Numerical Solution of Singular Lane-Emden Type Equations using Clique Polynomial

Raju Jummannavar, Ravikiran Mundewadi, Ashwini Yalnaik, Balachandra Hadimani*, Venkatesh Bhandage*

***Abstract*—In this paper, the clique polynomial method (CPM) is proposed for the numerical solution of Singular Lane-Emden type equations. A new operational matrix of integration concerning clique polynomials of the complete graph has been generated in its generalized representation. This operational matrix is applied to differential equations and is transformed into a system of algebraic equations that can be solved efficiently with the help of Newton's iterative solver. The efficiency of the developed method is revealed by considering illustrative examples, and the obtained results are compared favorably with the corresponding exact solution and errors.**

***Index Terms*—Clique polynomial, operational matrix, collocation point, Lane-Emden equations.**

I. Introduction

The numerical solution of singular initial value problems of second-order ordinary differential equations (ODEs) has been found attractive to many physicists and mathematicians in recent years. The Lane-Emden type equation is one of these categories and is found to have remarkable utilization. Many of the problems arising in astrophysics and mathematical physics can be molded into Lane-Emden-type equations. These are nonlinear differential equations with a singularity at the origin, which narrates the equilibrium density distribution in the self-gravitating sphere of polytrophic isothermal gas and plays a prominent role in the fundamentals of radiative cooling, stellar structure, and cluster modeling of galaxies. The name Lane-Emden is due to the astrophysicists Jonathan H. Lane and Robert Emden [23], as they were the first to study these equations in 1870. In general, the Lane-Emden equation is of the form

$$\lambda''(t)+\frac{\alpha}{t}\lambda'(t)+f(t,\lambda(t))=g(t)\,,0<t\le 1,\alpha\ge 0, \quad (1)$$

subject to initial conditions

$$\lambda(0)=\beta,\ \ \lambda'(0)=0, \quad (2)$$

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where $f(t,\lambda)$ is a real valued continuous function, $g(t)\in C[0,1]$, and α, β are constants. In [9], it was shown that an analytic solution for equation (1) is possible in the neighborhood of the singular point $t=0$ for the initial conditions (2). Choosing $\alpha=2$, $f(t,\lambda)=\lambda^m$, $g(t)=0$ and $\beta=1$ in equations (1) and (2), we have

$$\lambda''+\frac{2}{t}\lambda'+\lambda^m=0 \quad (3)$$

alternatively,

$$\frac{1}{t^2}\,\frac{d}{dt}\left(t^2\,\frac{d\lambda}{dt}\right)+\lambda^m=0 \quad (4)$$

subject to initial conditions

$$\lambda(0)=1,\ \ \lambda'(0)=0. \quad (5)$$

equation (3) with initial condition of equation (5) is called *standard form of Lane-Emden equation*. The parameter m in equation (3) is found to have physical significance for $0\le m\le 5$. Taking $\alpha=2$, $f(t,\lambda)=e^{\lambda}$, $g(t)=0$ and $\beta=0$ in equations (1) and (2), yields *isothermal gas spheres equation* as

$$\lambda''+\frac{2}{t}\lambda+e^{\lambda}=0 \quad (6)$$

subject to initial conditions

$$\lambda(0)=0,\ \ \lambda'(0)=0. \quad (7)$$

Various numerical techniques have evolved in the past to solve singular Lane-Emden-type equations like the sinc-collocation method [32], the second-kind Chebyshev operational matrix algorithm [12], ultraspherical wavelet spectral solutions [1], an implicit series solution [26], the new wavelet collocation method [2], the Jacobi rational pseudo-spectral method [13], the Pade approximation [37], the Jacobi-Gauss collocation method [8], the new ultraspherical wavelet collocation method [14], and new spectral solutions [3]. Along with these techniques, many researchers were attracted by the operational matrix of integration by considering the collocation method, in which the given differential equation is transformed into integral equations by considering the operational matrix, which is generated by the integration of polynomials, and then a system of algebraic equations is obtained from the given problem by reducing the integral operator. The generation of the operational matrix has been a major task in the field of numerical methods. In the literature, many operational matrix methods have been discussed, as follows: Laguerre polynomial operational matrix, Chebyshev polynomial operational matrix, Legendre polynomial operational matrix, Genocchi polynomial operational matrix, and Hermite polynomial operational matrix.

In numerical methods, polynomials play an important role. Dealing with graph theory, one can observe the existence of a variety of polynomials related to graphs in terms of graph invariants. Recently, a group of graph theory and numerical analysis researchers developed methods to solve numerical problems by using graph theoretic polynomials [27]–[29], which gave efficient results. Motivated by this work, we aim to develop a new operational matrix by using the concept of graph theory, which gives an alternative technique to find a highly approximate solution for the singular Lane-Emden type equations. The new generalized operational matrix of clique polynomial is discussed in section II.

II. Clique Polynomial and its properties

All graphs taken into consideration in this paper are simple, undirected and finite. A graph G contains a finite set (nonempty) of n vertices called the vertex set, denoted by $V(G)$, along with a pre-defined set of m unordered pairs of different vertices of the vertex set is called as the edge set, represented by $E(G)$. Whenever two vertices shares a common edge those vertices are said to be adjacent. A graph S is known as subgraph of graph G, if all the edges and vertices of S are contained in G. A graph where all pair of vertices are adjacent is known as complete graph and this graph on n vertices is represented by K_n. A k-clique of graph G is a complete subgraph of G with k vertices. For graph theoretical definitions, notations and related works we refer [10], [18].

In 1991, Hoede et al. [19] introduced the clique polynomial for a graph G, which attracted many researchers over the globe. The clique polynomial for a graph G in variable t, represented by $C(G;t)$, is defined as follows:

$$C\left(G;t\right)=\sum_{k=0}^{n}a_kt^k \tag{8}$$

where a_k is the number of distinct k-cliques in G. We write $C(G;t)=1$, whenever $V(G)=\emptyset$. Works related to clique polynomials can be seen in [15]–[17]. In the construction of the operational matrix of integration, we consider clique polynomial of complete graphs. For complete graph K_n, number of distinct k-cliques is $\binom{n}{k}=\frac{n!}{(n-k)!k!}$.
The clique polynomial of a complete graph K_n by

$$C(K_n;t)=\binom{n}{0}+\binom{n}{1}t+\binom{n}{2}t^2+\cdots+\binom{n}{n}t^n. \tag{9}$$

In particular,

$$\begin{aligned}C(K_0;t)&=1\\C(K_1;t)&=1+t\\C(K_2;t)&=1+2t+t^2.\end{aligned}$$

Some properties of clique polynomial of complete graphs are

(i) $\quad C\left(K_n;t\right)\big|_{t=1}=2^n$

(ii) $\quad \int_0^t C\left(K_n;t\right)dt=\frac{1}{n+1}\left[C\left(K_{n+1};t\right)-1\right], n\geq 1$

(iii) $\quad \int_0^1 C\left(K_n;t\right)dt=\frac{2^{n+1}-1}{n+1}$

(iv) $\quad \int_0^t C\left(K_m;t\right)C\left(K_n;t\right)dt$
$\quad =\frac{1}{m+n+1}\left[C\left(K_{m+n+1};t\right)-1\right]$

(v) $\quad \int_0^1 C\left(K_m;t\right)C\left(K_n;t\right)dt=\frac{2^{m+n+1}-1}{m+n+1}.$

III. Function approximation

Let $\{C(K_0;t),C(K_1;t),C(K_2;t),...,C(K_{n-1};t)\}\subset L^2[0,1]$ be a set of clique polynomial of complete graphs and $X=span\{C(K_0;t),C(K_1;t),C(K_2;t),...,C(K_{n-1};t)\}$.

Let $\lambda(t)$ be any function of $L^2[0,1]$. Clearly, X is a subspace of $L^2[0,1]$ whose dimension is finite value of n.

