Formulating Significant Identities Connecting Infinite Series of Eisenstein and Borweins' Cubic Theta Functions

Smitha Ganesh Bhat and Vidya Harekala Chandrashekara*

***Abstract*—The (p,k)-parametrization method, as introduced by Alaca, offers an innovative approach to derive a variety of new Eisenstein series identities that incorporate Borweins' theta functions. This paper presents new Eisenstein series identities that involve Borweins' cubic theta functions, achieved through the (p,k)-parameterization technique.**

***Index Terms*—Cubic Theta Functions, Eisenstein Series**

I. Introduction

In this article, we investigate the connections between Eisenstein series and Borweins' cubic theta functions using parameters introduced by Alaca. Eisenstein series are complex-valued functions that play a fundamental role in number theory and modular forms theory, while Borweins' cubic theta functions are special functions related to cubic forms and modular forms.

The parameters p and k introduced by Alaca serve as key elements in establishing these relationships. By examining these parameters, we uncover profound connections between these two mathematical objects.

What makes this exploration particularly noteworthy is that it goes beyond relying solely on computational methods. Instead, we demonstrate analytically that certain Ramanujan-type Eisenstein series, which are a special class of Eisenstein series with remarkable properties, can be represented as combinations of Borweins' cubic theta functions. This not only sheds light on the intricate interplay between these mathematical entities but also provides deeper insights into the underlying structures and symmetries within the realm of modular forms and related areas of mathematics.

In Section 2 of the document, we aim to lay down foundational knowledge and insights that are essential for understanding and advancing towards our main objectives. This section serves as a stepping stone or a starting point for the reader to grasp the concepts, theories, or methodologies that will be utilized or built upon in the subsequent sections of the article.

By providing these preliminary insights, we aim to create a solid groundwork upon which they can develop their arguments, theories, or analyses further. This may involve introducing key definitions, outlining relevant background information, discussing previous research or methodologies, or establishing theoretical frameworks.

Overall, Section 2 acts as a primer that sets the stage for the reader, enabling them to better comprehend the main goals and contributions of the research presented in the document.

In Section 3 of the article, we unveil a series of captivating identities, which bear resemblance to Earnest Xia's previous discoveries but are distinct and original in their own right. These identities represent novel mathematical relationships that we have uncovered through our research.

Among the findings presented in this section are the Ramanujan-Eisenstein series and Borwein's cubic theta functions, two significant mathematical constructs with intricate properties and applications. By revealing and substantiating these identities, we contribute to the body of mathematical knowledge, shedding new light on the connections between different mathematical entities and expanding our understanding of their interplay.

Overall, Section 3 serves as a platform for showcasing our discoveries and insights, offering valuable contributions to the field of mathematics and potentially inspiring further research and exploration in related areas.

II. Preliminaries

The origins of the arithmetic-geometric mean iteration can be traced back to the theories of elliptic functions and theta functions. The Borwein brothers [3], [4] derived the following multidimensional theta functions:

$$a(q) := \sum_{r,s=-\infty}^{\infty} q^{r^2+rs+s^2}.$$

$$b(q) := \sum_{r,s=-\infty}^{\infty} \omega^{r-s} q^{r^2+rs+s^2}.$$

$$c(q) := \sum_{r,s=-\infty}^{\infty} q^{\left(r+\frac{1}{3}\right)^2+\left(r+\frac{1}{3}\right)\left(s+\frac{1}{3}\right)+\left(s+\frac{1}{3}\right)^2}.$$

for $|q|<1$, where q represents complex numbers, and $\omega = exp(2\pi i/3)$ is the principal cube root of unity, the given expressions for two-dimensional theta functions reveal that when $q=0$, the values become $a(q)=1$, $b(q)=1$, and $c(q)=0$.

From Euler's binomial theorem, Borwein brothers have devised expressions for $b(q)$ and $c(q)$ in terms of infinite

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Smitha Ganesh Bhat is Assistant Professor-Senior Scale at Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India. (e-mail:smitha.holla@manipal.edu)

Vidya Harekala Chandrashekara is Assistant Professor-Senior Scale at Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India. (corresponding author to provide e-mail:vidyaashwath@gmail.com)

products, which are given by,

$$b(q) = \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty},$$
$$c(q) = \frac{3q^{\frac{1}{3}}(q^3;q^3)_\infty^3}{(q;q)_\infty},$$

where

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i).$$

Definition II.1. *Srinivasa Ramanujan in his second notebook [6] has provided the definitions for the Eisenstein Series $L(q)$ and $M(q)$ as follows:*

$$L(q) := 1 - 24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r} := 1 - 24\sum_{r=1}^{\infty}\delta_1(r)q^r,$$
$$M(q) := 1 + 240\sum_{r=1}^{\infty}\frac{r^3q^r}{1-q^r} := 1 + 240\sum_{r=1}^{\infty}\delta_3(r)q^r.$$

