neural networks are commonly adopted to tackle optimization problems. It is not surprising, in these contexts, that we should place a high value on their equilibriums. There are also findings of the neural dynamical systems that address more than just stability, as well as many other dynamical behaviors, such as periodicity, see [2–4]. Lately, several monographs are related to the aforementioned features of equilibrium points in CGNNs, also other dynamical behaviors, see [5–8].

Fractional calculus [9–14] has a history of over three hundred years. The wonderful of the derivative one is nonlocal, the other is its future state relies on both present and past states, which makes it more accurate to describe the problem compared to the classical derivative. Fractional equations have been employed to characterize lots of realistic problems in the present day, for instance, heat conduction [15], neural network [16], biological systems [17], robots [18]. Remarkably, fractional-order neural networks (FONNs) have a pivotal position in neural networks due to it provides an efficient method memory and genetic properties [10]. The utilization of FONNs is remarkable and dynamic characteristics have become very important research objects in recent years, such as synchronization [16], approximate periodicity [19, 20], Hopf bifurcation [21], stability [22, 23] and chaos [24], etc.

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Zadeh [25] in 1965 offered fuzzy logic, which takes into account uncertainty and ambiguity, and is one of the most widespread and critical problems in realistic modeling. Yang and Yang [26] then introduced a novel cellular neural networks involved fuzzy logic. It is a fuzzy neural network features fuzzy logic towards template inputs and/or outputs. Lately, with its advantages in the areas of picture processing, etc., fuzzy neural networks have drawn an increasing amount of attention, see [27–30].

The two principal motivations for the work in this paper, with the above discussion in mind, are as follows. First of all, the periodically movement in real-world applications is an intriguing and important dynamic property of neural networks, given that numerous living and cognitive activities are regular repetitions of actions such as heartbeat, movement, memory, etc. In this regard, there is an importance of examining the periodicity of neural networks to find out how they works. Up to now, various researchers have discussed the periodicity or almost periodicity of classical CGNNs [29, 31–36]. However, there are few literatures dealing with periodic oscillations to fractional-order neural networks (FONNs), resulting from the non-periodicity of fractional-order differential equations (FODEs) [37], which exhibit asymptotic periodicity alone, see [38–40]. Secondly, some literatures [38–42] have studied the Mittag-Leffler stability for FONNs free of time lag. It is important to note that, according to same methods as articles [38–42], the result that FONNs with time variable lags is the Mittag-Leffler stable cannot be yielded. Therefore, this article focuses on asymptotic periodicity and global asymptotic stability of fractional-order fuzzy CGNNs involved time-varying lags (FOCGNNs).

The remainders of this article are arranged below. Some useful preliminaries for fractional-order calculus and Mittag-Leffler function are reviewed in section 2. section 3 discusses that FOCGNNs (1) admits a sole $S$-asymptotical $\varpi$-periodic oscillation ($S$-APO$_\varpi$). In section 4, the global asymptotic stability of the FOCGNNs (1) is acquired in accordance with Laplace transform, the comparison principle and the stability theorem. In section 5, a numerical example is presented to illustrate the validity and feasibility of our work. We conclude the findings of this paper and look forward to the future work in section 6.

## II. Previous preparations

Notations: $\mathbb{R}^n$ stands for the family of real vectors in $n$-dimension, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, $\mathbb{Z}^+ = \{1, 2, \ldots\}$, $\mathbb{C}$ is complex set and $C^n(\Omega, \mathbb{R}^n)$ is a collection consisting of continuous and differentiable functions up to order $n$: $\Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$.

### A. Model description

For simplicity, let

$$p_i^{\hbar}(t)=\hbar_i\big(x_i(t)\big)=\int_0^{x_i(t)}\frac{1}{d_i(s)}\mathrm{d}s \text{ with } \hbar_i(0)=0,$$

$$\Lambda_i^{\hbar}\big(p_i(t)\big)=\lambda p_i(t)-a_i\big(p_i^{\hbar}(t)\big)$$

and $\hbar_i^{-1}$ stand for the inverse function of $\hbar_i$ for $\lambda>0$, $i=1,2,\ldots,n$.

This article considers the following nonlocal CGNNs:

$$\begin{aligned}{}^{c}D_0^{\gamma}p_i(t) &= -\lambda p_i(t)+\Lambda_i^{\hbar}\big(p_i(t)\big)\\ &\quad+\sum_{j=1}^{n}b_{ij}(t)g_j\big(p_j^{\hbar}(t-\sigma_j(t))\big)\\ &\quad+\bigvee_{j=1}^{n}\vartheta_{ij}g_j\big(p_j^{\hbar}(t-\sigma_j(t))\big)\\ &\quad+\bigwedge_{j=1}^{n}\nu_{ij}g_j\big(p_j^{\hbar}(t-\sigma_j(t))\big)+\bigvee_{j=1}^{n}T_{ij}\beta_j\\ &\quad+\bigwedge_{j=1}^{n}H_{ij}\beta_j+J_i(t),\quad t>0, \end{aligned}\tag{1}$$

with initial conditions

$$p_i(s)=\varphi_i(s),\quad s\in[-\sigma,0],$$

in which $\sigma=\max_{1\le j\le n}\sup_{t>0}\sigma_j(t)$, ${}^{c}D_0^{\gamma}$ denotes Caputo frctional derivative of the order $\gamma\in(0,1]$, $p_i$ is the $i$th state, $d_i>0$ shows an amplification function, $a_i(0)=0$, $g_j$ denotes the neuronal function, $b_{ij}$ represents the $ij$th strength, $J_i$ is the input, $\vartheta_{ij}$, $\nu_{ij}$, $P_{ij}$, $H_{ij}$ are the operating elements of fuzzy models, $i,j=1,2,\ldots,n$.

In terms of the discussion in our previous work [30], if $\gamma=1$ in CGNNs (1), then it is equivalent to

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -d_i\big(x_i(t)\big)\Big[a_i\big(x_i(t)\big)-\sum_{j=1}^{n}b_{ij}(t)g_j\big(x_j(t-\sigma_{ij})\big)\\ &\quad-\bigvee_{j=1}^{n}\vartheta_{ij}g_j\big(x_j(t-\sigma_{ij})\big)-\bigwedge_{j=1}^{n}\nu_{ij}g_j\big(x_j(t-\sigma_j)\big)\\ &\quad-\bigvee_{j=1}^{n}T_{ij}\beta_j-\bigwedge_{j=1}^{n}H_{ij}\beta_j-J_i(t)\Big],\quad t>0,\end{aligned}\tag{2}$$

where $i=1,2,\ldots,n$.

