Fault Diameter of Strong Product Graph of an Arbitrary Connected Graph and a Complete Graph

Yuxiang Yue, Feng Li

***Abstract*—Fault diameter is an important parameter to measure the reliability and effectiveness of interconnection networks. Strong product is an efficient method to construct large graphs from small graphs. In this paper, we study the fault diameter of strong product graph of an arbitrary connected graph and a complete graph. According to the classification of an arbitrary connected graph, we first determine the fault diameter of strong product graph of two complete graphs. Then we give the fault diameter of strong product graph of an incompletely connected graph and a complete graph, which can be denoted by the fault diameter of its incompletely factor graph. In addition, we also give a more general result about fault diameter.**

***Index Terms*—fault diameter, complete graph, incompletely connected graph, strong product graph.**

I. Introduction

In this paper, all graphs considered are simple and undirected graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The cardinality of vertex set is denoted by $|V(G)|$. Let R be a path in G, the length of R is $|V(R)|-1$ and is denoted by $L(R)$. Let x and y be any two vertices in G, the length of a shortest path between x and y in G is called the distance between x and y, which is denoted by $d(G;x,y)$. Then the diameter of G is the maximum length of all distances between any two vertices in G, denoted by $d(G)$. If there are two or more paths connecting x and y, and the internal vertices of these paths are not the same except for x and y, then these paths are called internally vertex disjoint paths. The maximum number of internally vertex disjoint paths between x and y in G, denoted as $\zeta(G;x,y)$.

If any vertex subset in G is deleted, this is equivalent to remove all vertices of the vertex subset and all edges incident with the vertex subset. The connectivity of G is the minimum cardinality of all vertex subsets in G, which are deleted from G to obtain an unconnected or a trivial graph, denoted by $\kappa(G)$. If G is a complete graph K_n, we can directly get $\kappa(K_n)=n-1$. Especially, if $\kappa(G)\geq w$, the graph G is called w-connected graph. We use $\delta(G)$ denote the minimum degree of G. A graph G is called maximally connected graph, if $\kappa(G)=\delta(G)$. The set of neighbors of a vertex x in G is denoted by $N_G(x)$. In addition, the definitions of strong product and fault diameter are given below.

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Definition 1. ([21]) *Let G be a w-connected graph, the fault vertex set of G is denoted by F with $|F|<w$. The $(w-1)$-fault diameter of a graph G is defined as*

$$D_w(G)=max\{d(G-F):F\subset V(G),|F|<w\}.$$

Remark 1. *In the worst case of failure, we can get that $|F|=w-1$. For any w-connected graph G, the relationship between diameter and fault diameter holds*

$$d(G)=D_1(G)\leq D_2(G)\leq\cdots\leq D_{w-1}(G)\leq D_w(G).$$

Definition 2. ([22]) *Let $G_1=(V(G_1),E(G_1))$, $G_2=(V(G_2),E(G_2))$, the strong product of G_1 and G_2 is the graph denoted as $G_1\otimes G_2$ with vertex set $V(G_1)\times V(G_2)$. Any two distinct vertices (x_1,x_2) and (y_1,y_2) in $G_1\otimes G_2$ are adjacent, if and only if $x_1=x_2$, $(y_1,y_2)\in E(G_2)$, or $y_1=y_2$, $(x_1,x_2)\in E(G_1)$, or $(x_1,x_2)\in E(G_1)$, $(y_1,y_2)\in E(G_2)$.*

Remark 2. *From the above definition, the strong product has the following results.*
(a) $G_1\otimes G_2\cong G_2\otimes G_1$.
(b) $(G_1\otimes G_2)\otimes G_3\cong G_1\otimes(G_2\otimes G_3)$.
(c) $\{x\}\otimes G\cong G\otimes\{x\}\cong G$.
(d) $|V(G_1\otimes G_2)|=|V(G_1)||V(G_2)|$.

For the strong product $\{x\}\otimes G$, it is usually denoted by a symbol xG. Similarly, the strong product $G\otimes\{x\}$ can also be denoted by a symbol Gx. In addition, for brevity, the vertices (x_1,x_2) are written as x_1x_2.

Since an arbitrary connected graph can be divided into complete graph and incompletely connected graph, we use $K_m\otimes K_n$ to denote the strong product graph of two complete graphs with orders $m,n\geq 1$. The example $K_2\otimes K_4$ is shown on Fig. 1, where $V(K_2)=\{x_1,x_2\}$ and $V(K_4)=\{y_1,y_2,y_3,y_4\}$.