Definition II.2. *For any complex c and d, Ramanujan[2, p.35] documented a general theta function,*

$$f(c,d) := \sum_{m=-\infty}^{\infty} c^{m(m+1)/2}d^{m(m-1)/2}$$
$$:= (-c;cd)_\infty(-d;cd)_\infty(cd;cd)_\infty,$$

where

$$(c;q)_\infty := \prod_{m=0}^{\infty}(1-cq^m), \qquad |q|<1.$$

The special case of theta function defined by Ramanujan[2, p.35],

$$\varphi(q) := f(q,q) = \sum_{m=-\infty}^{\infty} q^{m^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty.$$

In their remarkable article, Alaca et al. [1] have defined the $(p,\ k)$ parametrization of theta functions. These are highly significant in designing the duplication and triplication principle and further obtaining certain sum to product identities. The parameters p and k are defined as:

$$p := p(q) = \frac{\varphi^2(q) - \varphi^2(q^3)}{2\phi^2(q^3)}.$$
$$k := k(q) = \frac{\varphi^3(q^3)}{\varphi(q)}.$$

Since $\varphi(0) = 1$, it clear that $p(0) = 0$ and $k(0) = 1$.

Lemma II.3. *[1] For aforementioned Eisenstein series [3], [4], the representations of $M(q)$, $M(q^l)$, $L(q) - lL(q^l)$, $(l = 2,3,4,6,12)$ and also $L(-q^l) - rL(q^r)$, $l \in \{1,3\}$ and $r \in \{1,2,3\}$, in terms of the parameters p and k are given by,*

$$M(q) = (1 + 124p(1+p^6) + 964p^2(1+p^4) + 2788p^3(1 + p^2) + 3910p^4 + p^8)k^4,$$
$$M(q^2) = (1 + 4p(1+p^6) + 64p^2(1+p^4) + 178p^3(1+p^2) + 235p^4 + p^8)k^4,$$
$$M(q^3) = (1 + 4p(1+p^6) + 4p^2(1+p^4) + 28p^3(1+p^2) + 70p^4 + p^8)k^4,$$
$$M(q^6) = (1 + 4p(1+p^6) + 4p^2(1+p^4) - 2p^3(1+p^2) - 5p^4 + p^8)k^4,$$
$$M(q^{12}) = (1 + 4p(1+p) - 2p^3(1+p^2) - 5p^4 + p^6(1+p)/4 + p^8/16)k^4,$$
$$L(-q) - L(q) = 3(8p + 12p^2 + 6p^3 + p^4)k^2,$$
$$L_{1,2}(q) = (L(-q) - L(q))/48 = (p/2 + 3p^2/4 + 3p^3/8 + p^4/16)k^2,$$
$$L_{1,2}(q^3) = (L(-q^3) - L(q^3))/48 = p^3(2+p)k^2/16,$$
$$L(-q) - 2L(q^2) = -(1 - 10p - 12p^2 - 4p^3 - 2p^4)k^2,$$
$$L(q) - 2L(q^2) = -(1 + 14p(1+p^2) + 24p^2 + p^4)k^2,$$
$$L(q) - 3L(q^3) = -(1 + 8p(1+p^2) + 18p^2 + p^4)k^2,$$
$$L(q) - 6L(q^6) = -(5 + 22p(1+p^2) + 36p^2 + 5p^4)k^2,$$
$$L(q^2) - 3L(q^6) = -2(1 + 2p(1+p^2) + 3p^2 + p^4)k^2,$$
$$L(q^3) - 2L(q^6) = -(1 + 2p(1+p^2) + p^4)k^2,$$
$$L(q) - 4L(q^4) = -3(1 + 6p + 12p^2 + 8p^3)k^2,$$
$$L(q) - 12L(q^{12}) = -(11 + 34p + 36p^2 + 16p^3 + 2p^4)k^2.$$