Hence, there exists a unique finest approximation, say $\lambda^*(t)$, for $\lambda(t)$ in X such that $\forall\mu\in X$,

$$\|\lambda(t)-\lambda^*(t)\|_2\leq\|\lambda(t)-\mu(t)\|_2$$

which yields the fact that $\forall\mu\in X$,

$$\langle\lambda(t)-\lambda^*(t),\mu(t)\rangle=0 \tag{10}$$

where $\langle\cdot\rangle$ represents inner product. Since $\lambda^*(t)\in X$, then there exists unique coefficients $a_0,a_1,a_2,...,a_{n-1}$ such that

$$\lambda(t)\approx\lambda^*(t)=\sum_{i=0}^{n-1}a_iC\left(K_i;t\right)=A^TC(t)$$

in which A and $C(t)$ are $n\times 1$ matrices given as

$$A=[a_0,a_1,a_2,...,a_{n-1}]^T \tag{11}$$

and

$$C(t)=\left[C(K_0;t),C(K_1;t),C(K_2;t),...,C(K_{n-1};t)\right]. \tag{12}$$

Using equation (10), we have

$$\left\langle\lambda(t)-A^TC(t),C(K_i;t)\right\rangle=0,\quad i=0,1,2,...,n-1$$

which can be written for simplicity as

$$A^T\left\langle C(t),C(t)\right\rangle=\left\langle\lambda(t),C(t)\right\rangle$$

where $\langle C(t),C(t)\rangle$ is a matrix of size $n\times n$.
Let the inner product is defined as

$$B=\langle C(t),C(t)\rangle=\int_0^1 C(t)C(t)^Tdt,$$

by property (v) gives the entries of matrix B. Hence, all $\lambda(t)\in L^2[0,1]$ can be expanded by clique polynomials as

$$\lambda(t)=A^TC(t)\qquad\text{where}\qquad A=B^{-1}\left\langle\lambda(t),C(t)\right\rangle.$$

As the truncated clique polynomial expansion can be an approximate solution for Lane-Emden type equations, the error function $E(t)$ for $\lambda(t)$ is given by

$$E\left(t\right)=\left|\lambda\left(t\right)-A^TC\left(t\right)\right|.$$

Theorem 1 gives the error estimation due to the clique polynomial expansion.

Theorem 1. *Let $\lambda(t)\in C^n[0,1]$ and $A^TC(t)$ is an approximate solution by making use of clique polynomial. Then bound for error is given by*

$$\|E(t)\|\leq\left\|\frac{1}{n!2^{2n-1}}\max_{t\in[0,1]}\left|\lambda^{(n)}\left(t\right)\right|\right\|.$$

Proof:

$$\begin{aligned}\|E(t)\|^2&=\int_0^1\left(\lambda(t)-A^TC(t)\right)^2\,dt\\&\leq\int_0^1\left(\lambda(t)-P_n(t)\right)^2\,dt\end{aligned}$$

in which $P_n(t)$ represents the interpolating polynomial of degree n that approximates $\lambda(t)$ over $[0,1]$. By making use of the maximum error estimate for the polynomial on $[0,1]$, we have

$$\begin{aligned}\|E(t)\|^2 &\leq \int_0^1 \left(\frac{2}{n!4^n}\max_{t\in[0,1]}\left|\lambda^{(n)}(t)\right|\right)dt\\ &= \left\|\frac{1}{n!2^{2n-1}}\max_{t\in[0,1]}\left|\lambda^{(n)}(t)\right|\right\|^2,\end{aligned}$$

where maximum error bound for the interpolation has been used. ■

IV. Operational matrix of Integration

The clique polynomial of complete graphs are considered as basis for the construction of operational matrix. The first six basis functions are

$$\begin{aligned}C(K_0;t) &= 1\\ C(K_1;t) &= 1+t\\ C(K_2;t) &= 1+2t+t^2\\ C(K_3;t) &= 1+3t+3t^2+t^3\\ C(K_4;t) &= 1+4t+6t^2+4t^3+t^4\\ C(K_5;t) &= 1+5t+10t^2+10t^3+5t^4+t^5.\end{aligned}$$

Let the six basis function of clique polynomial is $C_6(t)$ $= [C(K_0;t), C(K_1;t), C(K_2;t), C(K_3;t), C(K_4;t), C(K_5;t)]^T$. On integrating the above basis over the limits from 0 to t and presenting it in the matrix form,

$$\int_0^t C(K_0;t)dt = t = [-1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0]\,C_6(t)$$

$$\begin{aligned}\int_0^t C(K_1;t)dt &= t+\frac{t^2}{2}\\ &= \left[-\frac{1}{2}\ \ 0\ \ \frac{1}{2}\ \ 0\ \ 0\ \ 0\right]C_6(t)\end{aligned}$$

$$\begin{aligned}\int_0^t C(K_2;t)dt &= t+t^2+\frac{t^3}{3}\\ &= \left[-\frac{1}{3}\ \ 0\ \ 0\ \ \frac{1}{3}\ \ 0\ \ 0\right]C_6(t)\end{aligned}$$

$$\begin{aligned}\int_0^t C(K_3;t)dt &= t+\frac{3t^2}{2}+t^3+\frac{t^4}{4}\\ &= \left[-\frac{1}{4}\ \ 0\ \ 0\ \ 0\ \ \frac{1}{4}\ \ 0\right]C_6(t)\end{aligned}$$

$$\begin{aligned}\int_0^t C(K_4;t)dt &= t+2t^2+2t^3+t^4+\frac{t^5}{5}\\ &= \left[-\frac{1}{5}\ \ 0\ \ 0\ \ 0\ \ 0\ \ \frac{1}{5}\right]C_6(t)\end{aligned}$$

$$\begin{aligned}\int_0^t C(K_5;t)dt &= t+\frac{5t^2}{2}+\frac{10t^3}{3}+\frac{5t^4}{2}+t^5+\frac{t^6}{6}\\ &= \left[-\frac{1}{6}\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\right]C_6(t)\\ &+\frac{1}{6}C(K_6;t)\end{aligned}$$

$$\text{Thus, } \int_0^t C_6(t)\ \ dt = P_{6\times 6}C_6(t)+\overline{C_6}(t)$$

where

$$P_{6\times6} = \begin{bmatrix}-1 & 1 & 0 & 0 & 0 & 0\\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0\\ -\frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0\\ -\frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0\\ -\frac{1}{5} & 0 & 0 & 0 & 0 & \frac{1}{5}\\ -\frac{1}{6} & 0 & 0 & 0 & 0 & 0\end{bmatrix};\overline{C_6}(t) = \begin{bmatrix}0\\0\\0\\0\\0\\\frac{1}{6}C(K_6;t)\end{bmatrix}.$$

Again appliying double integration on six basis functions we have

$$\int_0^t\int_0^t C(K_0;t)dtdt = \frac{t^2}{2} = \left[\frac{1}{2}\ \ -1\ \ \frac{1}{2}\ \ 0\ \ 0\ \ 0\right]C_6(t)$$

$$\begin{aligned}\int_0^t\int_0^t C(K_1;t)dtdt &= \frac{t^2}{2}+\frac{t^3}{6}\\ &= \left[\frac{1}{3} -\frac{1}{2}\ 0\frac{1}{6}\ \ 0\ 0\right]C_6(t)\end{aligned}$$

$$\begin{aligned}\int_0^t\int_0^t C(K_2;t)dtdt &= \frac{t^2}{2}+\frac{t^3}{3}+\frac{t^4}{12}\\ &= \left[\frac{1}{4}\ \ -\frac{1}{3}\ \ 0\ \ 0\ \ \frac{1}{12}\ \ 0\right]C_6(t)\end{aligned}$$

$$\begin{aligned}\int_0^t\int_0^t C(K_3;t)\ dtdt &= \frac{t^2}{2}+\frac{t^3}{2}+\frac{t^4}{4}+\frac{t^5}{20}\\ &= \left[\frac{1}{5}\ \ -\frac{1}{4}\ \ 0\ \ 0\ \ 0\ \ \frac{1}{20}\right]C_6(t)\end{aligned}$$

$$\begin{aligned}\int_0^t\int_0^t C(K_4;t)\ dtdt &= \frac{t^2}{2}+\frac{2t^3}{3}+\frac{t^4}{2}+\frac{t^5}{5}+\frac{t^6}{30}\\ &= \left[\frac{1}{6}\ \ -\frac{1}{5}\ \ 0\ \ 0\ \ 0\ \ 0\right]C_6(t)\\ &+\frac{1}{30}C(K_6;t)\end{aligned}$$

$$\begin{aligned}\int_0^t\int_0^t C(K_5;t)\ dtdt &= \frac{t^2}{2}+\frac{5t^3}{6}+\frac{5t^4}{6}+\frac{t^5}{2}+\frac{t^6}{6}+\frac{t^7}{42}\\ &= \left[\frac{1}{7}\ \ -\frac{1}{6}\ \ 0\ \ 0\ \ 0\ \ 0\right]C_6(t)\\ &+\frac{1}{42}C(K_7;t)\end{aligned}$$

$$\text{Thus, } \int_0^t\int_0^t C_6(t)\ \ dtdt = P'_{6\times6}C_6(t)+\overline{C}'_6(t)$$

where

$$P' = \begin{bmatrix}\frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0\\ \frac{1}{3} & -\frac{1}{2} & 0 & \frac{1}{6} & 0 & 0\\ \frac{1}{4} & -\frac{1}{3} & 0 & 0 & \frac{1}{12} & 0\\ \frac{1}{5} & -\frac{1}{4} & 0 & 0 & 0 & \frac{1}{20}\\ \frac{1}{6} & -\frac{1}{5} & 0 & 0 & 0 & 0\\ \frac{1}{7} & -\frac{1}{6} & 0 & 0 & 0 & 0\end{bmatrix};\overline{C}'_6(t) = \begin{bmatrix}0\\0\\0\\0\\\frac{1}{30}C(K_6;t)\\\frac{1}{42}C(K_7;t)\end{bmatrix}$$