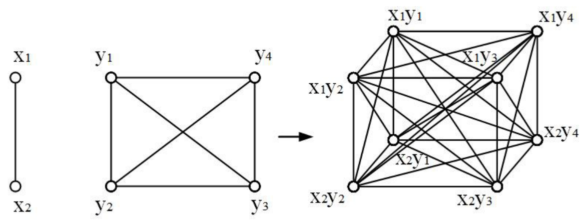

Fig. 1. The strong product graph $K_2\otimes K_4$

In addition, we use $H\otimes K_n$ to denote the strong product graph of an incompletely connected graph with order $m\geq$

2 and a complete graph with order $n \geq 1$. The example $P_3 \otimes K_3$ is shown on Fig. 2, where $V(P_3) = \{x_1, x_2, x_3\}$ and $V(K_3) = \{y_1, y_2, y_3\}$. In this paper, we mainly discuss the above two kinds of strong product graphs. For undefined symbols and terms, readers can refer to the literature [9].

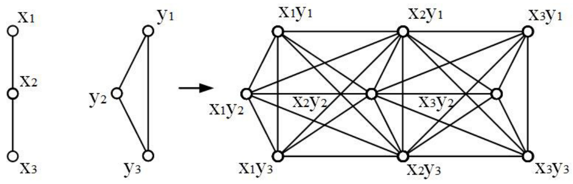

Fig. 2. The strong product graph $P_3 \otimes K_3$

The topological structure of any interconnection network is a graph, the vertex represents processor, the edge represents link, and the diameter represents the transmission delay of network. However, the processors are prone to failure if they work for a long time, this will affect the effectiveness of network information transmission. Krishnamoorthy and Krishnamurthy [10] proposed the concept of fault diameter for the first time to quantify this influence. They also determined the fault diameter of hypercube and gave an upper bound of the fault diameter of Cartesian product graph in the paper. For the general results, the upper bounds of the fault diameter of an arbitrary connected graph are given in [5, 6], and the relationship between fault diameter and edge fault diameter is given in [4]. For the specific results, the fault diameters of many well-known networks are determined [7, 11, 16]. The latest results are about the fault diameters of deformed hypercube networks, see the literatures [14, 15, 19].

The product graph is an important method to construct large graphs from small graphs. Recently, the researches on the product graph have attracted more and more attention [1, 12, 13, 20]. According to the definition of fault diameter, the connectivity must be given before determining the fault diameter of any graph. Especially, it is easy to determine the connectivity of Cartesian product graphs. So most researches on the fault diameter of product graphs focus on the fault diameter of Cartesian product graphs. In [21], the upper bound of fault diameter of Cartesian product graph of two graphs is given which is the correction of the result of [10]. Subsequently, the upper bound of fault diameter of Cartesian product graph of n graphs is given in [2]. There are also some results about Cartesian graph bundles in [3, 8]. But for other product graphs, such as the fault diameter of strong product graphs, there are no relevant results.

The strong product was first proposed by Sabidussi [17]. It is the union of Cartesian product and direct product [9]. However, it took a long time to determine the connectivity of strong product graphs. Yang and Xu [22] first gave the connectivity of the strong product graph of an incompletely connected graph and a complete graph. Through this result, they also gave the connectivity of the strong product graph of two maximally connected graphs. Then Špacapan [18] determined the connectivity of strong product graphs. The work on the fault diameter of strong product graphs can be carried out. In this paper, we study the fault diameter of strong product graph of an arbitrary connected graph and a complete graph. According to the classification of an arbitrary connected graph, we first determine the fault diameter of strong product graph of two complete graphs. Then we give the fault diameter of the strong product graph of an incompletely connected graph and a complete graph by constructing the worst case paths. Moreover, we also give a more general result about fault diameter through Menger Theorem.

II. Main results

Before determining the fault diameter, we first obtain the connectivity of the two kinds of strong product graphs. It is easy to know that the connectivity of a complete graph is $\kappa(K_n) = n - 1$. But for the connectivity of strong product graph of two complete graphs, we need the following lemma to provide a solution.