Lemma II.4. *The parametric representations of $a(q^r)$, $b(q^r)$, $c(q^r)$, $(r \in 1,2,4,6)$ and $a(-q)$, $b(-q)$, $c(-q)$ in terms of the parameters p and k deduced by Alaca et al. [1] are as follows*

$$a(-q) = (1 - 2p - 2p^2)k,$$
$$a(q) = (1 + 4p + p^2)k,$$
$$a(q^2) = (1 + p + p^2)k,$$
$$a(q^4) = (1 + p - \frac{1}{2}p^2)k,$$
$$a(q^6) = \frac{(p^2 + p + 1 + 2^{1/3}((1-p)(2+p)(1+2p))^{2/3})k}{3},$$
$$b(-q) = 2^{-\frac{1}{3}}((1-p)(1+2p)^4(2+p))^{\frac{1}{3}}k,$$
$$b(q) = 2^{-\frac{1}{3}}((1-p)^4(1+2p)(2+p))^{\frac{1}{3}}k,$$
$$b(q^2) = 2^{-2/3}((1-p)(1+2p)(2+p))^{\frac{2}{3}}k,$$
$$b(q^4) = 2^{-\frac{4}{3}}((1-p)(1+2p)(2+p)^4)^{\frac{1}{3}}k,$$
$$c(-q) = -2^{\frac{1}{3}}3(p(1+p))^{\frac{1}{3}}k,$$
$$c(q) = 2^{-\frac{1}{3}}3(p(1+p)^4)^{\frac{1}{3}}k,$$
$$c(q^2) = 2^{-\frac{2}{3}}3(p(1+p))^{\frac{2}{3}}k.$$
$$c(q^4) = 2^{-\frac{4}{3}}3(p^4(1+p))^{\frac{1}{3}}k,$$
$$c(q^6) = \frac{\left(p^2 + p + 1 - 2^{-2/3}((1-p)(2+p)(1+2p))^{2/3}\right)k}{3}.$$

,

III. Connections Involving Ramanujan's Eisenstein Series and Cubic Theta Functions

In Ramanujan's notebook [6], he recorded intriguing series involving variables L and M, presenting significant identities for infinite series incorporating theta functions. Building upon Ramanujan's work, Xia et al.[10] utilized computational methods to reveal elegant mathematical identities involving Eisenstein series and cubic theta functions, particularly focusing on expressions of the form $L(q)-rL(q^r)$, where r varied over the set $\{2,\ 3,\ 4,\ 6,\ 12\}$. Subsequent researchers, such as Shruti and Srivatsakumar B.R.[7], derived further identities and explored convolution sums. More recent contributions by Vidya H. C. and Ashwath Rao B.[8], as well as Vidya H. C. and Smitha G. Bhat[9], introduced identities like $L(-q^l)-L(q^l)$, where l belongs to $\{1,\ 3\}$. Additionally, Vidya H. C. and Ashwath Rao B. [5] established connections among theta functions, contributing to the ongoing exploration of these mathematical relationships.

In our paper, we have successfully derived precise mathematical relationships between Ramanujan-type Eisenstein series and cubic theta functions. Our focus particularly lies on Eisenstein series of the form $M(q^n)$, where n takes values from the set $\{1,\ 2,\ 3,\ 6,\ 12\}$. Importantly, we accomplished this without relying on computational tools, emphasizing a purely analytical approach. By establishing these identities, we contribute to a deeper understanding of the connections between these mathematical entities, offering insights that transcend computational methods and further enriching the field of mathematical analysis.

Theorem III.1. *The relation amongst an infinite series and theta functions holds:*

$$(i)\quad -\left[\frac{1}{2}+u+4v+\frac{w}{9}\right]+240\sum_{1}^{\infty}\left[\left(\frac{1}{40}-\frac{u}{5}-\frac{2v}{5}\right)\frac{r^3q^r}{1-q^r}-\left(\frac{3}{10}+\frac{4u}{5}-\frac{4w}{45}\right)\frac{r^3q^{2r}}{1-q^{2r}}-\left(\frac{9}{10}+\frac{18v}{5}+\frac{w}{5}\right)\frac{r^3q^{3r}}{1-q^{3r}}\right]$$
$$+u\left[-1+24\sum_{1}^{\infty}\left[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\right]\right]^2+v\left[-2+24\sum_{1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\right]\right]^2$$
$$+\left(\frac{3}{8}-\frac{2w}{9}\right)\left[-2+24\sum_{1}^{\infty}\left[\frac{3rq^{6r}}{1-q^{6r}}-\frac{rq^{2r}}{1-q^{2r}}\right]\right]^2+w\left[-1+24\sum_{1}^{\infty}\left[\frac{2rq^{6r}}{1-q^{6r}}-\frac{rq^{3r}}{1-q^{3r}}\right]\right]^2$$
$$=a(q)a^3(q^2). \tag{1}$$