Similarly on considering n basis functions, we can construct generalized OMI of order $n\times n$ as follows:

$$\int_0^t C(t)\ \ dt = P\ C(t)+\overline{C}(t) \tag{13}$$

where

$$P=\begin{bmatrix}-\frac{1}{(n-(n-1))} & \frac{1}{(n-(n-1))} & 0 & 0 & \dots & 0\\ -\frac{1}{(n-(n-2))} & 0 & \frac{1}{(n-(n-2))} & 0 & \dots & 0\\ -\frac{1}{(n-(n-3))} & 0 & 0 & \frac{1}{(n-(n-3))} & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots\\ -\frac{1}{(n-1)} & 0 & 0 & 0 & \dots & \frac{1}{(n-1)}\\ -\frac{1}{n} & 0 & 0 & 0 & \dots & 0\end{bmatrix}$$

$$C(t)=\begin{bmatrix}C(K_0;t)\\ C(K_1;t)\\ C(K_2;t)\\ \vdots\\ C(K_{n-2};t)\\ C(K_{n-1};t)\end{bmatrix} \text{ and } \overline{C}(t)=\begin{bmatrix}0\\0\\0\\ \vdots\\ 0\\ \frac{1}{n}C(K_n;t)\end{bmatrix}.$$

For the double integration with n basis functions the generalized operational matrix of integration as follows:

$$\int_0^t\int_0^t C(t) \quad dtdt = P'\ C(t)+\overline{C}'(t) \tag{14}$$

where,

$$P'=\begin{bmatrix}\frac{1}{(n-(n-2))} & -\frac{1}{(n-(n-1))} & \frac{1}{(n-(n-2))(n-1)} & 0 & \dots & 0\\ \frac{1}{(n-(n-3))} & -\frac{1}{(n-(n-2))} & 0 & \frac{1}{(n-(n-3))(n-2)} & \dots & 0\\ \frac{1}{(n-(n-4))} & -\frac{1}{(n-(n-3))} & 0 & 0 & \dots & \vdots\\ \frac{1}{n-1} & -\frac{1}{n-2} & 0 & 0 & \dots & \frac{1}{(n-1)(n-2)}\\ \frac{1}{n} & -\frac{1}{n-1} & 0 & 0 & \dots & 0\\ \frac{1}{n+1} & -\frac{1}{n} & 0 & 0 & \dots & 0\end{bmatrix}$$

$$C(t)=\begin{bmatrix}C(K_0;t)\\ C(K_1;t)\\ C(K_2;t)\\ \vdots\\ C(K_{n-2};t)\\ C(K_{n-1};t)\end{bmatrix};\overline{C}'(t)=\begin{bmatrix}0\\0\\0\\ \vdots\\ 0\\ \frac{1}{(n-1)n}C(K_n;t)\\ \frac{1}{n(n+1)}C(K_{n+1};t)\end{bmatrix}$$

V. Description of the Clique polynomial method

Now newly obtained OMI is used along with collocation method for solving nonlinear singular IVPs. Let us assume,

$$\lambda''(t)=A^T C(t) \tag{15}$$

in which $C(t)$ is the vector defined in equation (12) and A is an unknown vector that should be computed. On the double integration of equation (15) w.r.t t from 0 to t along with the initial conditions given in equation (2) and from relations of equations (13) and (14) we get,

$$\lambda'(t)=\beta_2+A^T\left(P\ C(t)+\overline{C}(t)\right)$$

$$\lambda(t)=\beta_1+t\beta_2+A^T\left(P'\ C(t)+\overline{C'}(t)\right)$$

where $\lambda(t)$, $\lambda'(t)$, $\lambda''(t)$ are expressed in terms of polynomial basis. On substituting $\lambda(t)$, $\lambda'(t)$, $\lambda''(t)$ in equation (1) we have,

$$A^T C(t)+p(t)\left(\beta_2+A^T\left(P\ C(t)+\overline{C}(t)\right)\right)$$

$$+q(t)\left(\beta_1+t\beta_2+A^T\left(P'\ C(t)+\overline{C'}(t)\right)\right)=r(t,\lambda)$$

On discretizing by making use of the collocation points $t_i=\frac{2i-1}{2n}, 1\le i\le n$, a system of n equations with n unknowns is obtained, which are of the form

$$A^T C(t_i)+p(t_i)\left(\beta_2+A^T\left(P\ C(t_i)+\overline{C}(t_i)\right)\right)+q(t_i)$$

$$\left(\beta_1+t_i\beta_2+A^T\left(P'\ C(t_i)+\overline{C'}(t_i)\right)\right)=r(t_i,\lambda(t_i))$$

On solving this system, we get the unknown vector A by the help of Matlab through Newton iterative method. Now the approximate solution is obatined easily on substituting A in the respective expression of $\lambda(t)$.

VI. Numerical experiments

Example 1: Consider the Lane-Emden equation [38],

$$\lambda'' + \frac{2}{t}\lambda' + \lambda = 6 + 12t + t^2 + t^3;\ 0 < t \leq 1 \tag{16}$$

subject to the conditions

$$\lambda(0) = 0, \lambda'(0) = 0$$

whose exact solution is $\lambda(t) = t^2 + t^3$. On solving equation (16) by using the present method(CPM) at $n = 4$, we get the coefficients $a_0 = -4$, $a_1 = 6$ and $a_2 = a_3 = 0$. Substituting these in the corresponding expression of $\lambda(t)$, we have the solution same as that of exact solution.

Example 2: Consider the nonlinear Lane-Emden equation [6],

$$\lambda'' + \frac{2}{t}\lambda' + \lambda^3 = 6 + t^6;\ 0 < t \leq 1 \tag{17}$$

subject to the conditions

$$\lambda(0) = 0, \lambda'(0) = 0$$

whose exact solution is $\lambda(t) = t^2$. On solving equation (17) by using the present method(CPM) at $n = 4$, we get the coefficients $a_0 = 2$, $a_1 = a_2 = a_3 = 0$. Substituting these in the corresponding expression of $\lambda(t)$, we have the solution same as that of exact solution.

Example 3: Consider the nonlinear Emden–Fowler type equation [35],

$$\lambda'' + \frac{6}{t}\lambda' + 14\lambda = -4\lambda ln\,\lambda;\ t > 0 \tag{18}$$

subject to the conditions

$$\lambda(0) = 1, \lambda'(0) = 0$$

which has the exact solution as $\lambda(t) = e^{-t^2}$. On solving equation (18) by using the present method(CPM), Table I gives the numerical solution for different values of n, Fig. 1 gives the comparison of numerical solution with the exact solution for $n = 10$ and Table II represents the absolute error analysis. Fig. 2 and Table III shows the comparison of the error analysis of the Clique polynomial method with existing methods [4], [33].

Example 4: Consider the nonlinear Emden–Fowler type equation [20],

$$\lambda'' + \frac{2}{t}\lambda' = -exp\left(\lambda\right);\ t > 0 \tag{19}$$

subject to the conditions

$$\lambda(0) = 0, \lambda'(0) = 0$$

which has the analytical solution as

$$\lambda(t) = -\frac{1}{6}t^2 + \frac{1}{120}t^4 - \frac{1}{1890}t^6 + \frac{61}{1632960}t^8 - \frac{629}{224532000}t^{10}$$

On solving equation (19) by using the present method(CPM), Fig. 3 gives the comparison of numerical solution with the exact solution for $n = 9$. Fig. 4 and Table IV shows the comparsion of absolute error analysis of CPM with existing method [11].

Example 5: Consider the nonlinear Emden–Fowler type equation [20],

$$\lambda'' + \frac{2}{t}\lambda' + 8e^{\lambda} + 4e^{\frac{\lambda}{2}} = 0 \tag{20}$$

subject to the conditions

$$\lambda(0) = 0, \lambda'(0) = 0$$

which has the analytical solution $\lambda(t) = -2ln\left(1 + t^2\right)$. On solving equation (20) by using the present method, Fig. 5 gives the comparison of numerical solution with the exact solution for $n = 9$. Fig. 6 and Table V shows the comparison of absolute error analysis of CPM with existing methods [7], [39].