### B. Some definitions and lemmas

**Definition II.1** ([10]). *For $g\in C^n([t_0,\infty),\mathbb{R}^n)$, the fractional derivative of $f$ in sense of Caputo with $\gamma$-order can be given by*

$${}^{c}D_{t_0}^{\gamma}g(t)=\frac{1}{\Gamma(n-\gamma)}\int_{t_0}^{t}\frac{g^{(n)}(s)}{(t-s)^{\gamma-n+1}}\mathrm{d}s$$

*for $0<n-1<\gamma<n, n\in\mathbb{Z}^+$.*

**Definition II.2** ([10]). *The types of Mittag-Leffler mappings can be described by*

$$E_\gamma(z):=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\gamma k+1)},\quad E_{\gamma,\beta}(z):=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\gamma k+\beta)},$$

*where $z\in\mathbb{C}$, $\gamma,\beta>0$.*

**Lemma II.1** ([10]). $\frac{\mathrm{d}}{\mathrm{d}z}[z^\gamma E_{\gamma,\gamma+1}(\kappa z^\gamma)]=z^{\gamma-1}E_{\gamma,\gamma}(\kappa z^\gamma)$, *where $\gamma,\kappa,z\in\mathbb{C}$.*

**Lemma II.2** ([42]). $\lim_{t\to\infty}t^\gamma E_{\gamma,\gamma+1}(-\kappa t^\gamma)=\frac{1}{\kappa}$ *and* $t^\gamma E_{\gamma,\gamma+1}(-\kappa t^\gamma)\le\frac{1}{\kappa}$ *for $\kappa>0$, $\gamma\in(0,1]$, $t\le 0$.*

**Lemma II.3** ([20]). *If $c$, $\kappa>0$ and $\gamma\in(0,1]$, then*

$$\lim_{t\to\infty}E_\gamma(-\kappa t^\gamma)=0,$$

$$\lim_{t\to\infty}\int_0^c(t-s)^{\gamma-1}E_{\gamma,\gamma}[-\kappa(t-s)^\gamma]\mathrm{d}s=0.$$

## III. $S$-APO$_\varpi$ OF FOCGNNs

Let $\|x\|_1=\max_{1\le i\le n}|x_i|$ for any $x=(x_1,x_2,\ldots,x_n)^\top\in\mathbb{R}^n$. Set $\bar{g}=\sup_{t\ge0}|g(t)|$ and $\underline{g}=\inf_{t\ge0}|g(t)|$ for bounded function $g$ defined on $[0,+\infty)$.

**Definition III.1** ([38, 39]). *Assume that $g\in C([t_0,+\infty),\mathbb{R}^n)$ and there is a positive constant $\varpi$ ensuring that $\lim_{t\to+\infty}\|g(t+\varpi)-g(t)\|_1=0$, then $g$ is $S$-APO$_\varpi$.*

In FOCGNNs (1), let the assumptions below hold.

$(H_1)$ $b_{ij}, J_i$ are $S$-APO$_\varpi$ and $\sigma_j(t+\varpi)=\sigma_j(t)$ with $\varpi>0$, $i,j=1,2,\ldots,n$.

$(H_2)$ There exists $L_j^g>0$ satisfying $|g_j(x)-g_j(y)|\le L_j^g|x-y|$, $\forall x,y\in\mathbb{R}$, $j=1,2,\ldots,n$.

$(H_3)$ $0<\nu_i=\frac{1}{\lambda}\Big[L_i^\lambda+\sum_{j=1}^{n}\Big(\bar{b}_{ij}+|\vartheta_{ij}|+|\nu_{ij}|\Big)L_j^g\bar{d}_j\Big]<1$, $i=1,2,\ldots,n$.

Let $\mathbb{S}_\varpi$={$z\in C([0,+\infty),\mathbb{R}^n)$ : $z$ is $S$-APO$_\varpi$ with $\varphi_i(s), s\in[-\sigma,0]$}. Then $\mathbb{S}_\varpi$ is Banach space with norm $\|z\|_\infty=\sup_{t\le0}\max_{1\le i\le n}|p_i(t)|$.

Consider the system:

$$\begin{cases}{}^{c}D_0^{\gamma}z(t)=-az(t)+b(t)f(z(t-\sigma)),\quad t>0,\\ z(s)=\varphi(s),\quad s\in[-\sigma,0],\end{cases}\tag{3}$$

where $a\in\mathbb{R}$ is a positive constant, $b(t)\in C(\mathbb{R},\mathbb{R}^{n\times n})$ is $S$-asymptotically $\varpi$-periodic function, there exists $L^g>0$ such that $g(x)\in C(\mathbb{R}^n,\mathbb{R}^n)$ satisfies the following condition

$$\|g(x)-g(y)\|_1\le L^g\|x-y)\|_1,\quad\forall x,y\in\mathbb{R}^n.$$

For each $\phi(t)\in\mathbb{S}_\varpi$, it gets $g(\phi(t))\in\mathbb{S}_\varpi$. Then, we research

$$\begin{cases}{}^{c}D_0^{\gamma}z(t)=-az(t)+b(t)g(\phi(t-\sigma)),\quad t>0,\\ z(s)=\varphi(s),\quad s\in[-\sigma,0].\end{cases}\tag{4}$$

Via [10], (4) is depicted as

$$\begin{cases}z(t)=z^{\phi(t)}=\varphi(0)E_\gamma(-at^\gamma)\\ \quad+\int_0^t(t-s)^{\gamma-1}E_{\gamma,\gamma}[-a(t-s)^\gamma]\\ \quad\times b(s)g(\phi(s-\sigma))\mathrm{d}s,\quad t>0\\ z(s)=z^{\phi(s)}=\varphi(s),\quad s\in[-\sigma,0].\end{cases}\tag{5}$$

Based on Eqs. (5), let $P:\phi\to z^\phi$ (*i.e.*, $P\phi=z^\phi$), $\forall\phi\in\mathbb{S}_\varpi$. If operator $P$ owns a sole fixed point $\phi^*\in\mathbb{S}_\varpi$, then $\phi^*=P\phi^*=z^{\phi^*}$. From Eqs. (5), $\phi^*$ is the unique $S$-APO$_\varpi$ of Eqs. (3).

According to FOCGNNs (1) and Eqs. (4), the following system should be considered

$$\begin{cases} {}^{c}D_{0}^{\gamma}p_i(t) = -\lambda p_i(t) + \Lambda_i^{\hbar}\big(\phi_i(t)\big) \\ \quad + \sum_{j=1}^{n} b_{ij}(t)g_j\big(\phi_j^{\hbar}(t-\sigma_j(t))\big) \\ \quad + \bigvee_{j=1}^{n} \vartheta_{ij}g_j\big(\phi_j^{\hbar}(t-\sigma_j(t))\big) \\ \quad + \bigwedge_{j=1}^{n} \nu_{ij}g_j\big(\phi_j^{\hbar}(t-\sigma_j(t))\big) \\ \quad + \bigvee_{j=1}^{n} T_{ij}\beta_j + \bigwedge_{j=1}^{n} H_{ij}\beta_j + J_i(t), \quad t>0, \\ p_i(s) = \varphi_i(s), \quad s\in[-\sigma,0], \quad i=1,2,\dots,n \end{cases}$$

for any $\phi = (\phi_1,\phi_2,\dots,\phi_n)^{\top} \in \mathbb{S}_{\varpi}$.