$$(ii)\quad -\left[\frac{1}{8}+u+4v+\frac{w}{9}\right]+240\sum_{1}^{\infty}\left[\left(-\frac{u}{5}-\frac{2v}{5}\right)\frac{r^3q^r}{1-q^r}-\left(\frac{1}{8}+\frac{4u}{5}-\frac{4w}{45}\right)\frac{r^3q^{2r}}{1-q^{2r}}-\left(\frac{18v}{5}-\frac{w}{5}\right)\frac{r^3q^{3r}}{1-q^{3r}}+\frac{81}{80}\frac{r^3q^{6r}}{1-q^{6r}}\right]$$
$$+u\left[-1+24\sum_{1}^{\infty}\left[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\right]\right]^2+v\left[-2+24\sum_{1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\right]\right]^2+\left(\frac{9}{32}-\frac{2w}{9}\right)\left[-2+24\sum_{1}^{\infty}\left[\frac{3rq^{6r}}{1-q^{6r}}-\frac{rq^{2r}}{1-q^{2r}}\right]\right]^2$$
$$+w\left[-1+24\sum_{1}^{\infty}\left[\frac{2rq^{6r}}{1-q^{6r}}-\frac{rq^{3r}}{1-q^{3r}}\right]\right]^2=a(q^2)b^3(q^2). \tag{2}$$

$$(iii)\quad \left[1-u-4v+\frac{w}{9}\right]+240\sum_{1}^{\infty}\left[\left(-\frac{1}{80}-\frac{u}{5}-\frac{2v}{5}\right)\frac{r^3q^r}{1-q^r}-\left(\frac{4u}{5}-\frac{4w}{45}\right)\frac{r^3q^{2r}}{1-q^{2r}}+\left(\frac{81}{80}-\frac{18v}{5}-\frac{w}{5}\right)\frac{r^3q^{3r}}{1-q^{3r}}\right]+u\left[-1+24\sum_{1}^{\infty}\left[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\right]\right]^2$$
$$+v\left[-2+24\sum_{1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\right]\right]^2-\frac{2w}{9}\left[-2+24\sum_{1}^{\infty}\left[\frac{3rq^{6r}}{1-q^{6r}}-\frac{rq^{2r}}{1-q^{2r}}\right]\right]^2$$
$$+w\left[-1+24\sum_{1}^{\infty}\left[\frac{2rq^{6r}}{1-q^{6r}}-\frac{rq^{3r}}{1-q^{3r}}\right]\right]^2=a(q)b^3(q). \tag{3}$$

$$(iv)\quad -\left[1+u+4v-\frac{w}{9}\right]+240\sum_{1}^{\infty}\left[\left(\frac{3}{40}-\frac{u}{5}-\frac{2v}{5}\right)\frac{r^3q^r}{1-q^r}-\left(\frac{2}{5}+\frac{4u}{5}-\frac{4w}{45}\right)\frac{r^3q^{2r}}{1-q^{2r}}-\left(\frac{27}{40}+\frac{18v}{5}+\frac{w}{5}\right)\frac{r^3q^{3r}}{1-q^{3r}}\right]+u\left[-1+24\sum_{1}^{\infty}\left[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\right]\right]^2$$
$$+v\left[-2+24\sum_{1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\right]\right]^2+\left(\frac{1}{2}-\frac{2w}{9}\right)\left[-2+24\sum_{1}^{\infty}\left[\frac{3rq^{6r}}{1-q^{6r}}-\frac{rq^{2r}}{1-q^{2r}}\right]\right]^2$$
$$+w\left[-1+24\sum_{1}^{\infty}\left[\frac{2rq^{6r}}{1-q^{6r}}-\frac{rq^{3r}}{1-q^{3r}}\right]\right]^2$$