Lemma 1. *Let K_m and K_n be two complete graphs with orders $m, n \geq 1$. Then*

$$K_m \otimes K_n = K_{mn}.$$

Proof. Let $G = K_m \otimes K_n$, $x_h y_g$ and $x_p y_q$ be any two vertices in G, where $x_h, x_p \in V(K_m)$ and $y_g, y_q \in V(K_n)$. By the definition of strong product, there are three cases can be discussed.

Case 1. $x_h = x_p$.

Since $x_h = x_p$ and $(y_g, y_q) \in E(K_n)$, the two vertices $x_h y_g$ and $x_h y_q$ are adjacent in G, we can get $(x_h y_g, x_h y_q) \in E(G)$.

Case 2. $y_g = y_q$.

Since $(x_h, x_p) \in E(K_m)$ and $y_g = y_q$, the two vertices $x_h y_g$ and $x_p y_g$ are adjacent in G, we can get $(x_h y_g, x_p y_g) \in E(G)$.

Case 3. $x_h \neq x_p$, $y_g \neq y_q$.

Since $(x_h, x_p) \in E(K_m)$ and $(y_g, y_q) \in E(K_n)$, the two vertices $x_h y_g$ and $x_p y_q$ are adjacent in G, we can get $(x_h y_g, x_p y_q) \in E(G)$.

So any two vertices $x_h y_g$ and $x_p y_q$ are adjacent in G. By the (d) of Remark 2, we have

$$|V(G)| = |V(K_m)||V(K_n)| = mn.$$

From this, G is the complete graph K_{mn}. □

Through (a) and (b) of Remark 2, the strong product is commutative and associative, we can still extend Lemma 1 to t-dimension.

Corollary 1. *Let $K_{v_1}, K_{v_2}, \cdots, K_{v_t}$ be t complete graphs with number $t \geq 2$ and orders $v_1, v_2, \cdots, v_t \geq 1$. Then*

$$K_{v_1} \otimes K_{v_2} \otimes \cdots \otimes K_{v_t} = K_{\prod_{i=1}^{t} v_i}.$$

Lemma 2. *Let K_n be a complete graph with order $n \geq 1$, and F be the fault vertex set in K_n with $|F| = d$. Then*

$$K_n - F = K_{n-d}.$$

Proof. Before removing the fault vertex set F, there is always an edge between any two vertices in the complete graph K_n. Since the removed edges are all incident with the vertices in F, then any two vertices are still adjacent in $K_n - F$. For $|F| = d$, then

$$|V(K_n - F)| = n - d.$$

From this, $K_n - F$ is the complete graph K_{n-d}. □

Through the previous two lemmas, the strong product graph of two complete graphs is still a complete graph. The complete graph is still a complete graph if any vertex subset is deleted.

Theorem 3. *Let K_m and K_n be two complete graphs with orders $m, n \geq 1$. For any $1 \leq w \leq mn - 1$. Then*

$$D_w(K_m \otimes K_n) = 1.$$

Proof. Let F be the fault vertex set with $|F| = w - 1$. By Lemma 1, we can get $K_m \otimes K_n = K_{mn}$, then $\kappa(K_m \otimes K_n) = mn - 1$. For any $1 \leq w \leq mn-1$, $K_m \otimes K_n - F$ is connected. By Lemma 2, for any fault vertex set F, the diameter of $K_m \otimes K_n - F$ is

$$d(K_m \otimes K_n - F) = d(K_{mn} - F) = d(K_{mn-w+1}) = 1.$$

From this, we have $D_w(K_m \otimes K_n) = 1$. □

The Theorem 3 determines the fault diameter of $K_m \otimes K_n$. By Corollary 1, we can directly extend the theorem to t-dimension.

Corollary 2. *Let $K_{v_1}, K_{v_2}, \cdots, K_{v_t}$ be t complete graphs with number $t \geq 2$ and orders $v_1, v_2, \cdots, v_t \geq 1$. For $1 \leq w \leq \prod_{i=1}^{t} v_i - 1$. Then*

$$D_w(K_{v_1} \otimes K_{v_2} \otimes \cdots \otimes K_{v_t}) = 1.$$

Before determining the fault diameter of $H \otimes K_n$, we first obtain its connectivity. The following lemma provides a solution.

Lemma 4. ([22]) *Let H be an incompletely connected graph with the connectivity $k \geq 1$, and K_n be a complete graph with order $n \geq 1$. Then*

$$\kappa(H \otimes K_n) = nk.$$

Under the determined connectivity, we prove the following theorem by constructing isomorphic subgraphs and the worst case paths.