Example 6: Consider the nonlinear Lane-Emden equation [4],

$$\lambda'' + \frac{8}{t}\lambda' + t\lambda = t^5 - t^4 + 44t^2 - 30t;\ 0 < t \leq 1 \tag{21}$$

subject to the conditions

$$\lambda(0) = 0, \lambda'(0) = 0$$

which has the analytical solution as $\lambda(t) = t^4 - t^3$.

On solving equation (21) by using the present method, Fig. 7 gives the comparison of numerical solution with the exact solution for $n = 4$. Fig. 8 and Table VI shows the comparison of absolute error analysis of CPM with existing methods [4], [34].

Example 7: Consider the nonlinear Lane-Emden equation [5],

$$\lambda'' + \frac{1}{t}\lambda' + \lambda^5 = 0\ \ 0 < t < 1 \tag{22}$$

subject to the conditions

$$\lambda(0) = 1, \lambda'(0) = 0$$

which has the exact solution as $\lambda(t) = (1 + \frac{t^2}{3})^{-\frac{1}{2}}$. On solving equation (22) by using the present method, Fig. 9 gives the comparison of numerical solution with the exact solution for $n = 10$. Fig. 10 and Table VII shows the the comparison of absolute error analysis of CPM with existing methods [5], [34].

Example 8: Consider the nonlinear Lane-Emden equation [24],

$$\lambda'' + \frac{2}{t}\lambda' - 2\left(2t^2 + 3\right)\lambda = 0;\ t \geq 0 \tag{23}$$

subject to the conditions

$$\lambda(0) = 1, \lambda'(0) = 0$$

which has the exact solution as $\lambda(t) = e^{t^2}$.

On solving equation (23) by using the present method, Fig. 11 gives the comparison of numerical solution with the exact solution for $n = 11$. Fig. 12 and Table VIII shows the the comparison of absolute error analysis of CPM with existing methods [5], [34].

Example 9: Consider the nonlinear Lane-Emden equation [36],

$$\lambda'' + \frac{2}{t}\lambda' + sin(\lambda) = 0;\ \ t \geq 0 \tag{24}$$

subject to the conditions

$$\lambda(0) = 1,\ \lambda'(0) = 0.$$

A series solution obtained by Wazwaz [36] by using ADM is:

$$\lambda(t) \simeq 1 - \frac{1}{6}k_1 t^2 + \frac{1}{120}k_1 k_2 t^4 + k_1\left(\frac{1}{3024}k_1^2 - \frac{1}{5040}k_2^2\right) t^6 + k_1 k_2\left(-\frac{113}{3265920}k_1^2 + \frac{1}{362880}k_2^2\right) t^8 + k_1 \left(\frac{1781}{898128000}k_1^2 k_2^2 - \frac{1}{39916800}k_2^4 - \frac{19}{23950080}k_1^4\right) t^{10}.$$

where $k_1 = sin(1)$ and $k_2 = cos(1)$. On solving equation (24) by using the present method, Fig. 13 gives the comparison of numerical solution with the exact solution for $n = 11$. Fig. 14 and Table IX shows the comparison of the numerical solution with exact solution and absolute error analysis of CPM with existing methods [30], [31].

Example 10: Consider the nonlinear Lane-Emden equation [36],

$$\lambda'' + \frac{2}{t}\lambda' + sinh(\lambda) = 0;\ \ t \geq 0 \tag{25}$$

subject to the conditions

$$\lambda(0) = 1,\ \lambda'(0) = 0.$$

A series solution obtained by Wazwaz [36] by using ADM is:

$$\lambda(t) \simeq 1 - \frac{\left(e^2 - 1\right) t^2}{12e} + \frac{\left(e^4 - 1\right) t^4}{480e^2} - \frac{\left(2e^6 + 3e^2 - 3e^4 - 2\right) t^6}{30240e^3} + \frac{\left(61e^8 - 104e^6 + 104e^2 - 61\right) t^8}{26127360e^4}.$$

where $k_1 = sin(1)$ and $k_2 = cos(1)$. On solving equation (25) by using the present method, Fig. 15 gives the comparison of numerical solution with the exact solution for $n = 11$. Fig. 16 and Table X shows the comparison of the numerical solution with exact solution and absolute error analysis of CPM with existing method [31].

VII. Conclusion

In this paper, we made an effort to generate new operational matrix of integration by the help of graph theoretic concepts to solve nonlinear singular Lane-Emden type equations for various physical conditions, that play an important role in the growth of new research in both graph theory and numerical analysis, and advantageous for new researchers of both the fields. This method is examined for few examples and obtained results are satisfactory comparing to the existing numerical solutions. At the end, we can conclude the outputs of this analysis as follows:

1. This technique delivers better accuracy, stability and convergence as compared to existing methods [4], [5], [7], [11], [24], [30], [31], [33], [34], [36], [39].
2. One can easily implement this method into computer programs and can be extended for higher order differential equations by slightly modifying it.
3. The present method is easy as we have generalised OMI.
4. The present scheme can be applied to any initial value problems.
5. In the future, we can apply this technique to solve real-world problems in diverse fields, including physical models, fluid dynamics and engineering problems.

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TABLE I: Numerical solution of Clique polynomial method of Example 3.

t	Exact solution	At n=7	At n=8	At n=9	At n=10
0	1.000000000000000	1.000000000000000	1.000000000000000	1.000000000000000	1.000000000000000
0.1	0.990049833749168	0.990049607407242	0.990049823472374	0.990049836579394	0.990049833934539
0.2	0.960789439152323	0.960789558649421	0.960789448707850	0.960789437212064	0.960789439018945
0.3	0.913931185271228	0.913931322221168	0.913931189289573	0.913931184801275	0.913931185262270
0.4	0.852143788966211	0.852143804114549	0.852143791404681	0.852143787918283	0.852143788938259
0.5	0.778800783071405	0.778800859148348	0.778800788105785	0.778800781908282	0.778800782988195
0.6	0.697676326071031	0.697676390483394	0.697676326516178	0.697676324220880	0.697676325938526
0.7	0.612626394184416	0.612626378666123	0.612626400142216	0.612626391471709	0.612626394550580
0.8	0.527292424043049	0.527292568530821	0.527292420932174	0.527292420041049	0.527292424988971
0.9	0.444858066222941	0.444857929279300	0.444858077094580	0.444858060637102	0.444858068001453
1	0.367879441171442	0.367879390045079	0.367879442960602	0.367879435958953	0.367879443378122

TABLE II: Absolute error of Clique polynomial method of Example 3

t	At n=7	At n=8	At n=9	At n=10
0	0	0	0	0
0.1	2.2634e-07	1.0277e-08	2.8302e-09	1.8537e-10
0.2	1.1950e-07	9.5555e-09	1.9403e-09	1.3338e-10
0.3	1.3695e-07	4.0183e-09	4.6995e-10	8.9577e-12
0.4	1.5148e-08	2.4385e-09	1.0479e-09	2.7953e-11
0.5	7.6077e-08	5.0344e-09	1.1631e-09	8.3210e-11
0.6	6.4412e-08	4.4515e-10	1.8502e-09	1.3250e-10
0.7	1.5518e-08	5.9578e-09	2.7127e-09	3.6616e-10
0.8	1.4449e-07	3.1109e-09	4.0020e-09	9.4592e-10
0.9	1.3694e-07	1.0872e-08	5.5858e-09	1.7785e-09
1	5.1126e-08	1.7892e-09	5.2125e-09	2.2067e-09

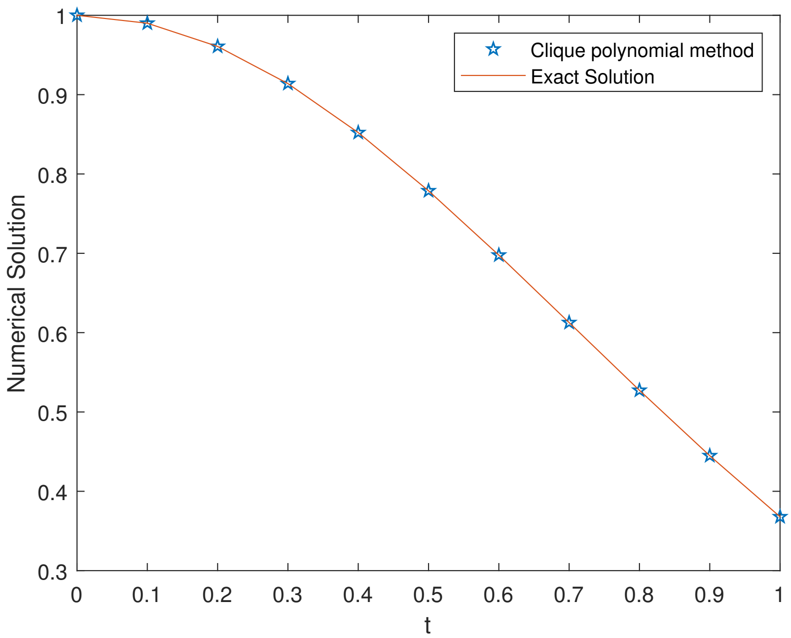

Fig. 1: Numerical solution of CPM with exact solution for $n = 10$ of Example 3.