$$= a^3(q)a(q^2). \tag{4}$$

$$(v)\quad \Big[1-u-4v+\frac{w}{9}\Big]+240\sum_{1}^{\infty}\Big[\Big(\frac{1}{10}-\frac{u}{5} - \frac{2v}{5}\Big)\frac{r^3q^r}{1-q^r}-\Big(\frac{4u}{5}-\frac{4w}{45}\Big)\frac{r^3q^{2r}}{1-q^{2r}} + \Big(\frac{9}{10}-\frac{18v}{5}-\frac{w}{5}\Big)\frac{r^3q^{3r}}{1-q^{3r}}\Big] + u\Big[-1+24\sum_{1}^{\infty}\Big[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\Big]\Big]^2 + v\Big[-2+24\sum_{1}^{\infty}\Big[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Big]\Big]^2 - \frac{2w}{9}\Big[-2+24\sum_{1}^{\infty}\Big[\frac{3rq^{6r}}{1-q^{6r}}-\frac{rq^{2r}}{1-q^{2r}}\Big]\Big]^2 + w\Big[-1+24\sum_{1}^{\infty}\Big[\frac{2rq^{6r}}{1-q^{6r}}-\frac{rq^{3r}}{1-q^{3r}}\Big]\Big]^2 = [3a(q^3)-2b(q)]. \tag{5}$$

Proof:

$$C_1M(q)+C_2M(q^2)+C_3M(q^3)+C_4M(q^{12})+C_5\{L(q) - 2L(q^2)\}^2+C_6\{L(q)-3L(q^3)\}^2+C_7\{L(q^2)-3L(q^6)\}^2 + C_8\{L(q^3)-2L(q^6)\}^2 = a(q)a^3(q^2). \tag{6}$$

Subsequently upon (p,k) parametrization of the above expression using Lemma II.3 and by equating the coefficients of $k^4, pk^4, p^2k^4,\ p^3k^4, p^4k^4, p^5k^4, p^6k^4, p^7k^4$ and p^8k^4 on either sides, we obtain

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 4 & 4 & 1\\ 124 & 4 & 4 & 4 & 28 & 64 & 16 & 4\\ 964 & 64 & 4 & 4 & 244 & 400 & 40 & 4\\ 2788 & 178 & 28 & -2 & 700 & 1216 & 64 & 4\\ 3910 & 235 & 70 & -5 & 970 & 1816 & 76 & 10\\ 2788 & 178 & 28 & -2 & 700 & 1216 & 64 & 4\\ 964 & 64 & 4 & \frac{1}{4} & 244 & 400 & 40 & 4\\ 124 & 4 & 4 & \frac{1}{4} & 28 & 64 & 16 & 4\\ 1 & 1 & 1 & \frac{1}{16} & 1 & 4 & 4 & 1 \end{pmatrix}\begin{pmatrix} C_1\\ C_2\\ C_3\\ C_4\\ C_5\\ C_6\\ C_7\\ C_8\end{pmatrix} = \begin{pmatrix} 1\\ 7\\ 19\\ 34\\ 40\\ 34\\ 19\\ 7\\ 1\end{pmatrix}.$$

We note that, the system results in an infinitely many solutions,

$$C_1=\Big(\frac{1}{40}-\frac{u}{5}-\frac{2v}{5}\Big),\ C_2=\Big(-\frac{3}{10}-\frac{4u}{5}+\frac{4w}{45}\Big),$$
$$C_3=\Big(-\frac{9}{40}-\frac{18v}{5}-\frac{w}{5}\Big),\ C_4=0,$$
$$C_5=u,\ C_6=v,\ C_7=\Big(\frac{3}{8}-\frac{2w}{9}\Big),\ C_8=w.$$

Substituting these values in (6) yields, (1). Modifying the right-hand side of (1) and utilizing (6) yields the following equations [i.] through [iv.].

$$[i.]\ -\Big(\frac{u}{5}+\frac{2v}{5}\Big)M(q)-\Big(\frac{1}{8}+\frac{4u}{5}+\frac{4w}{45}\Big)M(q^2) - \Big(\frac{18v}{5}+\frac{w}{5}\Big)M(q^3)+u\{L(q)-2L(q^2)\}^2 + v\{L(q)-3L(q^3)\}^2+\Big(\frac{9}{32}-\frac{2w}{9}\Big)\{L(q^2)-3L(q^6)\}^2 + w\{L(q^3)-2L(q^6)\}^2 = a(q^2)b^3(q^2).$$

$$[ii.]\ -\Big(\frac{1}{80}-\frac{u}{5}-\frac{2v}{5}\Big)M(q)-\Big(\frac{4u}{5}-\frac{4w}{45}\Big)M(q^2) + \Big(\frac{81}{80}-\frac{18v}{5}-\frac{w}{5}\Big)M(q^3)+u\{L(q)-2L(q^2)\}^2 + v\{L(q)-3L(q^3)\}^2-\Big(\frac{2w}{9}\Big)\{L(q^2)-3L(q^6)\}^2 + w\{L(q^3)-2L(q^6)\}^2 = a(q)b^3(q)$$