Define the following operator

$$\begin{aligned} P:\phi \to z^{\phi}, \quad \forall \phi\in\mathbb{S}_{\varpi}: \\ P\phi = \big((P\phi)_1,(P\phi)_2,\dots,(P\phi)_n\big)^{\top} \\ = (z_1^{\phi}, z_2^{\phi},\dots,z_n^{\phi})^{\top} = z^{\phi}, \end{aligned} \tag{6}$$

where

$$\begin{cases} (P\phi)_i(t) = p_i^{\phi} = \varphi_i(0)E_{\gamma}(-\lambda t^{\gamma}) \\ \quad + \int_0^t (t-s)^{\gamma-1}E_{\gamma,\gamma}[-\lambda(t-s)^{\gamma}] \\ \quad \times\Big[\Lambda_i^{\hbar}\big(\phi_i(s)\big) + \sum_{j=1}^{n} b_{ij}(s) \\ \quad \times g_j\big(\phi_j^{\hbar}(s-\sigma_j(s))\big) \\ \quad + \bigvee_{j=1}^{n}\vartheta_{ij}g_j\big(\phi_j^{\hbar}(s-\sigma_j(s))\big) \\ \quad + \bigwedge_{j=1}^{n}\nu_{ij}g_j\big(\phi_j^{\hbar}(s-\sigma_j(s))\big) \\ \quad + \bigvee_{j=1}^{n}T_{ij}\beta_j + \bigwedge_{j=1}^{n}H_{ij}\beta_j + J_i(s)\Big]\mathrm{d}s, \quad t>0, \\ (P\phi)_i(s) = p_i^{\phi(s)} = \varphi_i(s), \quad s\in[-\sigma,0], \end{cases} \tag{7}$$

where $i=1,2,\dots,n$. Same as the discussion of Eqs. (5), we can conclude that if $\phi^*\in\mathbb{S}_{\varpi}$ is the sole fixed point of operator $P$, then $\phi^* = P\phi^* = z^{\phi^*}$ is the sole $S$-APO$_{\varpi}$ of Eqs. (1).

**Remark III.1.** *If $\gamma=1$ in system (1), then it is a classical integer-order model:*

$$\begin{aligned} \frac{\mathrm{d}p_i(t)}{\mathrm{d}t} &= -a_i\big(p_i^{\hbar}(t)\big) + \sum_{j=1}^{n} b_{ij}(t)g_j\big(p_j^{\hbar}(t-\sigma_j(t))\big) \\ &\quad + \bigvee_{j=1}^{n}\vartheta_{ij}g_j\big(p_j^{\hbar}(t-\sigma_j(t))\big) \\ &\quad + \bigwedge_{j=1}^{n}\nu_{ij}g_j\big(p_j^{\hbar}(t-\sigma_j(t))\big) + \bigvee_{j=1}^{n}T_{ij}\beta_j \\ &\quad + \bigwedge_{j=1}^{n}H_{ij}\beta_j + J_i(t), \ \ t>0, i=1,2,\dots,n, \end{aligned}$$

*which is widely researched in literatures [28, 30, 33, 43–45], such as stability [28], exponential stability [30, 43], global stability [33], synchronization [30, 44, 45]. Therefore, the results in this paper enrich these studies in monographs [28, 30, 33, 43–45] to a certain extent.*

**Theorem III.1.** *System (1) owns a sole $S$-APO$_{\varpi}$, if $(H_1)$-$(H_3)$ and the following assumption are fulfilled.*

$(H_4)$ *It holds that $|\Lambda_i^{\hbar}(x) - \Lambda_i^{\hbar}(y)| \le L_i^{\lambda}|x-y|$ for some $L_i^{\lambda}>0$, $\forall x,y\in\mathbb{R}$, $i=1,2,\dots,n$.*

**Proof:** In the first place, one shows $P:\mathbb{S}_{\varpi}\to\mathbb{S}_{\varpi}$. For any $\phi=(\phi_1,\phi_2,\dots,\phi_n)^{\top}\in\mathbb{S}_{\varpi}$, $\epsilon>0$, it has $t>t_1>0$ so that

$$\begin{gathered} |\phi_i(t+\varpi)-\phi_i(t)|<\epsilon, \\ |\phi_i(t+\varpi-\sigma_i(t+\varpi))-\phi_i(t-\sigma_i(t))| \\ = |\phi_i(t+\varpi-\sigma_i(t))-\phi_i(t-\sigma_i(t))|<\epsilon, \\ |b_{ij}(t+\varpi)-b_{ij}(t)|<\epsilon, \\ |J_i(t+\varpi)-J_i(t)|<\epsilon, \quad i,j=1,2,\dots,n. \end{gathered}$$

Besides, $\|\phi\|_{\infty}<+\infty$ since $\phi\in\mathbb{S}_{\varpi}$.

According to Eqs. (7), for $t>0$, it follows

$$\begin{aligned} (P\phi)_i(t+\varpi) &= \varphi_i(0)E_{\gamma}(-\lambda(t+\varpi)^{\gamma}) \\ &\quad + \int_0^{t+\varpi}(t+\varpi-s)^{\gamma-1} \\ &\quad \times E_{\gamma,\gamma}[-\lambda(t+\varpi-s)^{\gamma}]\Big[\Lambda_i^{\hbar}\big(\phi_i(s)\big) \\ &\quad + \sum_{j=1}^{n}b_{ij}(s)g_j\big(\phi_j^{\hbar}(s-\sigma_j(s))\big) \\ &\quad + \bigvee_{j=1}^{n}\vartheta_{ij}g_j\big(\phi_j^{\hbar}(s-\sigma_j(s))\big) \\ &\quad + \bigwedge_{j=1}^{n}\nu_{ij}g_j\big(\phi_j^{\hbar}(s-\sigma_j(s))\big) \\ &\quad + \bigvee_{j=1}^{n}T_{ij}\beta_j + \bigwedge_{j=1}^{n}H_{ij}\beta_j + J_i(s)\Big]\mathrm{d}s \\ &= \varphi_i(0)E_{\gamma}(-\lambda(t+\varpi)^{\gamma}) \\ &\quad + \int_{-\varpi}^{t}(t-s)^{\gamma-1}E_{\gamma,\gamma}[-\lambda(t-s)^{\gamma}] \\ &\quad \times\Big[\Lambda_i^{\hbar}\big(\phi_i(s+\varpi)\big) \\ &\quad + \sum_{j=1}^{n}b_{ij}(s+\varpi)g_j\big(\phi_j^{\hbar}(s+\varpi-\sigma_j(s))\big) \\ &\quad + \bigvee_{j=1}^{n}\vartheta_{ij}g_j\big(\phi_j^{\hbar}(s+\varpi-\sigma_j(s))\big) \\ &\quad + \bigwedge_{j=1}^{n}\nu_{ij}g_j\big(\phi_j^{\hbar}(s+\varpi-\sigma_j(s))\big) \\ &\quad + \bigvee_{j=1}^{n}T_{ij}\beta_j + \bigwedge_{j=1}^{n}H_{ij}\beta_j \\ &\quad + J_i(s+\varpi)\Big]\mathrm{d}s. \end{aligned} \tag{8}$$