TABLE III: Abs. Error comparison of CPM with existing methods [4], [33] of Example 3.

t	Abs. Error of Method [4]	Abs. Error of Method [33] for $k = 1$, $M = 10$	Abs. Error of CPM for $n = 10$
0.1	4.8969e-06	4.5978e-09	1.8537e-10
0.2	6.8492e-06	1.5175e-09	1.3338e-10
0.3	8.0369e-07	1.2560e-08	8.9577e-12
0.4	8.3868e-06	1.2911e-08	2.7953e-11
0.5	1.2870e-05	1.6847e-08	8.3210e-11
0.6	5.3244e-05	2.1010e-08	1.3250e-10
0.7	2.0690e-04	1.3582e-08	3.6616e-10
0.8	5.9394e-04	2.7184e-08	9.4592e-10
0.9	1.4199e-03	1.2134e-08	1.7785e-09
1	3.0777e-03	4.3449e-09	2.2067e-09

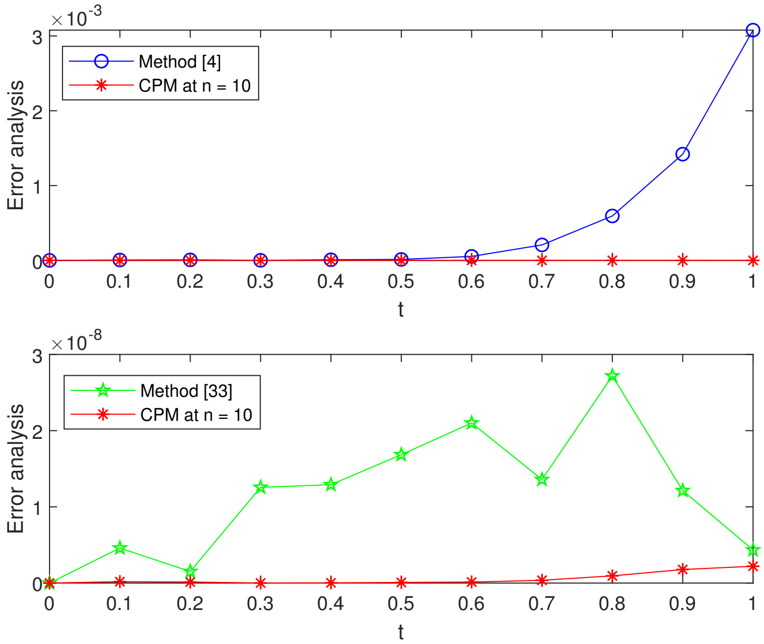

Fig. 2: Comparison of Error Analysis of Example 3.

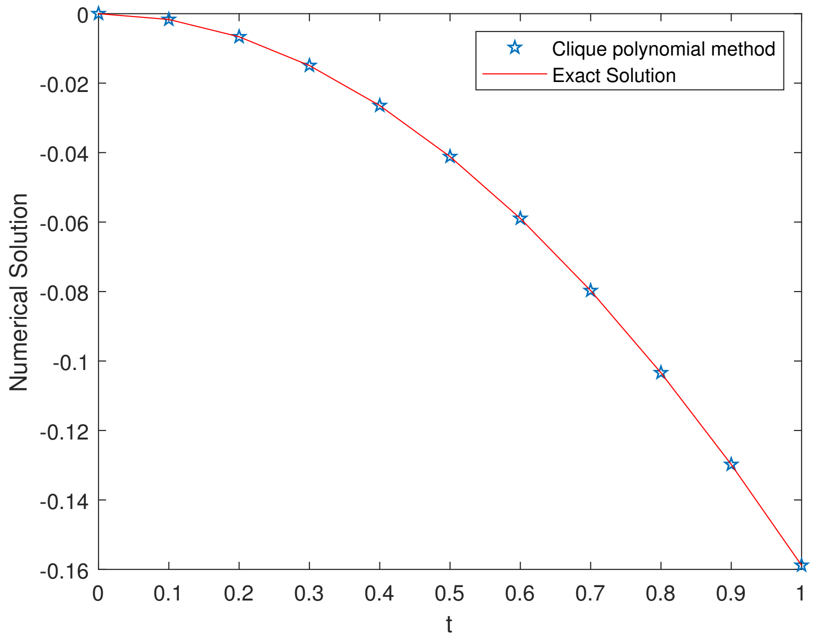

Fig. 3: Numerical solution of CPM with exact solution for $n = 9$ of Example 4.

TABLE IV: Abs. Error comparison of CPM with existing method [11] of Example 4.

t	Exact solution	Abs. Error of Method [11]	Abs. Error of CPM for $n = 9$
0.1	-0.001665833862061	1.8661e-06	3.0114e-12
0.2	-0.006653367100370	9.0671e-06	5.8920e-13
0.3	-0.014932883276828	1.2616e-05	2.7155e-12
0.4	-0.026455476286396	1.3376e-05	5.1626e-11
0.5	-0.041153956831229	3.3431e-06	4.6177e-10
0.6	-0.058944072043512	1.9927e-05	2.7194e-09
0.7	-0.079725992283819	9.0922e-06	1.1952e-08
0.8	-0.103386010960547	2.7010e-05	4.2211e-08
0.9	-0.129798398750175	3.3987e-06	1.2550e-07
1	-0.158827353673418	3.1646e-05	3.2385e-07

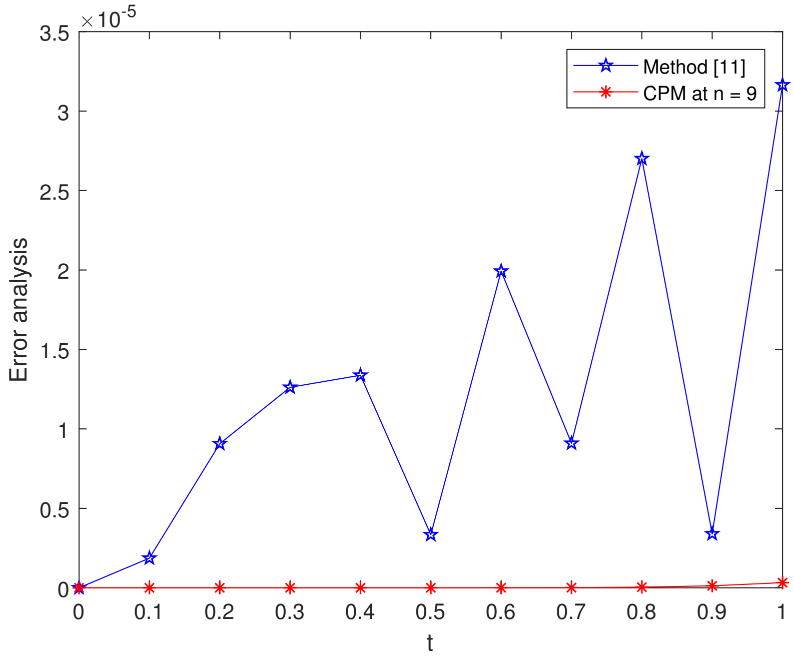

Fig. 4: Comparison of Error Analysis of Example 4.