$$[iii.]\ \Big(\frac{3}{40}-\frac{u}{5}-\frac{2v}{5}\Big)M(q)-\Big(\frac{2}{5}+\frac{4u}{5}-\frac{4w}{45}\Big)M(q^2) - \Big(\frac{27}{40}+\frac{18v}{5}+\frac{w}{5}\Big)M(q^3)+u\{L(q)-2L(q^2)\}^2+ v\{L(q)-3L(q^3)\}^2+\Big(\frac{1}{2}-\frac{2w}{9}\Big)\{L(q^2)-3L(q^6)\}^2 + w\{L(q^3)-2L(q^6)\}^2 = a^3(q)a(q^2).$$

$$[iv.]\ \Big(\frac{1}{10}-\frac{u}{5}-\frac{2v}{5}\Big)M(q)-\Big(\frac{4u}{5}-\frac{4w}{45}\Big)M(q^2) + \Big(\frac{9}{10}+\frac{18v}{5}+\frac{w}{5}\Big)M(q^3)+u\{L(q)-2L(q^2)\}^2 + v\{L(q)-3L(q^3)\}^2-\Big(\frac{2w}{9}\Big)\{L(q^2)-3L(q^6)\}^2 + w\{L(q^3)-2L(q^6)\}^2 = [3a(q^3)-2b(q)]^2.$$

Simplifying the aforementioned equations results in equations (2) through (5). ■

Theorem III.2. *The relation amongst an infinite series and theta functions holds:*

$$(i)\quad \Big[1-u-v\Big]+240\sum_{1}^{\infty}\Big[\Big(\frac{1}{10}-\frac{u}{5}\Big)\frac{r^3q^r}{1-q^r} - \Big(\frac{4u}{5}\Big)\frac{r^3q^{2r}}{1-q^{2r}}+\Big(\frac{9}{10}-\frac{v}{5}\Big)\frac{r^3q^{3r}}{1-q^{3r}} - \Big(\frac{4v}{5}\Big)\frac{r^3q^{6r}}{1-q^{6r}}\Big]+u\Big[-1+24\sum_{1}^{\infty}\Big[\frac{2rq^{2r}}{1-q^{2r}} - \frac{rq^r}{1-q^r}\Big]\Big]^2+v\Big[-1+24\sum_{1}^{\infty}\Big[\frac{2rq^{6r}}{1-q^{6r}} - \frac{rq^{3r}}{1-q^{3r}}\Big]\Big]^2 = 3[a(q^3)-2b(q)]^2. \tag{7}$$

$$(ii)\ \Big[1-u-v\Big]+240\sum_{1}^{\infty}\Big[-\Big(\frac{u}{5}\Big)\frac{r^3q^r}{1-q^r}-\Big(\frac{1}{80}+\frac{4u}{5}\Big)\frac{r^3q^{2r}}{1-q^{2r}}-\Big(\frac{v}{5}\Big)\frac{r^3q^{3r}}{1-q^{3r}}+\Big(\frac{81}{80}-\frac{4v}{5}\Big)\frac{r^3q^{6r}}{1-q^{6r}}\Big]\Big]+u\Big[-1+24\sum_{1}^{\infty}\Big[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\Big]\Big]^2+v\Big[-1+24\sum_{1}^{\infty}\Big[\frac{2rq^{6r}}{1-q^{6r}}-\frac{rq^{3r}}{1-q^{3r}}\Big]\Big]^2=a(q^2)b^3(q^2). \tag{8}$$

$$(iii)\ \Big[1-u-v\Big]+240\sum_{1}^{\infty}\Big[-\Big(\frac{1}{80}+\frac{u}{5}\Big)\frac{r^3q^r}{1-q^r}-\Big(\frac{4u}{5}\Big)\frac{r^3q^{2r}}{1-q^{2r}}+\Big(\frac{81}{80}-\frac{v}{5}\Big)\frac{r^3q^{3r}}{1-q^{3r}}-\Big(\frac{4v}{5}\Big)\frac{r^3q^{6r}}{1-q^{6r}}\Big]\Big]+u\Big[-1+24\sum_{1}^{\infty}\Big[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\Big]\Big]^2+v\Big[-1+24\sum_{1}^{\infty}\Big[\frac{2rq^{6r}}{1-q^{6r}}-\frac{rq^{3r}}{1-q^{3r}}\Big]\Big]^2=a(q)b^3(q). \tag{9}$$