Theorem 5. *Let H be an incompletely connected graph with the connectivity $k \geq 1$, and K_n be a complete graph with order $n \geq 1$. For any $1 \leq w \leq nk$. Then*

$$D_w(H \otimes K_n) = D_{\lceil \frac{w}{n} \rceil}(H).$$

Proof. Let $G = H \otimes K_n$ with $V(H) = \{x_1, \cdots, x_m\}$ and $V(K_n) = \{y_1, \cdots, y_n\}$. Since H is an incompletely connected graph, we have $m \geq 3$. Let $x_h y_g$ and $x_p y_q$ be any two vertices in G, where $x_h, x_p \in V(H)$ and $y_g, y_q \in V(K_n)$. Let F be the fault vertex set in G with $|F| = w - 1$. By Lemma 4, we can get $\kappa(H \otimes K_n) = nk$. So for any $1 \leq w \leq nk$, $G - F$ is connected. We first discuss the upper bound of the fault diameter of G. According to the positional relationship between the two vertices $x_h y_g$ and $x_p y_q$, it can be divided into three cases.

Case 1. $x_h = x_p$.

Without loss of generality, we discuss the distance between any two vertices $x_h y_g$ and $x_h y_q$ in $x_h K_n$ after failure. By the (c) of Remark 2, $x_h K_n \cong K_n$, then $x_h K_n$ is a complete graph and any two vertices are adjacent in $x_h K_n$. By Lemma 2, $x_h K_n - F \cap V(x_h K_n)$ is still a complete graph, then any two vertices $x_h y_g$ and $x_h y_q$ are still adjacent in $G - F$. Since $m \geq 3$, then $d(H) \geq 1$, we have $d(G - F; x_h y_g, x_h y_q) = 1 \leq d(H)$.

Case 2. $y_g = y_q$.

Consider n disjoint subgraphs Hy_j in G for $j = 1, 2, \cdots, n$. Since the vertex sets of n disjoint subgraphs Hy_j is a partition of the vertex set of G, we have

$$V(G) = V(Hy_1) \cup V(Hy_2) \cup \cdots \cup V(Hy_n).$$

The cardinality of any fault vertex set F holds $1 \leq |F| = w - 1 \leq nk - 1$. Since there are n disjoint subgraphs, even in the worst case, there is at least one subgraph with no more than $\lceil \frac{w}{n} \rceil - 1$ fault vertices. Without loss of generality, we assume that there is a subgraph Hy_j satisfies

$$1 \leq |F \cap V(Hy_j)| = \lceil \frac{w}{n} \rceil - 1 \leq k - 1.$$

According to whether the two vertices $x_h y_j$ and $x_p y_j$ in Hy_j are fault vertices, we can discuss three subcases.

Subcase 2.1. $x_h y_j, x_p y_j \notin F$.

Since the connectivity of H is k, without loss of generality, we let

$$N_H(x_h) = \{a_1, a_2, \cdots, a_k\}.$$

$$N_H(x_p) = \{b_1, b_2, \cdots, b_k\}.$$

By Lemma 1, we can get $K_2 \otimes K_n = K_{2n}$. Each vertex in $V(x_h K_n)$ is adjacent to all vertices in $V(a_1 K_n) \cup V(a_2 K_n) \cup \cdots \cup V(a_k K_n)$, then each vertex in $V(x_p K_n)$ is adjacent to all vertices in $V(b_1 K_n) \cup V(b_2 K_n) \cup \cdots \cup V(b_k K_n)$. Consider in the subgraph $Hy_j - F \cap V(Hy_j)$, there is at least a path between $x_h y_j$ and $x_p y_j$ of length no more than $D_{\lceil \frac{w}{n} \rceil}(H)$. Without loss of generality, we assume that R_1 is the path of length $D_{\lceil \frac{w}{n} \rceil}(H)$ between $x_h y_j$ and $x_p y_j$ in $Hy_j - F \cap V(Hy_j)$ by the neighbors $a_1 y_j$ and $b_1 y_j$.

$$R_1 : x_h y_j \rightarrow a_1 y_j \rightarrow \cdots \rightarrow b_1 y_j \rightarrow x_p y_j.$$