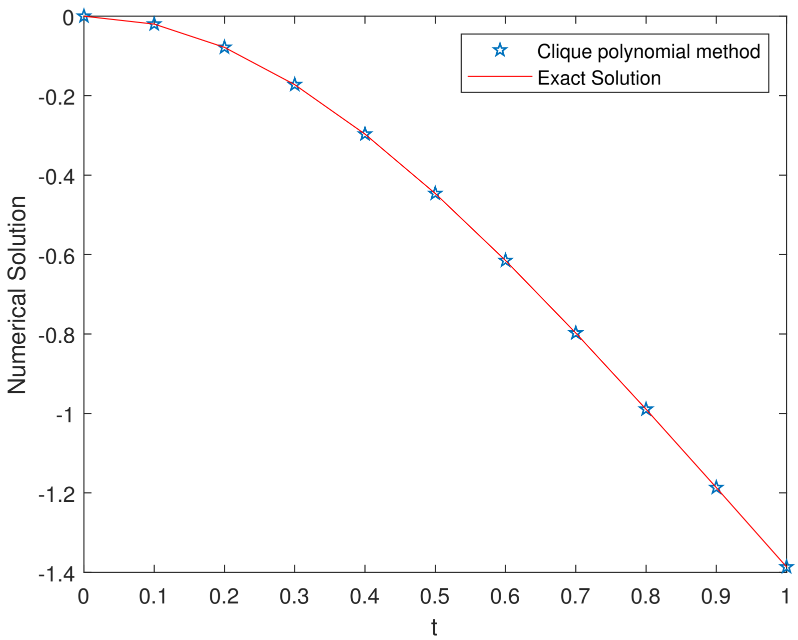

Fig. 5: Numerical solution of CPM with exact solution for $n = 9$ of Example 5.

TABLE V: Abs. Error comparison of CPM with existing methods [7], [39] of Example 5.

t	Exact solution	Abs. Error of Method [39]	Abs. Error of Method [7]	Abs. Error of CPM for $n = 9$
0.1	-0.019900661706336	1.5794e-4	2.0000e-5	2.9124e-07
0.2	-0.078441426306563	7.7327e-5	2.4999e-5	1.8296e-08
0.3	-0.172355392482105	1.2624e-4	2.4999e-5	1.6217e-07
0.4	-0.296840010236547	4.6110e-5	1.8999e-5	3.5197e-07
0.5	-0.446287102628420	2.0336e-4	2.4000e-5	5.5866e-07
0.6	-0.614969399495921	3.8512e-4	2.9000e-5	7.3844e-07
0.7	-0.797552239914736	1.2793e-3	2.5999e-5	8.1436e-07
0.8	-0.989392483672214	4.3819e-3	1.4699e-4	8.6119e-07
0.9	-1.186653690555469	1.1972e-2	8.6199e-4	5.4904e-07
1	-1.386294361119891	2.6964e-2	3.2150e-3	1.2236e-07

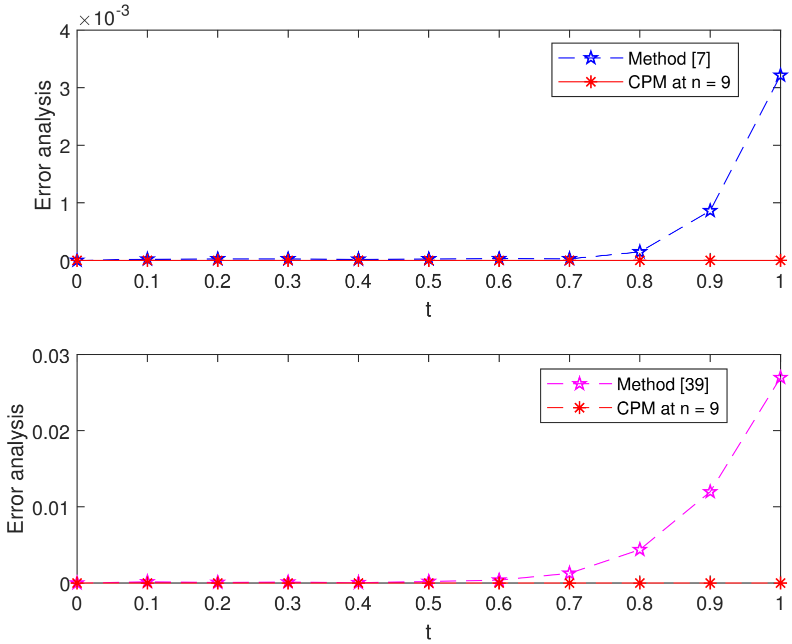

Fig. 6: Comparison of Error Analysis of Example 5.

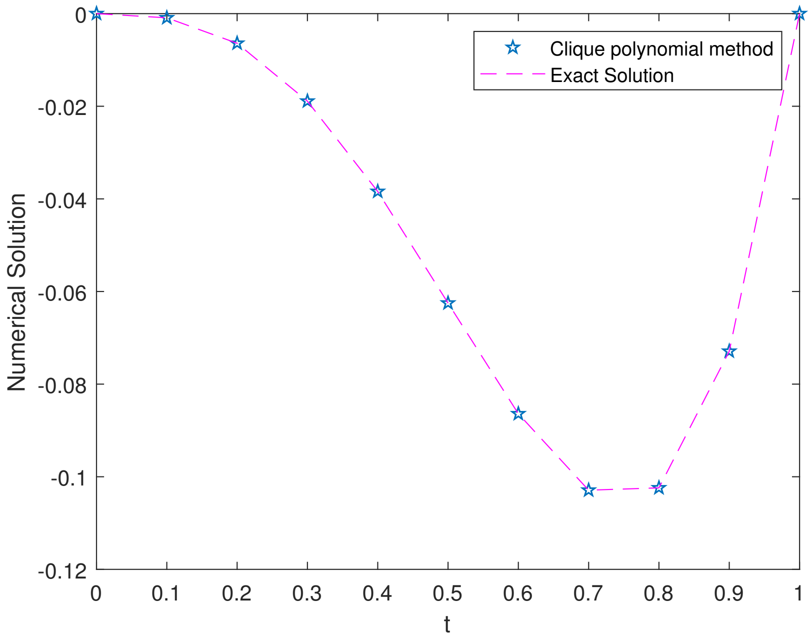

Fig. 7: Numerical solution of CPM with exact solution for $n = 4$ of Example 6.

TABLE VI: Abs. Error comparison of CPM with existing methods [4], [34] of Example 6.

t	Exact solution	Abs. Error of Method [34]	Abs. Error of Method [4]	Abs. Error of CPM for $n = 4$
0.1	-0.000900000000000	1.1e-05	1.6e-11	1.3878e-17
0.2	-0.006400000000000	1.0e-06	1.5e-11	2.9490e-17
0.3	-0.018900000000000	3.3e-05	0	1.8388e-16
0.4	-0.038400000000000	4.8e-05	1.0e-11	4.0939e-16
0.5	-0.062500000000000	2.3e-05	4.0e-11	6.6613e-16
0.6	-0.086400000000000	2.7e-05	1.0e-11	9.2981e-16
0.7	-0.102900000000000	5.4e-05	1.0e-10	1.2074e-15
0.8	-0.102400000000000	1.2e-05	2.0e-10	1.6792e-15
0.9	-0.072900000000000	8.4e-05	3.3e-10	2.6645e-15

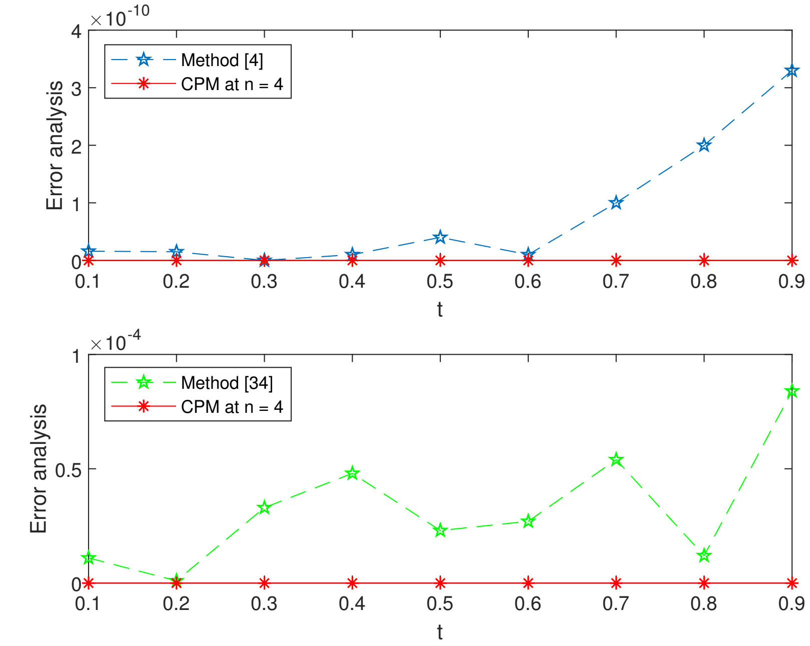

Fig. 8: Comparison of Error Analysis of Example 6.