$$(iv)\ \Big[1-u-v\Big]+240\sum_{1}^{\infty}\Big[\Big(\frac{3}{40}-\frac{u}{5}\Big)\frac{r^3q^r}{1-q^r}-\Big(\frac{1}{5}+\frac{4u}{5}\Big)\frac{r^3q^{2r}}{1-q^{2r}}-\Big(\frac{27}{40}+\frac{v}{5}\Big)\frac{r^3q^{3r}}{1-q^{3r}}+\Big(\frac{9}{5}-\frac{4v}{5}\Big)\frac{r^3q^{6r}}{1-q^{6r}}\Big]\Big]+u\Big[-1+24\sum_{1}^{\infty}\Big[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\Big]\Big]^2+v\Big[-1+24\sum_{1}^{\infty}\Big[\frac{2rq^{6r}}{1-q^{6r}}-\frac{rq^{3r}}{1-q^{3r}}\Big]\Big]^2=a^3(q)a(q^2). \tag{10}$$

$$(v)\ \Big[1-u-v\Big]+240\sum_{1}^{\infty}\Big[\Big(\frac{1}{40}-\frac{u}{5}\Big)\frac{r^3q^r}{1-q^r}-\Big(\frac{3}{20}+\frac{4u}{5}\Big)\frac{r^3q^{2r}}{1-q^{2r}}-\Big(\frac{9}{40}-\frac{v}{5}\Big)\frac{r^3q^{3r}}{1-q^{3r}}+\Big(\frac{27}{20}-\frac{4v}{5}\Big)\frac{r^3q^{6r}}{1-q^{6r}}\Big]\Big]+u\Big[-1+24\sum_{1}^{\infty}\Big[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\Big]\Big]^2+v\Big[-1+24\sum_{1}^{\infty}\Big[\frac{2rq^{6r}}{1-q^{6r}}-\frac{rq^{3r}}{1-q^{3r}}\Big]\Big]^2=a(q)a^3(q^2). \tag{11}$$

Proof:

$$\begin{aligned}&C_1M(q)+C_2M(q^2)+C_3M(q^3)+C_4M(q^6)\\&+C_5\{L(q)-2L(q^2)\}^2+C_6\{3L(q^3)-4L(q^4)\}^2\\&+C_7\{L(q^3)-2L(q^6)\}^2+C_8\{L(q^4)-3L(q^{12})\}^2\\&=3[a(q^3)-2b(q)]^2.\end{aligned} \tag{12}$$

Subsequently upon (p,k) parametrization of the above expression using Lemma II.3 and by equating the coefficients of $k^4, pk^4, p^2k^4,\ p^3k^4, p^4k^4, p^5k^4, p^6k^4, p^7k^4$ and p^8k^4 on either sides, we obtain

$$\begin{pmatrix}1&1&1&1&1&1&1&4\\124&4&4&4&28&4&4&16\\964&64&4&4&244&4&4&16\\2788&178&28&-2&700&16&4&-8\\3910&235&70&-5&970&28&10&-14\\2788&178&28&-2&700&-8&4&4\\964&64&4&4&244&64&4&4\\124&4&4&4&28&-32&4&-2\\1&1&1&1&1&4&1&\frac{1}{4}\end{pmatrix}\begin{pmatrix}C_1\\C_2\\C_3\\C_4\\C_5\\C_6\\C_7\\C_8\end{pmatrix}=\begin{pmatrix}1\\16\\100\\304\\454\\304\\100\\16\\1\end{pmatrix}.$$

We note that, the system results in a unique solution,

$$C_1=\Big(\frac{1}{10}-\frac{u}{5}\Big),\ C_2=-\Big(\frac{4u}{5}\Big),\ C_3=\Big(\frac{9}{10}-\frac{v}{5}\Big),$$
$$C_4=-\Big(\frac{4v}{5}\Big),\ C_5=u,\ C_6=0,\ C_7=v,\ C_8=0.$$

Substituting these values in (12) yields, (7). Altering the right-hand side of (7) and subsequently using (12) results in equations (i) to (iv).