So,

$$\begin{aligned} &(P\phi)_i(t+\varpi)-(P\phi)_i(t) \\ &= \varphi_i(0)E_{\gamma}(-\lambda(t+\varpi)^{\gamma}) \\ &\quad -\varphi_i(0)E_{\gamma}(-\lambda t^{\gamma}) \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t (t-s)^{\gamma-1}E_{\gamma,\gamma}[-\lambda(t-s)^\gamma] \\
&\times\Big[\Lambda_i^\hbar\big(\phi_i(s+\varpi)\big)-\Lambda_i^\hbar\big(\phi_i(s)\big) \\
&+\sum_{j=1}^n b_{ij}(s+\varpi)g_j\big(\phi_j^\hbar(s+\varpi-\sigma_j(s))\big) \\
&-\sum_{j=1}^n b_{ij}(s)g_j\big(\phi_j^\hbar(s-\sigma_j(s))\big) \\
&+\bigvee_{j=1}^n \vartheta_{ij}g_j\big(\phi_j^\hbar(s+\varpi-\sigma_j(s))\big) \\
&-\bigvee_{j=1}^n \vartheta_{ij}g_j\big(\phi_j^\hbar(s-\sigma_j(s))\big) \\
&+\bigwedge_{j=1}^n \nu_{ij}g_j\big(\phi_j^\hbar(s+\varpi-\sigma_j(s))\big) \\
&-\bigwedge_{j=1}^n \nu_{ij}g_j\big(\phi_j^\hbar(s-\sigma_j(s))\big) \\
&+J_i(s+\varpi)-J_i(s)\Big]\mathrm{d}s \\
&+\int_{-\varpi}^0 (t-s)^{\gamma-1}E_{\gamma,\gamma}[-\lambda(t-s)^\gamma] \\
&\times\Big[\Lambda_i^\hbar\big(\phi_i(s+\varpi)\big) \\
&+\sum_{j=1}^n b_{ij}(s+\varpi)g_j\big(\phi_j^\hbar(s+\varpi-\sigma_j(s))\big) \\
&+\bigvee_{j=1}^n \vartheta_{ij}g_j\big(\phi_j^\hbar(s+\varpi-\sigma_j(s))\big) \\
&+\bigwedge_{j=1}^n \nu_{ij}g_j\big(\phi_j^\hbar(s+\varpi-\sigma_j(s))\big) \\
&+\bigvee_{j=1}^n T_{ij}\beta_j+\bigwedge_{j=1}^n H_{ij}\beta_j+J_i(s+\varpi)\Big]\mathrm{d}s \\
=\;&\lambda_{i1}(t)+\lambda_{i2}(t)+\lambda_{i3}(t)+\lambda_{i4}(t) \\
&+\lambda_{i5}(t)+\lambda_{i6}(t)+\lambda_{i7}(t)+\lambda_{i8}(t) \\
&+\lambda_{i9}(t)+\lambda_{i10}(t)+\lambda_{i11}(t)+\lambda_{i12}(t) \\
&+\lambda_{i13}(t)+\lambda_{i14}(t). \qquad (9)
\end{aligned}
$$

where

$$\lambda_{i1}(t)=\varphi_i(0)\{E_\gamma[-\lambda(t+\varpi)^\gamma]-E_\gamma(-\lambda t^\gamma)\},$$

$$
\begin{aligned}
\lambda_{i2}(t)=&\int_0^t (t-s)^{\gamma-1}E_{\gamma,\gamma}[-\lambda(t-s)^\gamma] \\
&\{\Lambda_i^\hbar[\phi_i(s+\varpi)]-\Lambda_i^\hbar[\phi_i(s)]\}\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{i3}(t)=&\int_0^t (t-s)^{\gamma-1}E_{\gamma,\gamma}[-\lambda(t-s)^\gamma] \\
&\times\sum_{j=1}^n [b_{ij}(s+\varpi)-b_{ij}(s)] \\
&\times g_j\big[\phi_j^\hbar(s+\varpi-\sigma_j(s))\big]\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{i4}(t)=&\int_0^t (t-s)^{\gamma-1}E_{\gamma,\gamma}[-\lambda(t-s)^\gamma]\sum_{j=1}^n b_{ij}(s) \\
&\times\{g_j[\phi_j^\hbar(s+\varpi-\sigma_j(s))]-g_j[\phi_j^\hbar(s-\sigma_j(s))]\}\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{i5}(t)=&\int_0^t (t-s)^{\gamma-1}E_{\gamma,\gamma}[-\lambda(t-s)^\gamma] \\
&\times\Big\{\bigvee_{j=1}^n \vartheta_{ij}g_j\big[\phi_j^\hbar(s+\varpi-\sigma_j(s))\big] \\
&-\bigvee_{j=1}^n \vartheta_{ij}g_j\big[\phi_j^\hbar(s-\sigma_j(s))\big]\Big\}\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{i6}(t)=&\int_0^t (t-s)^{\gamma-1}E_{\gamma,\gamma}[-\lambda(t-s)^\gamma] \\
&\times\Big\{\bigwedge_{j=1}^n \nu_{ij}g_j\big[\phi_j^\hbar(s+\varpi-\sigma_j(s))\big] \\
&-\bigwedge_{j=1}^n \nu_{ij}g_j\big[\phi_j^\hbar(s-\sigma_j(s))\big]\Big\}\mathrm{d}s,
\end{aligned}
$$

$$\lambda_{i7}(t)=\int_0^t (t-s)^{\gamma-1}E_{\gamma,\gamma}[-\lambda(t-s)^\gamma]\big[J_i(s+\varpi)-J_i(s)\big]\mathrm{d}s,$$

$$\lambda_{i8}(t)=\int_{-\varpi}^0 (t-s)^{\gamma-1}E_{\gamma,\gamma}[-\lambda(t-s)^\gamma]\Lambda_i^\hbar\big(\phi_i(s+\varpi)\big)\mathrm{d}s,$$