Since the two vertices $x_h y_g$ and $a_1 y_j$ are adjacent, the two vertices $x_p y_g$ and $b_1 y_j$ are adjacent, we can construct a path R_2 between $x_h y_g$ and $x_p y_g$ in $G - F$.

$$R_2 : x_h y_g \rightarrow a_1 y_j \xrightarrow{R_1 - \{x_h y_j, x_p y_j\}} b_1 y_j \rightarrow x_p y_g,$$

with $L(R_2) = L(R_1) = D_{\lceil \frac{w}{n} \rceil}(H)$. From this, there is at least a path between $x_h y_g$ and $x_p y_g$ of length no more than $D_{\lceil \frac{w}{n} \rceil}(H)$ in $G - F$, we have $d(G - F; x_h y_g, x_p y_g) \leq D_{\lceil \frac{w}{n} \rceil}(H)$.

Subcase 2.2. $x_h y_j \notin F$, $x_p y_j \in F$ or $x_h y_j \in F$, $x_p y_j \notin F$.

Without loss of generality, we assume that $x_h y_j \in F$ and $x_p y_j \notin F$. The neighbors of $x_h y_j$ in Hy_j are

$$N_{Hy_j}(x_h y_j) = \{a_1 y_j, a_2 y_j, \cdots, a_k y_j\}.$$

Since $x_h y_g$ is adjacent to the vertices of $V(N_{Hy_j}(x_h y_j))$ and not adjacent to the vertices of $V(Hy_j) \backslash V(N_{Hy_j}(x_h y_j))$. Remove the vertex $x_h y_j$ and the edges incident with $x_h y_j$ in Hy_j, then consider the edge set

$$E_1 = \{(x_h y_g, a_i y_j) : i = 1, 2, \cdots, k\}.$$

Combine the vertex $x_h y_g$, the edges of E_1 and the subgraph $Hy_j - \{x_h y_j\}$ into a new subgraph $H^{'}$. The new subgraph $H^{'}$ has the same number of vertices and edges as subgraph

Hy_j, and retains the adjacency of the subgraph Hy_j. From this, the two subgraphs $H^{'}$ and Hy_j are isomorphic. For the subgraph $H^{'}$, we have

$$|F \cap V(H^{'})| = \lceil \frac{w}{n} \rceil - 2 \leq k - 2.$$

In the subgraph $H^{'} - F \cap V(H^{'})$, even in the worst case, there is at least a path between $x_h y_g$ and $x_p y_j$ of length no more than $D_{\lceil \frac{w}{n} \rceil - 1}(H)$. Without loss of generality, we assume that R_3 is the path of length $D_{\lceil \frac{w}{n} \rceil - 1}(H)$ between $x_h y_g$ and $x_p y_j$ in $H^{'} - F \cap V(H^{'})$ by the neighbor $b_1 y_j$.

$$R_3 : x_h y_g \rightarrow \cdots \rightarrow b_1 y_j \rightarrow x_p y_j.$$

Since the two vertices $x_p y_g$ and $b_1 y_j$ are adjacent, we can construct a path R_4 between $x_h y_g$ and $x_p y_g$ in $G - F$.

$$R_4 : x_h y_g \xrightarrow{R_3 - x_p y_j} b_1 y_j \rightarrow x_p y_g,$$

with $L(R_4) = L(R_3) = D_{\lceil \frac{w}{n} \rceil - 1}(H)$. From this, there is at least a path between $x_h y_g$ and $x_p y_g$ of length no more than $D_{\lceil \frac{w}{n} \rceil - 1}(H)$ in $G - F$, we have $d(G - F; x_h y_g, x_p y_g) \leq D_{\lceil \frac{w}{n} \rceil - 1}(H)$.

Subcase 2.3. $x_h y_j, x_p y_j \in F$.