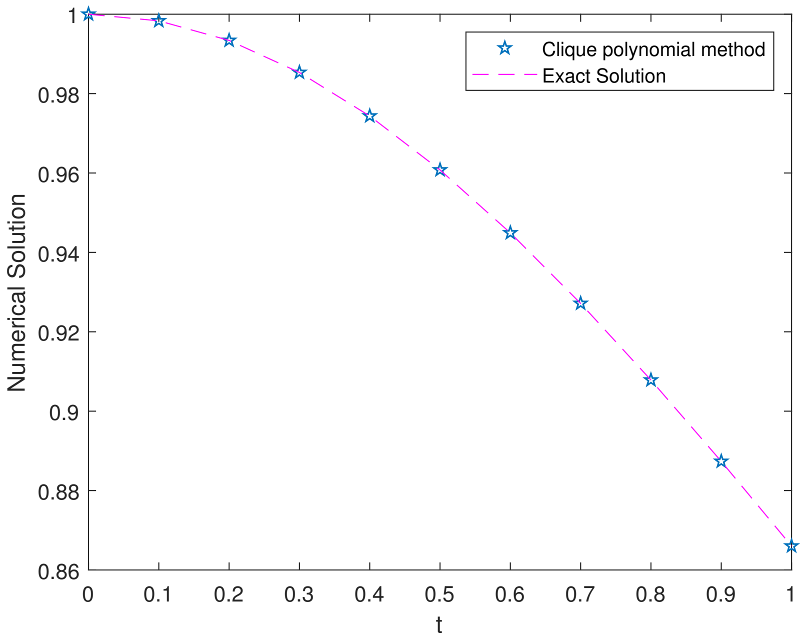

Fig. 9: Numerical solution of CPM with exact solution for $n = 10$ of Example 7.

TABLE VII: Abs. Error comparison of CPM with existing methods [5], [34] of Example 7.

t	Exact solution	Abs. Error of Method [34]	Abs. Error of Method [5]	Abs. Error of CPM for $n = 10$
0	1.000000000000000	0	9.0475e-08	0
0.1	0.998337488459583	1.54e-04	1.0363e-07	1.0752e-10
0.2	0.993399267798783	4.22e-04	1.4015e-07	6.8730e-11
0.3	0.985329278164293	6.04e-04	1.9359e-07	2.4310e-10
0.4	0.974354703692446	6.03e-04	2.5437e-07	5.4277e-10
0.5	0.960768922830523	4.12e-04	3.1179e-07	9.4219e-10
0.6	0.944911182523068	9.20e-05	3.5579e-07	1.9231e-09
0.7	0.927145540823120	2.54e-04	3.7844e-07	3.1365e-09
0.8	0.907841299003204	5.00e-04	3.7467e-07	4.1049e-09
0.9	0.887356509416114	5.10e-04	3.4252e-07	5.2451e-09
1	0.866025403784439	1.59e-04	2.8283e-07	5.4743e-09

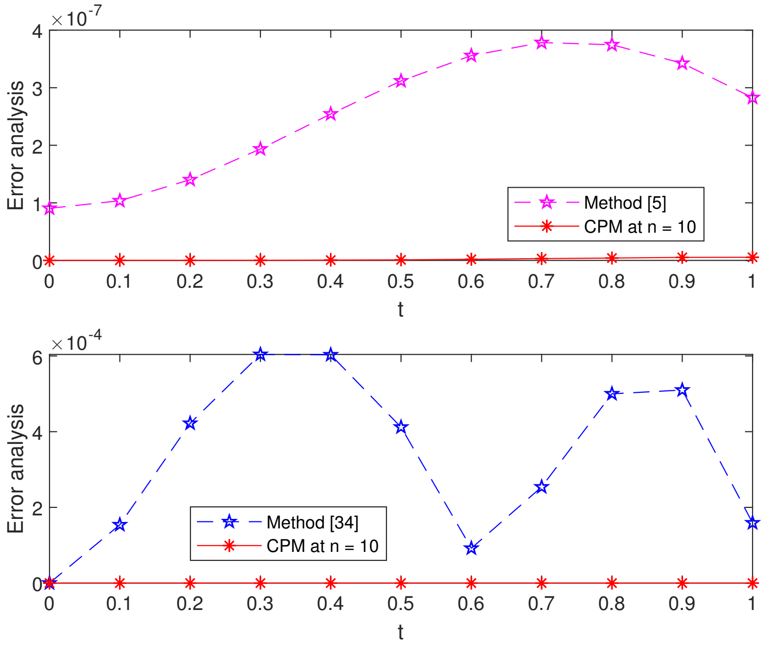

Fig. 10: Comparison of Error Analysis of Example 7.

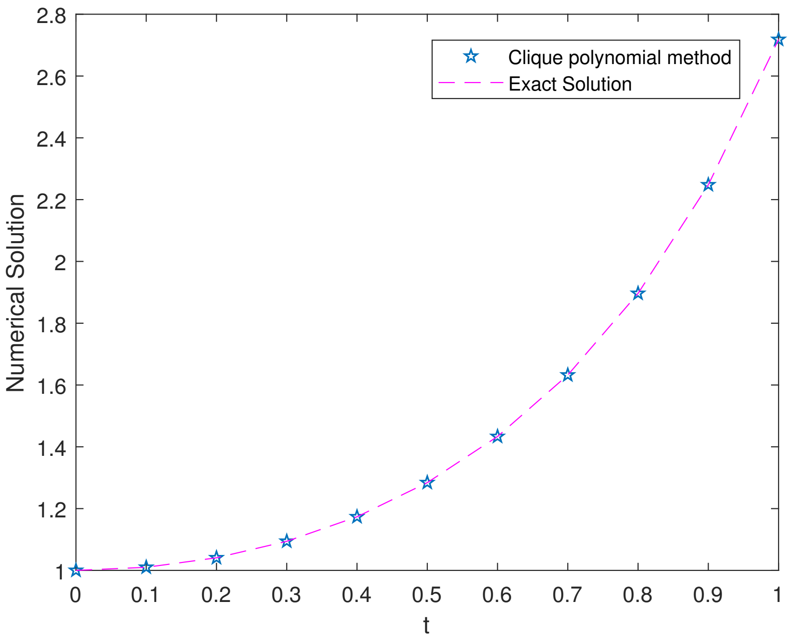

Fig. 11: Numerical solution of CPM with exact solution for $n = 11$ of Example 8.

TABLE VIII: Abs. Error comparison of CPM with existing methods [24], [34] of Example 8.

t	Exact solution	Abs. Error of Method [24]	Abs. Error of Method [34]	Abs. Error of CPM for $n = 11$
0	1.000000000000000	0	0	0
0.1	1.010050167084168	6.50e-04	2.58e-04	1.5668e-09
0.2	1.040810774192388	1.29e-03	2.67e-04	1.5831e-09
0.3	1.094174283705210	3.26e-04	7.00e-05	1.9419e-09
0.4	1.173510870991810	1.37e-02	3.00e-05	2.6106e-09
0.5	1.284025416687741	2.57e-03	3.00e-06	7.2435e-10
0.6	1.433329414560340	2.13e-03	2.00e-05	4.1777e-09
0.7	1.632316219955379	8.52e-03	7.00e-05	9.9940e-09
0.8	1.896480879304952	4.08e-03	1.15e-04	1.7006e-08
0.9	2.247907986676472	8.71e-03	1.28e-04	1.0678e-08
1	2.718281828459046	3.48e-03	4.70e-05	1.6416e-08

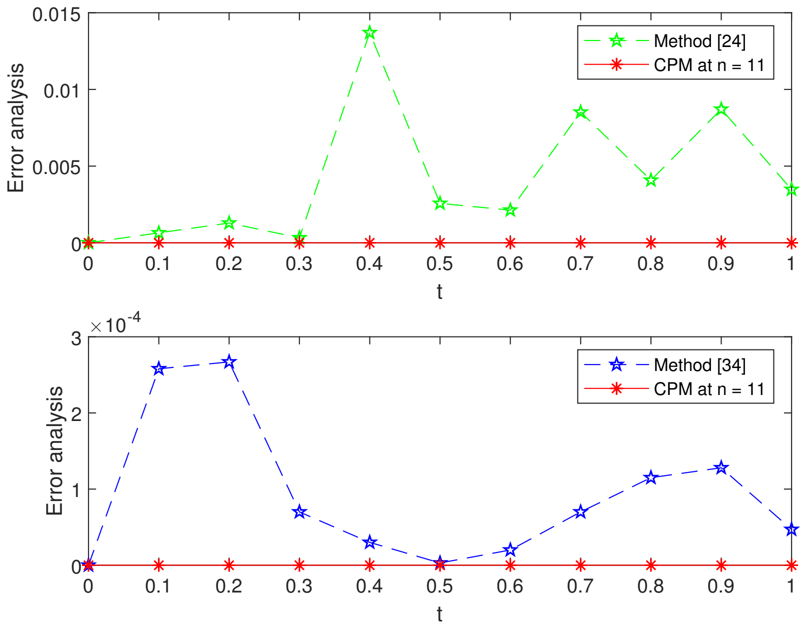

Fig. 12: Comparison of Error Analysis of Example 8.