$$
\begin{aligned}
\lambda_{i9}(t)=&\int_{-\varpi}^0 (t-s)^{\gamma-1}E_{\gamma,\gamma}[-\lambda(t-s)^\gamma] \\
&\times\sum_{j=1}^n b_{ij}(s+\varpi)g_j\big(\phi_j^\hbar(s+\varpi-\sigma_j(s))\big)\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{i10}(t)=&\int_{-\varpi}^0 (t-s)^{\gamma-1}E_{\gamma,\gamma}[-\lambda(t-s)^\gamma] \\
&\times\bigvee_{j=1}^n \vartheta_{ij}g_j\big(\phi_j^\hbar(s+\varpi-\sigma_j(s))\big)\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{i11}(t)=&\int_{-\varpi}^0 (t-s)^{\gamma-1}E_{\gamma,\gamma}[-\lambda(t-s)^\gamma] \\
&\times\bigwedge_{j=1}^n \nu_{ij}g_j\big(\phi_j^\hbar(s+\varpi-\sigma_j(s))\big)\mathrm{d}s,
\end{aligned}
$$

$$\lambda_{i12}(t)=\int_{-\varpi}^0 (t-s)^{\gamma-1}E_{\gamma,\gamma}[-\lambda(t-s)^\gamma]\bigvee_{j=1}^n T_{ij}\beta_j\mathrm{d}s,$$

$$\lambda_{i13}(t)=\int_{-\varpi}^0 (t-s)^{\gamma-1}E_{\gamma,\gamma}[-\lambda(t-s)^\gamma]\bigwedge_{j=1}^n H_{ij}\beta_j\mathrm{d}s,$$

$$
\begin{aligned}
\lambda_{i14}(t)=&\int_{-\varpi}^0 (t-s)^{\gamma-1}E_{\gamma,\gamma}[-\lambda(t-s)^\gamma] \\
&\times J_i(s+\varpi)\mathrm{d}s,
\end{aligned}
$$

where $t>0$, $i=1,2,\ldots,n$.

For any $\epsilon > 0$, there is $t_2 > t_1$ satisfying

$$|\lambda_{i1}(t)| < \epsilon, \quad \forall t > t_2, \quad i = 1, 2, \ldots, n. \tag{10}$$

There is a fact that $E_{\gamma,\gamma}[-\lambda t^\gamma] \geq 0$ for $t \geq 0$. According to assumption $(H_4)$, it obtains

$$\begin{aligned} &|\lambda_{i2}(t)| \\ \leq &\Big| \int_0^{t_1} (t-s)^{\gamma-1} E_{\gamma,\gamma}[-\lambda(t-s)^\gamma] \{\Lambda_i^\hbar[\phi_i(s+\varpi)] \\ &-\Lambda_i^\hbar[\phi_i(s)]\} \mathrm{d}s \Big| + \Big| \int_{t_1}^t (t-s)^{\gamma-1} E_{\gamma,\gamma}[-\lambda(t-s)^\gamma] \\ &\times \{\Lambda_i^\hbar[\phi_i(s+\varpi)] - \Lambda_i^\hbar[\phi_i(s)]\} \mathrm{d}s \Big| \\ \leq &\ 2L_i^\lambda \|\phi\|_\infty \int_0^{t_1} (t-s)^{\gamma-1} E_{\gamma,\gamma}[-\lambda(t-s)^\gamma] \mathrm{d}s \\ &+ L_i^\lambda \epsilon \int_{t_1}^t (t-s)^{\gamma-1} E_{\gamma,\gamma}[-\lambda(t-s)^\gamma] \mathrm{d}s \\ \leq &\ 2L_i^\lambda \|\phi\|_\infty \int_0^{t_1} (t-s)^{\gamma-1} E_{\gamma,\gamma}[-\lambda(t-s)^\gamma] \mathrm{d}s + L_i^\lambda \epsilon \\ &\times (t-t_1)^\gamma E_{\gamma,\gamma+1}[-\lambda(t-t_1)^\gamma], \end{aligned}$$

$\forall t > t_1, i = 1, 2, \ldots, n$. By Lemmas II.2-II.3, there exists $t_3 > t_2$ so that

$$|\lambda_{i2}(t)| < \frac{2L_i^\lambda}{\lambda} \epsilon, \quad \forall t > t_3, i = 1, 2, \ldots, n. \tag{11}$$

In line with Corollary 1 in paper [26], there exists $t_4 > t_3$ such that

$$|\lambda_{i3}(t)| < \frac{2}{\lambda} \sum_{j=1}^n (L_j^g \bar{d}_j \|\phi\|_\infty + |g_j(0)|) \epsilon, \tag{12}$$

$$|\lambda_{i4}(t)| < \frac{2}{\lambda} \sum_{j=1}^n \bar{b}_{ij} L_i^g \bar{d}_j \epsilon, \tag{13}$$

$$|\lambda_{i5}(t)| < \frac{2}{\lambda} \sum_{j=1}^n \vartheta_{ij} L_i^g \bar{d}_j \epsilon, \tag{14}$$

$$|\lambda_{i6}(t)| < \frac{2}{\lambda} \sum_{j=1}^n \nu_{ij} L_i^g \bar{d}_j \epsilon, \tag{15}$$

$$|\lambda_{i7}(t)| < \frac{2}{\lambda} \epsilon, \quad |\lambda_{i8}(t)| < L_i^\lambda \|\phi\|_\infty \epsilon, \tag{16}$$

$$|\lambda_{i9}(t)| < \sum_{j=1}^n |b_{ij}| [L_j^g \bar{d}_j \|\phi\|_\infty + |g_j(0)|] \epsilon, \tag{17}$$

$$\begin{aligned} |\lambda_{i10}(t)| < &\Bigg[ \sum_{j=1}^n |\vartheta_{ij}| L_j^g \bar{d}_j \|\phi\|_\infty \\ &+ \bigvee_{j=1}^n |\vartheta_{ij}| |g_j(0)| \Bigg] \epsilon, \end{aligned} \tag{18}$$

$$\begin{aligned} |\lambda_{i11}(t)| < &\Bigg[ \sum_{j=1}^n |\nu_{ij}| L_j^g \bar{d}_j \|\phi\|_\infty \\ &+ \bigwedge_{j=1}^n |\nu_{ij}| |g_j(0)| \Bigg] \epsilon, \end{aligned} \tag{19}$$

$$|\lambda_{i12}(t)| < \bigvee_{j=1}^n |T_{ij}| |\beta_j| \epsilon, \tag{20}$$

$$|\lambda_{i13}(t)| < \bigwedge_{j=1}^n |H_{ij}| |\beta_j| \epsilon, \tag{21}$$

$$|\lambda_{i14}(t)| < \bar{J}_i \epsilon, \quad t > t_4, \quad i = 1, 2, \ldots, n. \tag{22}$$

From Eqs. (10) to Eqs. (22), a sufficiently large $M$ exists to guarantee that

$$|(P\phi)_i(t+\varpi) - (P\phi)_i(t)| < M\epsilon, \quad t > t_4, i = 1, 2, \ldots, n,$$

namely, $P\phi \in \mathbb{S}_\varpi$.