Consider the neighbors of $x_p y_j$ in Hy_j are

$$N_{Hy_j}(x_p y_j) = \{b_1 y_j, b_2 y_j, \cdots, b_k y_j\}.$$

Since $x_p y_g$ is adjacent to the vertices of $V(N_{Hy_j}(x_p y_j))$ and not adjacent to the vertices of $V(Hy_j) \backslash V(N_{Hy_j}(x_p y_j))$. Remove $x_h y_j$, $x_p y_j$ and the edges incident with $x_h y_j$ or $x_p y_j$ in Hy_j, then consider the edge set

$$E_2 = \{(x_p y_g, b_i y_j) : i = 1, 2, \cdots, k\}.$$

With the adjacency of $x_h y_g$ in the Subcase 2.2, we combine the vertex $x_h y_g$, the vertex $x_p y_g$, the edges of E_1, the edges of E_2 and the sugraph $Hy_j - \{x_h y_j, x_p y_j\}$ into a new subgraph $H^{''}$. The new subgraph $H^{''}$ has the same number of vertices and edges as subgraph Hy_j, and retains the adjacency of the subgraph Hy_j. From this, the two subgraphs $H^{''}$ and Hy_j are isomorphic. For the subgraph $H^{''}$, we have

$$|F \cap V(H^{''})| = \lceil \frac{w}{n} \rceil - 3 \leq k - 3.$$

In the subgraph $H^{''} - F \cap V(H^{''})$, even in the worst case, there is at least a path between $x_h y_g$ and $x_p y_g$ of length no more than $D_{\lceil \frac{w}{n} \rceil - 2}(H)$. From this, there is also at least a path between $x_h y_g$ and $x_p y_g$ of length no more than $D_{\lceil \frac{w}{n} \rceil - 2}(H)$ in $G - F$, we have $d(G - F; x_h y_g, x_p y_g) \leq D_{\lceil \frac{w}{n} \rceil - 2}(H)$.

Case 3. $x_h \neq x_p$, $y_g \neq y_q$.

As in the Case 2, there is also at least one subgraph $Hy_j (j = 1, 2, \cdots, n)$ with no more than $\lceil \frac{w}{n} \rceil - 1$ fault vertices, we assume that there is also a subgraph Hy_j satisfies $1 \leq |F \cap V(Hy_j)| = \lceil \frac{w}{n} \rceil - 1 \leq k - 1$. The following three subcases can be discussed.

Subcase 3.1. $x_h y_j, x_p y_j \notin F$.

Since the two vertices $x_h y_g$ and $a_1 y_j$ are adjacent, the two vertices $x_p y_q$ and $b_1 y_j$ are adjacent, we can construct a path R_5 between $x_h y_g$ and $x_p y_q$ in $G - F$ on the basis of R_1.

$$R_5 : x_h y_g \rightarrow a_1 y_j \xrightarrow{R_1 - \{x_h y_j, x_p y_j\}} b_1 y_j \rightarrow x_p y_q,$$

with $L(R_5) = L(R_1) = D_{\lceil \frac{w}{n} \rceil}(H)$. Similarly, we have $d(G - F; x_h y_g, x_p y_q) \leq D_{\lceil \frac{w}{n} \rceil}(H)$.

Subcase 3.2. $x_h y_j \notin F$, $x_p y_j \in F$ or $x_h y_j \in F$, $x_p y_j \notin F$.

Without loss of generality, we assume that $x_h y_j \in F$ and $x_p y_j \notin F$. As in the Subcase 2.2, we can also construct the subgraph $H^{'}$, then $|F \cap V(H^{'})| = \lceil \frac{w}{n} \rceil - 2 \leq k - 2$. Since the two vertices $x_p y_q$ and $b_1 y_j$ are adjacent, we can construct a path R_6 between $x_h y_g$ and $x_p y_q$ in $G - F$ on the basis of R_3.

$$R_6 : x_h y_g \xrightarrow{R_3 - x_p y_j} b_1 y_j \rightarrow x_p y_q,$$

with $L(R_6) = L(R_3) = D_{\lceil \frac{w}{n} \rceil - 1}(H)$. Similarly, we have $d(G - F; x_h y_g, x_p y_q) \leq D_{\lceil \frac{w}{n} \rceil - 1}(H)$.

Subcase 3.3. $x_h y_j, x_p y_j \in F$.