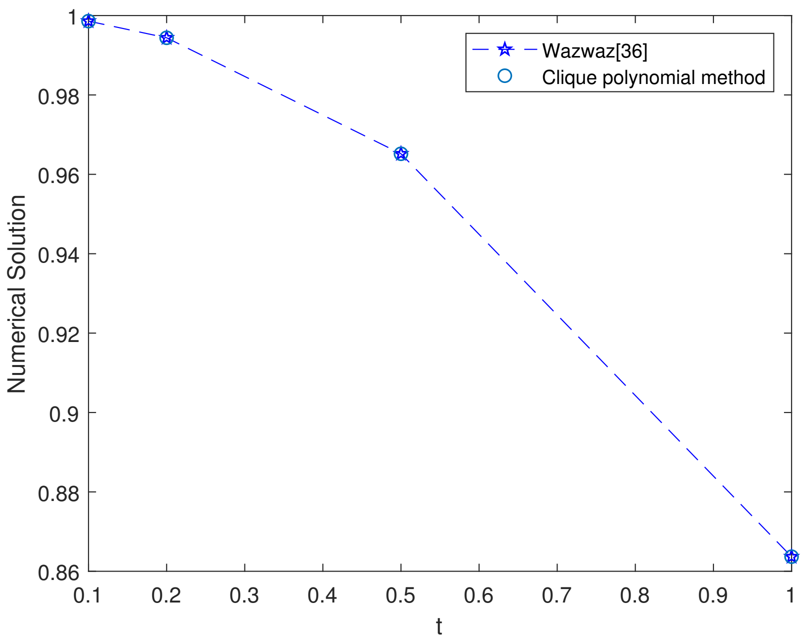

Fig. 13: Numerical solution of CPM for $n = 11$ and Wazwaz [36] of Example 9.

TABLE IX: Comparison of CPM with series solution by Wazwaz [36] and existing methods [30], [31] of Example 9.

t	Wazwaz [36]	Clique Polynomial Method for $n = 11$	Abs. Error of Method [31]	Abs. Error of Method [30]	Abs. Error of CPM for $n = 11$
0.1	0.998597927380692	0.998597927380769	7.21e-06	3.25e-07	8.3055e-12
0.2	0.994396264861727	0.994396264880525	1.00e-05	1.28e-06	4.1174e-11
0.5	0.965177751419767	0.965177780164668	1.04e-05	7.53e-06	2.8757e-08
1.0	0.863673745507962	0.863681125552833	7.03e-06	2.35e-05	7.3797e-06

TABLE X: Comparison of the CPM with series solution by Wazwaz [36] and existing method [31] of Example 10.

t	Wazwaz [36]	Clique Polynomial Method for $n = 11$	Abs. Error of Method [31]	Abs. Error of CPM for $n = 11$
0.1	0.998042841444808	0.998042841446416	7.10e-05	4.0035e-13
0.2	0.992189434813177	0.992189434811669	8.64e-05	1.1616e-12
0.5	0.951961101900241	0.951961092744380	7.65e-05	9.1529e-09
1.0	0.818251666904664	0.818242928484652	5.31e-05	8.7384e-06

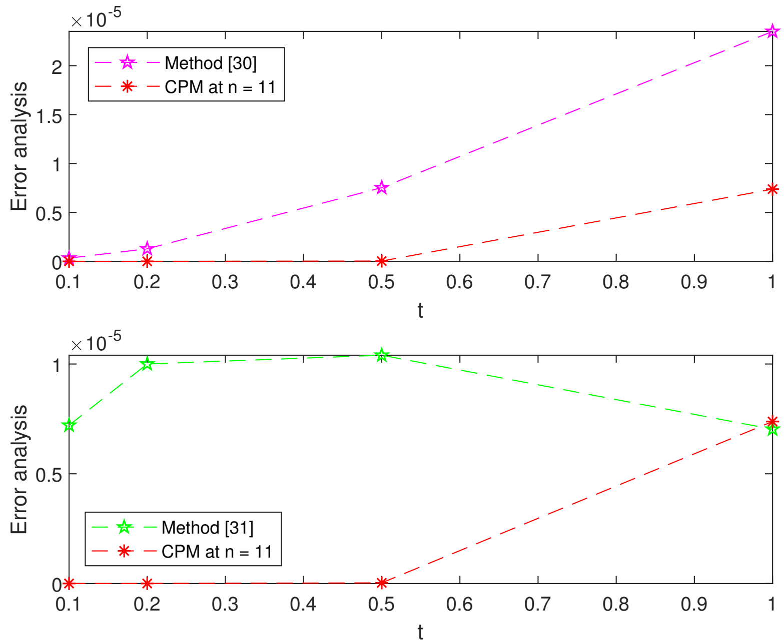

Fig. 14: Comparison of Error Analysis of Example 9.

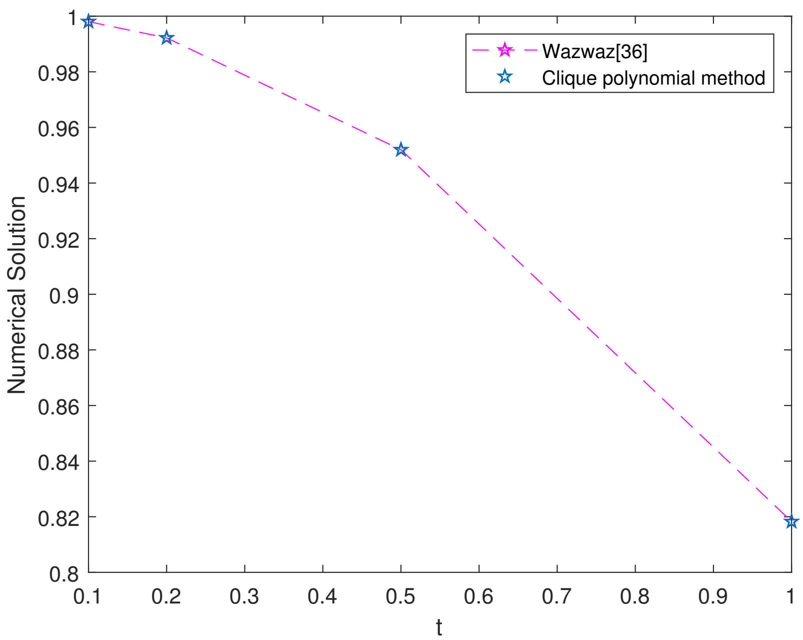

Fig. 15: Numerical solution of CPM for $n = 11$ and Wazwaz [36] of Example 10.

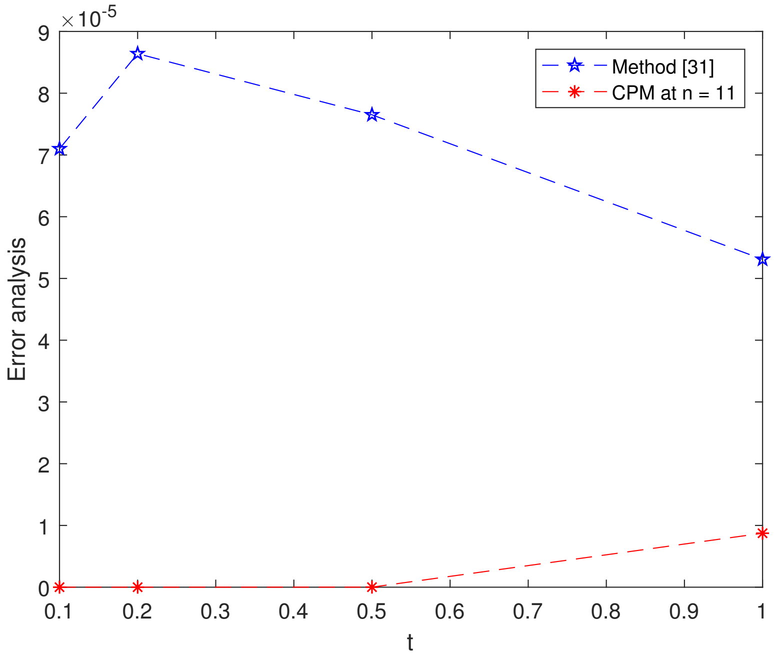

Fig. 16: Comparison of Error Analysis of Example 10.

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