Subsequently, the contractility for operator $P$ will be stated. For $\phi, \psi \in \mathbb{S}_\varpi$, by Eqs. (7) and Lemmas II.1-II.2, it gets

$$\begin{aligned} &(P\phi)_i(t) - (P\psi)_i(t) \\ = &\int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}[-\lambda(t-s)^\gamma] \\ &\times \Big[ \Lambda_i^\hbar(\phi_i(s)) - \Lambda_i^\hbar(\psi_i(s)) \\ &+ \sum_{j=1}^n b_{ij}(s) g_j(\phi_j^\hbar(s-\sigma_j(s))) \\ &- \sum_{j=1}^n b_{ij}(s) g_j(\psi_j^\hbar(s-\sigma_j(s))) \\ &+ \bigvee_{j=1}^n \vartheta_{ij} g_j(\phi_j^\hbar(s-\sigma_j(s))) \\ &- \bigvee_{j=1}^n \vartheta_{ij} g_j(\psi_j^\hbar(s-\sigma_j(s))) \\ &+ \bigwedge_{j=1}^n \nu_{ij} g_j(\phi_j^\hbar(s-\sigma_j(s))) \\ &- \bigwedge_{j=1}^n \nu_{ij} g_j(\psi_j^\hbar(s-\sigma_j(s))) \Big] \mathrm{d}s \\ \leq &\Big[ L_i^\lambda + \sum_{j=1}^n \Big( \bar{b}_{ij} + |\vartheta_{ij}| + |\nu_{ij}| \Big) \\ &\times L_j^g \bar{d}_j \Big] \|\phi - \psi\|_\infty \\ &\times \int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}[-\lambda(t-s)^\gamma] \mathrm{d}s \\ \leq &\frac{1}{\lambda} \Big[ L_i^\lambda + \sum_{j=1}^n \Big( \bar{b}_{ij} + |\vartheta_{ij}| + |\nu_{ij}| \Big) \\ &\times L_j^g \bar{d}_j \Big] \|\phi - \psi\|_\infty, \quad t \geq 0 \end{aligned} \tag{23}$$

for $i = 1, 2, \ldots, n$. By means of assumption $(H_4)$, it gets

$$\begin{aligned} \|P\phi(t) - P\psi(t)\|_\infty \leq &\max_{1 \leq i \leq n} \frac{1}{\lambda} \Big[ L_i^\lambda + \sum_{j=1}^n \Big( \bar{b}_{ij} + |\vartheta_{ij}| \\ &+ |\nu_{ij}| \Big) L_j^g \bar{d}_j \Big] \|\phi - \psi\|_\infty \end{aligned}$$

$$= \max_{1\le i\le n} \nu_i \|\phi - \psi\|_\infty.$$

So $P$ has the feature of contraction, which admits only one $\phi^* = P\phi^*$ and $\phi^*$ is $S$-APO$_\varpi$ of model (1). This completes the proof. ∎

**Remark III.2.** *In view of assumption $(H_4)$ and the expression $\Lambda_i^\hbar\big(p_i(t)\big) = \lambda p_i(t) - a_i\big(p_i^\hbar(t)\big)(i = 1,2,\dots,n)$, we can see that there is a close relationship between $L_i^\lambda(i = 1,2,\dots,n)$ and $\lambda$. To be precise, we need $a_i$ is a nondecreasing function as $d_i > 0$ for $i = 1,2,\dots,n$.*

**Remark III.3.** *The assumption $(H_3)$ in literature [48] demands the boundedness of function $g_j(j = 1,2,\dots,n)$, which is very restricted. Whereas, we remove it in this paper. For this reason, compared with literature [48], the present text possesses apparent advantages.*

**Remark III.4.** *In light of assumption $(H_3)$ in Theorem III.1, it is worth noting that the Lipschitz constants $L_j^g$ and $L_i^\lambda(i,j = 1,2,\dots,n)$ respectively in conditions $(H_2)$ and $(H_4)$ and the coefficients of Eqs. (1) are preferably smaller positive constants, in contrast, $\lambda$ is a larger positive constant.*

**Remark III.5.** *Assumptions $(H_2)$-$(H_4)$ in Theorem III.1 imply that the uniqueness of solution to Eqs. (1) remains unaffected by time-varying delays. However, in order to achieve the $S$-APO$_\varpi$ for Eqs. (1), delay $\sigma_j(t)(j = 1,2,\dots,n)$ is required to be periodic. Naturally, there is an open problem whether the conclusion is valid in case delay $\sigma_j(t)(j = 1,2,\dots,n)$ is $S$-asymptotically $\varpi$-periodic.*

## IV. Global asymptotical stability (GAS)

**Lemma IV.1** ([47]). *Let us study the following FODEs*

$$\begin{cases} {}^cD_0^\gamma w_i(t) \le -a_i w_i(t) + b_i \sum_{j=1}^n w_j[t-\sigma_j(t)], \quad t>0, \\ w_i(t) = \phi_i(t) \ge 0, \quad t\in[-\sigma,0], \quad \sigma = \max_{1\le i\le n}\sup_{t>0}|\sigma_j(t)|, \end{cases}$$

*and*

$$\begin{cases} {}^cD_0^\gamma p_i(t) = -a_i p_i(t) + b_i \sum_{j=1}^n p_j[t-\sigma_j(t)], \quad t>0, \\ p_i(t) = \phi_i(t) \ge 0, \quad t\in[-\sigma,0], \end{cases} \tag{24}$$

*where $w_i, p_i \ge$ are continuous on $[0,+\infty)$, $i = 1,2,\dots,n$. If $a_i > 0$ and $b_i > 0$, then $w_i(t) \le p_i(t)$, $\forall t \ge 0$, $i = 1,2,\dots,n$.*

**Lemma IV.2** ([20]). *Assume that $\dot\sigma_j^+ = \sup_{t\ge0}\dot\sigma_j(t) < 1(i = 1,2,\dots,n)$ and $\min_{1\le i\le n} a_i > \max_{1\le j\le n}\sum_{i=1}^n \frac{b_i}{1-\dot\sigma_j^+}$ in system (24), then system (24) is GAS.*

### A. Stability result for FOCGNNs

Set

$$L^g = \max_{1\le j\le n} L_j^g, \quad L^\lambda = \max_{1\le i\le n} L_i^\lambda, \quad \bar b_{i*} = \max_{1\le j\le n}\bar b_{ij},$$

$$\vartheta_{i*} = \max_{1\le j\le n}|\vartheta_{ij}|, \quad \nu_{i*} = \max_{1\le j\le n}|\nu_{ij}|, \quad \bar d = \max_{1\le j\le n}\bar d_j$$

for $i = 1,2,\dots,n$.