Since $x_p y_q$ is adjacent to the vertices of $V(N_{Hy_j}(x_p y_j))$ and not adjacent to the vertices of $V(Hy_j) \backslash V(N_{Hy_j}(x_p y_j))$. Remove $x_h y_j$, $x_p y_j$ and the edges incident with $x_h y_j$ or $x_p y_j$ in Hy_j, then consider the edge set

$$E_3 = \{(x_p y_q, b_i y_j) : i = 1, 2, \cdots, k\}.$$

With the adjacency of $x_h y_g$ in the Subcase 2.2, we combine the vertex $x_h y_g$, the vertex $x_p y_q$, the edges of E_1, the edges of E_3 and the sugraph $Hy_j - \{x_h y_j, x_p y_j\}$ into a new subgraph $H^{'''}$. The new subgraph $H^{'''}$ has the same number of vertices and edges as subgraph Hy_j, and retains the adjacency of the subgraph Hy_j. From this, the two subgraphs $H^{'''}$ and Hy_j are isomorphic. For the subgraph $H^{'''}$, we have

$$|F \cap V(H^{'''})| = \lceil \frac{w}{n} \rceil - 3 \leq k - 3.$$

Similarly, even in the worst case, there is at least a path between $x_h y_g$ and $x_p y_q$ of length no more than $D_{\lceil \frac{w}{n} \rceil - 2}(H)$ in the subgraph $H^{'''} - F \cap V(H^{'''})$. From this, there is also at least a path between $x_h y_g$ and $x_p y_q$ of length no more than $D_{\lceil \frac{w}{n} \rceil - 2}(H)$ in $G - F$, we have $d(G - F; x_h y_g, x_p y_q) \leq D_{\lceil \frac{w}{n} \rceil - 2}(H)$.

Through the above analysis, we can conclude $D_w(G) \leq D_{\lceil \frac{w}{n} \rceil}(H)$.

Consider the lower bound of the vertex fault diameter of G by giving fault vertex sets specifically, let F_H be a fault vertex set in H such that the diameter of $H - F_H$ is $D_{\lceil \frac{w}{n} \rceil}(H)$. Without loss of generality, we let

$$F_H = \{x_1, \cdots, x_{\lceil \frac{w}{n} \rceil - 1}\}.$$

If $mod(\frac{w-1}{n}) = 0$, we specifically give fault vertex set F_1 obtained on the basis of F_H.

$$F_1 = \{x_1 y_1, \cdots, x_1 y_n, x_2 y_1, \cdots, x_2 y_n, \cdots,$$
$$x_{\lceil \frac{w}{n} \rceil - 1} y_1, \cdots, x_{\lceil \frac{w}{n} \rceil - 1} y_n\}.$$

If $mod(\frac{w-1}{n}) \neq 0$, we specifically give fault vertex set F_2 obtained on the basis of F_H.

$$F_2 = \{x_1 y_1, \cdots, x_1 y_n, x_2 y_1, \cdots, x_2 y_n, \cdots,$$
$$x_{\lceil \frac{w}{n} \rceil} y_1, \cdots, x_{\lceil \frac{w}{n} \rceil} y_{mod(\frac{w-1}{n})}\}.$$

From this, we can get

$$d(G - F_1) = d(G - F_2) = D_{\lceil \frac{w}{n} \rceil}(H).$$

Therefore, we have $D_w(G) \geq D_{\lceil \frac{w}{n} \rceil}(H)$. □

From the Theorem 5, we can deduce some important results. Let $w = 1$, we can get the relationship of the diameters of $H \otimes K_n$ and H.

Corollary 3. *Let H be an incompletely connected graph with the connectivity $k \geq 1$, and K_n be a complete graph with order $n \geq 1$. Then*

$$d(H \otimes K_n) = d(H).$$

If the fault diameter of H is given. we can directly obtain the fault diameter of $H \otimes K_n$ by Theorem 5. As the basic graph, the fault diameters of path, cycle and wheel graph can be easily obtained, we have the following corollaries directly.

Corollary 4. *Let P_m be a path with order $m > 2$, and K_n be a complete graph with order $n \geq 1$. For any $1 \leq w \leq n$. Then*

$$D_w(P_m \otimes K_n) = m - 1.$$

Corollary 5. *Let C_m be a cycle with order $m > 3$, and K_n be a complete graph with order $n \geq 1$. For any $1 \leq w \leq 2n$. Then*

$$D_w(C_m \otimes K_n) = \begin{cases} \lfloor \frac{m}{2} \rfloor, & for\ 1 \leq w \leq n; \\ m-2, & for\ n < w \leq 2n. \end{cases}$$

Corollary 6. *Let W_{1+m} be a wheel graph with $m \geq 3$, and K_n be a complete graph with order $n \geq 1$. For any $1 \leq w \leq 3n$. Then*

$$D_w(W_{1+m} \otimes K_n) = \begin{cases} 2, & for\ 1 \leq w \leq n; \\ \lfloor \frac{m}{2} \rfloor, & for\ n < w \leq 2n; \\ m-2, & for\ 2n < w \leq 3n. \end{cases}$$

By Corollary 1, we can still extend the Theorem 5 to t-dimension, the fault diameter of the strong product graph of an incompletely connected graph and $t-1$ complete graphs is given.