**Lemma IV.3** ([38]). *Let $x \in C^1([0,+\infty),\mathbb{R})$. Then ${}^cD_0^\gamma|x(t)| \le \mathrm{sgn}(x(t)){}^cD_0^\gamma x(t)$ for $t\ge0$ and $\gamma\in(0,1)$.*

**Theorem IV.1.** *Model (1) is GAS when $(H_2)$, $(H_4)$ and*

$$(H_5)\quad \lambda > L^\lambda + \max_{1\le j\le n}\sum_{i=1}^n \frac{(\bar b_{i*}+\vartheta_{i*}+\nu_{i*})L^g\bar d}{1-\dot\sigma_j^+}.$$

*are fulfilled.*

**Proof:** Let $u = (u_1,u_2,\dots,u_n)^\top$ and $v = (v_1,v_2,\dots,v_n)^\top$ solve model (1) and $q_i = u_i - v_i, i = 1,2,\dots,n$. So

$$\begin{aligned} {}^cD_0^\gamma q_i(t) &= -\lambda q_i(t) + \Lambda_i^\hbar\big(u_i(t)\big) - \Lambda_i^\hbar\big(v_i(t)\big) \\ &+ \sum_{j=1}^n b_{ij}(t) g_j\big(u_j^\hbar(t-\sigma_j(t))\big) \\ &- \sum_{j=1}^n b_{ij}(t) g_j\big(v_j^\hbar(t-\sigma_j(t))\big) \\ &+ \bigvee_{j=1}^n \vartheta_{ij} g_j\big(u_j^\hbar(t-\sigma_j(t))\big) \\ &- \bigvee_{j=1}^n \vartheta_{ij} g_j\big(v_j^\hbar(t-\sigma_j(t))\big) \\ &+ \bigwedge_{j=1}^n \nu_{ij} g_j\big(u_j^\hbar(t-\sigma_j(t))\big) \\ &- \bigwedge_{j=1}^n \nu_{ij} g_j\big(v_j^\hbar(t-\sigma_j(t))\big), \quad t>0. \end{aligned} \tag{25}$$

By Lemma IV.3 and Eqs. (25), it has

$$\begin{aligned} &{}^cD_0^\gamma|q_i(t)| \\ =\ &\mathrm{sgn}(q_i(t)){}^cD_0^\gamma q_i(t) \\ =\ &-\lambda q_i(t)\mathrm{sgn}(q_i(t)) \\ &+\mathrm{sgn}(q_i(t))\Big[\Lambda_i^\hbar\big(u_i(t)\big) - \Lambda_i^\hbar\big(v_i(t)\big)\Big] + \mathrm{sgn}(q_i(t)) \\ &\times\sum_{j=1}^n b_{ij}(t)\Big[g_j\big(u_j^\hbar(t-\sigma_j(t))\big) - g_j\big(v_j^\hbar(t-\sigma_j(t))\big)\Big] \\ &+\mathrm{sgn}(q_i(t)) \\ &\times\bigvee_{j=1}^n \vartheta_{ij}\Big[g_j\big(u_j^\hbar(t-\sigma_j(t))\big) - g_j\big(v_j^\hbar(t-\sigma_j(t))\big)\Big] \\ &+\mathrm{sgn}(q_i(t)) \\ &\times\bigwedge_{j=1}^n \nu_{ij}\Big[g_j\big(u_j^\hbar(t-\sigma_j(t))\big) - g_j\big(v_j^\hbar(t-\sigma_j(t))\big)\Big] \\ \le\ &(-\lambda+L^\lambda)|q_i(t)| + \sum_{j=1}^n(\bar b_{i*}+\vartheta_{i*}+\nu_{i*}) \\ &\times L^g\bar d|q_j(t-\sigma_j(t))|, \quad t>0, i=1,2,\dots,n. \end{aligned} \tag{26}$$

Then

$$\begin{cases} {}^cD_0^\gamma \rho_i(t) = (-\lambda+L^\lambda)\rho_i(t) + \sum_{j=1}^n(\bar b_{i*}+\vartheta_{i*}+\nu_{i*}) \\ \qquad\qquad \times L^g\bar d\rho_j(t-\sigma_j(t)), \quad t>0, \\ \rho_i(s) = |q_i(s)| \ge 0, \quad s\in[-\sigma,0], i = 1,2,\dots,n. \end{cases}$$

By $(H_5)$, Lemmas IV.1 and IV.2, $\lim_{t\to\infty}|q_i(t)| \le \lim_{t\to\infty}\rho(t) = 0$, $i = 1,2,\dots,n$. It obtains GAS of Eqs. (1). The proof is end. ∎

Summarize Theorems III.1 and IV.1, one obtains

**Theorem IV.2.** *System* (1) *admits a unique globally asymptotically stable S-APO*$_\varpi$ *if* $(H_1)$-$(H_5)$ *are fulfilled.*

**Remark IV.1.** *Hypothesis* $(H_5)$ *illustrates that the global asymptotic stability of Eqs.* (1) *depends on delay* $\sigma$, *concretely, it demands* $\dot{\sigma}_j^+ < 1$ *for* $j = 1, 2, \dots, n$.

**Remark IV.2.** *According to Theorem III.1 and Theorem IV.1, we can see that activation functions* $g_j(\cdot)$ *and* $\Lambda_i(\cdot)$ *are very important.*

**Remark IV.3.** *For the effectiveness of condition* $(H_5)$, *it is well worth paying attention to the following aspects. On one hand, the coefficients* $(b_{ij}, \vartheta_{ij}, \nu_{ij}, d_i)$ *and Lipschitz constants* $(L_j^g, L_i^\lambda)(i, j = 1, 2, \dots, n)$ *should be selected smaller positive constants. For another, the delay* $\dot{\sigma}_j^+ (j = 1, 2, \dots, n)$ *term should optimally be close to* 1 *and* $\lambda$ *should be a greater positive constant.*

## V. Numerical examples

**Example V.1.** *We discuss*

$$
\begin{aligned}
{}^cD_0^{0.4} z_i(t) &= -\lambda z_i(t) + \Lambda_i^\hbar(z_i(t)) \\
&\quad + \sum_{j=1}^{2} b_{ij}(t) g_j\big(z_j^\hbar(t - \sigma_j(t))\big) \\
&\quad + \bigvee_{j=1}^{2} \vartheta_{ij} g_j\big(z_j^\hbar(t - \sigma_j(t))\big) \\
&\quad + \bigwedge_{j=1}^{2} \nu_{ij} g_j\big(z_j^\hbar(t - \sigma_j(t))\big) + \bigvee_{j=1}^{2} T_{ij}\beta_j \\
&\quad + \bigwedge_{j=1}^{2} H_{ij}\beta_j + J_i(t), \quad t > 0, \qquad (27)
\end{aligned}
$$