Corollary 7. *Let H be a incompletely connected graph with the connectivity $k \geq 1$, and $K_{v_1}, K_{v_2}, \cdots, K_{v_{t-1}}$ be $t-1$ complete graphs with number $t \geq 2$ and orders $v_1, v_2, \cdots, v_{t-1} \geq 1$. For $1 \leq w \leq \prod_{i=1}^{t-1} v_i k$. Then*

$$D_w(H \otimes K_{v_1} \otimes K_{v_2} \otimes \cdots \otimes K_{v_{t-1}}) = D_{\left\lceil \frac{w}{\prod_{i=1}^{t-1} v_i} \right\rceil}(H).$$

Since any complete graph is a maximally connected graph, we give the upper bound of the fault diameter of strong product graph of two maximally connected graphs by Menger Theorem.

Lemma 6. ([22]) *Let G_1 and G_2 be two maximally connected graphs with orders $n_1, n_2 \geq 2$, respectively. Then*

$$\kappa(G_1 \otimes G_2) = min\{\delta_1 n_2, \delta_2 n_1, \delta_1 + \delta_2 + \delta_1\delta_2\}.$$

Lemma 7. *(Menger Theorem) Let G be a connected and undirected graph, x and y are two different vertices in G. If $x, y \notin E(G)$, then $\zeta(G; x, y) = \kappa(G; x, y)$.*

Theorem 8. *Let G_1 and G_2 be two maximally connected graphs, orders $n_1, n_2 \geq 2$, minimum degrees $\delta_1, \delta_2 \geq 1$. If $G = G_1 \otimes G_2$, for any $1 \leq w \leq \kappa(G)$, then*

$$D_w(G) \leq max\{\left\lfloor \frac{n_1 n_2 - w - 1}{\delta_1 n_2 - w + 1} \right\rfloor + 1, \left\lfloor \frac{n_1 n_2 - w - 1}{\delta_2 n_1 - w + 1} \right\rfloor + 1,$$
$$\left\lfloor \frac{n_1 n_2 - w - 1}{\delta_1 + \delta_2 + \delta_1\delta_2 - w + 1} \right\rfloor + 1\}.$$

Proof. Let F be the fault vertex set in G with $|F| = w-1$, x and y are two different vertices in $G - F$. Without loss of generality, we assume $d(G - F) = h$. When $h \leq 1$, $G - F$ is a complete graph, the distance between x and y in $G - F$ is 1. When $h \geq 2$, we assume the distance between x and y in $G - F$ is $d(G - F; x, y) = h$.

By Menger Theorem, there are at least $\kappa(G) - w + 1$ internally vertex disjoint paths between x and y in $G - F$. The number of internal vertices in each path is at least $h-1$. Since the number of vertices in the strong product graph G satisfies $|V(G) = n_1 n_2|$. After the vertex failure occurred in G, we have $(\kappa(G) - w + 1)(h - 1) + 2 \leq n_1 n_2 - w - 1$. Since $\kappa(G) = min\{\delta_1 n_2, \delta_2 n_1, \delta_1 + \delta_2 + \delta_1\delta_2$, we can get

$$h \leq \left\lfloor \frac{n_1 n_2 - w - 1}{min\{\delta_1 n_2, \delta_2 n_1, \delta_1 + \delta_2 + \delta_1\delta_2\} - w + 1} \right\rfloor + 1.$$

From this, the theorem is proved. □

III. Conclusion

In this paper, we first determine the fault diameter of strong product graph of two complete graphs. Then we determine the fault diameter of strong product graph of an incompletely connected graph and a complete graph. Through the results, we find that the strong product graph of an arbitrary connected graph and a complete graph has small fault diameter and retains the same fault diameter as its incompletely factor graph. The strong product graph of an arbitrary connected graph and a complete graph provides a new and efficient method to construct large and reliable networks through small networks. Moreover, we also give the upper bound of the fault diameter of strong product graph of two maximally connected graphs. This provides direction for solving the general situation of the fault diameter of strong product graph.

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