*where* $z_i(s) = \varphi_i(s)$ *for* $s \in [-1.02, 0]$ *with* $\sigma_i(t) = 1 + 0.02\cos t$, $J_i(t) = 1 + \sin\sqrt{3}t$, $\beta_i = 0.1$, $a_i(z_i(t)) = 2z_i(t)$, $d_i(s) = \frac{1}{1+0.01\sin s}$,

$$
\begin{aligned}
b_{ij}(t) &= \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix} \\
&= \begin{pmatrix} 0.2 + 0.1\sin t & 0.3\cos\sqrt{5}t \\ 0.3\sin\sqrt{5}t & 0.2 + 0.1\cos t \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\vartheta_{ij}(t) &= \begin{pmatrix} \vartheta_{11}(t) & \vartheta_{12}(t) \\ \vartheta_{21}(t) & \vartheta_{22}(t) \end{pmatrix} = \begin{pmatrix} 0.1 & 0.2 \\ 0.2 & 0.4 \end{pmatrix} \\
&= \begin{pmatrix} T_{11}(t) & T_{12}(t) \\ T_{21}(t) & T_{22}(t) \end{pmatrix} = T_{ij}(t),
\end{aligned}
$$

$$
\begin{aligned}
\nu_{ij}(t) &= \begin{pmatrix} \nu_{11}(t) & \nu_{12}(t) \\ \nu_{21}(t) & \nu_{22}(t) \end{pmatrix} = \begin{pmatrix} 0.4 & 0.5 \\ 0.1 & 0.1 \end{pmatrix} \\
&= \begin{pmatrix} H_{11}(t) & H_{12}(t) \\ H_{21}(t) & H_{22}(t) \end{pmatrix} = H_{ij}(t)
\end{aligned}
$$

*for* $i, j = 1, 2$ *and*

$$
g_1(z_1(t)) = 0.1|\sin(z_1(t))|, \quad g_2(z_2(t)) = \frac{0.02 z_2^2(t)}{1 + z_2^2(t)}.
$$

*Taking* $\lambda = 12$, *it follows from a direct calculation and MATLAB tool that* $L_1^g = 0.1$, $L_2^g = 0.04$, $L^\lambda = \max_{1 \le i \le 2} L_i^\lambda \approx 10.0198$. *Evidently, assumptions* $(H_1)$, $(H_2)$ *and* $(H_4)$ *hold. Besides, for* $i = 1$, *one has*

$$
\frac{1}{\lambda}\left[L_1^\lambda + \sum_{j=1}^{n} \Big(\bar{b}_{1j} + |\vartheta_{1j}| + |\nu_{1j}|\Big) L_j^g \bar{d}_j\right] \in (0.8617, 0.8650)
$$

*and if* $i = 2$, *then*

$$
\frac{1}{\lambda}\left[L_1^\lambda + \sum_{j=1}^{n} \Big(\bar{b}_{2j} + |\vartheta_{2j}| + |\nu_{2j}|\Big) L_j^g \bar{d}_j\right] \in (0.9267, 0.9300).
$$

*Furthermore,*

$$
L^\lambda + \max_{1 \le j \le n} \sum_{i=1}^{n} \frac{(\bar{b}_{i*} + \vartheta_{i*} + \nu_{i*}) L^g \bar{d}}{1 - \dot{\sigma}_j^+} \approx 10.2925 < 12.
$$

*As a consequence, system* (27) *possesses a single S-asymptotic* $\varpi$*-periodic oscillation, as well as global asymptotic stability, see Figures* 1-4.

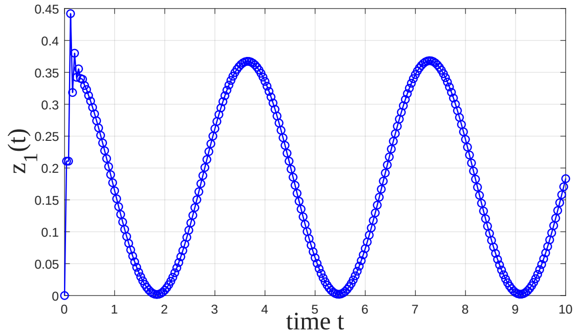


Fig. 1: State variable $z_1(t)$ of Eqs. (27)

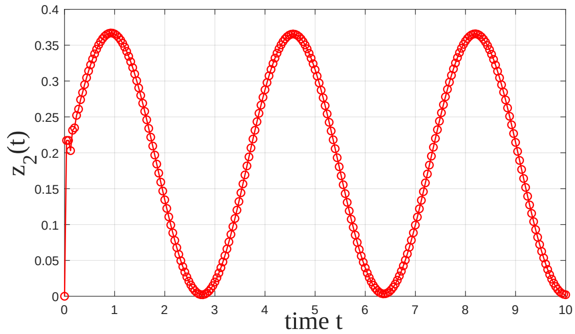


Fig. 2: State variable $z_2(t)$ of Eqs. (27)

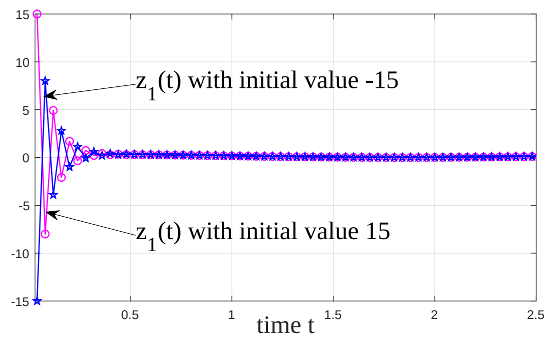


Fig. 3: Global asymptotic stability of $z_1(t)$ of Eqs. (27)

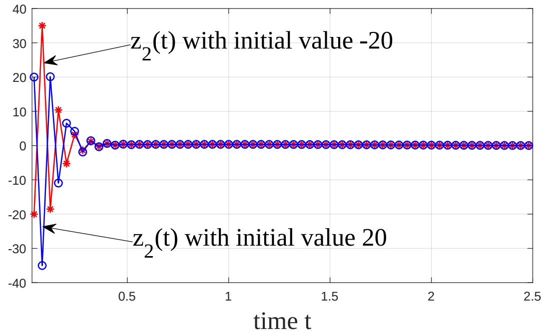


Fig. 4: Global asymptotic stability of $z_2(t)$ of Eqs. (27)

## VI. Conclusions and Future Works

The present paper learns $S$-APO$_\varpi$ in FOCGNNs via exploiting several crucial pivotal properties, in accordance with the Volterra integral expression corresponding to FODEs depicted by Mittag-Leffler function $E_{\gamma,\beta}(z)$, which is a generalized formulation of the Volterra integral equation corresponding to the integer-order differential equations expressed by the exponential function $e^z$. Furthermore, FOCGNNs is GAS. However, there are several outstanding topics that are worth addressing in the future, highlighted below.

1) The other cases, such as $\gamma_i \in (1, 2]\,(i = 1, 2, \dots, n)$, should be discussed.
2) It is meaningful to take the Mittag-Leffler Euler differences for Reimann-Liouville nonlocal derivatives into consideration.
3) Some other dynamical behaviors of FOCGNNs (1) can be taken into further consideration, such as, Mittag-Leffler stability, (pseudo) almost period, (pseudo) almost automorphism, etc.
4) We can also research dynamical behaviors of impulsive or stochastic FOCGNNs.
5) Given that biological models are not only affected by time but also spatially related, it is of interest to consider fractional-order parabolic CGNNs.

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