$$\begin{aligned}(i)\ &-\Big(\frac{u}{5}\Big)M(q)-\Big(\frac{1}{80}+\frac{4u}{5}\Big)M(q^2)-\Big(\frac{v}{5}\Big)M(q^3)\\&+\Big(\frac{81}{80}-\frac{4v}{5}\Big)M(q^6)+u\{L(q)-2L(q^2)\}^2\\&+v\{L(q^3)-2L(q^6)\}^2=a(q^2)b^3(q^2).\end{aligned}$$

$$\begin{aligned}(ii)\ &-\Big(\frac{1}{80}+\frac{u}{5}\Big)M(q)-\Big(\frac{4u}{5}\Big)M(q^2)\\&+\Big(\frac{81}{80}-\frac{v}{5}\Big)M(q^3)-\Big(\frac{4v}{5}\Big)M(q^6)\\&+u\{L(q)-2L(q^2)\}^2+v\{L(q^3)-2L(q^6)\}^2\\&=a(q)b^3(q).\end{aligned}$$

$$\begin{aligned}(iii)\ &\Big(\frac{3}{40}-\frac{u}{5}\Big)M(q)-\Big(\frac{1}{5}+\frac{4u}{5}\Big)M(q^2)\\&-\Big(\frac{27}{40}+\frac{v}{5}\Big)M(q^3)+\Big(\frac{9}{5}-\frac{4v}{5}\Big)M(q^6)\\&+u\{L(q)-2L(q^2)\}^2+v\{L(q^3)-2L(q^6)\}^2\\&=a^3(q)a(q^2).\end{aligned}$$

$$(iv)\ \left(\frac{1}{40}-\frac{u}{5}\right)M(q)-\left(\frac{3}{20}+\frac{4u}{5}\right)M(q^2) -\left(\frac{9}{40}+\frac{v}{5}\right)M(q^3)+\left(\frac{27}{20}-\frac{4v}{5}\right)M(q^6) +u\{L(q)-2L(q^2)\}^2+v\{L(q^3)-2L(q^6)\}^2 =a(q)a^3(q^2).$$

After applying simplification techniques to the preceding equations, we obtain the set of equations labeled as (8) through (11). ■

Theorem III.3. *The connection between an infinite series and theta functions is as follows:*

$$(i)\ \Big[1-4u-v\Big]+240\sum_{1}^{\infty}\Bigg[\left(\frac{3}{40}-\frac{2u}{5}\right)\frac{r^3q^r}{1-q^r} -\frac{1}{5}\frac{r^3q^{2r}}{1-q^{2r}}-\left(\frac{27}{40}+\frac{18u}{5}+\frac{v}{5}\right)\frac{r^3q^{3r}}{1-q^{3r}}+\left(\frac{9}{5} -\frac{4v}{5}\right)\frac{r^3q^{6r}}{1-q^{6r}}\Bigg]+u\Bigg[-2+24\sum_{1}^{\infty}\Bigg[\frac{3rq^{3r}}{1-q^{3r}} -\frac{rq^r}{1-q^r}\Bigg]\Bigg]^2+v\Bigg[-1+24\sum_{1}^{\infty}\Bigg[\frac{2rq^{6r}}{1-q^{6r}} -\frac{rq^{3r}}{1-q^{3r}}\Bigg]\Bigg]^2=a^3(q)a(q^2). \tag{13}$$

$$(ii)\ \Big[1-4u-v\Big]+240\sum_{1}^{\infty}\Bigg[-\left(\frac{2u}{5}\right)\frac{r^3q^r}{1-q^r} -\left(\frac{1}{80}\right)\frac{r^3q^{2r}}{1-q^{2r}}-\left(\frac{18u}{5}+\frac{v}{5}\right)\frac{r^3q^{3r}}{1-q^{3r}} +\left(\frac{81}{80}-\frac{4v}{5}\right)\frac{r^3q^{6r}}{1-q^{6r}}\Bigg]+u\Bigg[-2+24\sum_{1}^{\infty}\Bigg[\frac{3rq^{3r}}{1-q^{3r}} -\frac{rq^r}{1-q^r}\Bigg]\Bigg]^2+v\Bigg[-1+24\sum_{1}^{\infty}\Bigg[\frac{2rq^{6r}}{1-q^{6r}} -\frac{rq^{3r}}{1-q^{3r}}\Bigg]\Bigg]^2=a(q^2)b^3(q^2). \tag{14}$$

$$(iii)\ \Big[1-4u-v\Big]+240\sum_{1}^{\infty}\Bigg[-\left(\frac{1}{80}+\frac{2u}{5}\right)\frac{r^3q^r}{1-q^r} +\left(\frac{81}{80}-\frac{18u}{5}-\frac{v}{5}\right)\frac{r^3q^{3r}}{1-q^{3r}}-\left(\frac{4v}{5}\right)\frac{r^3q^{6r}}{1-q^{6r}}\Bigg] +u\Bigg[-2+24\sum_{1}^{\infty}\Bigg[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Bigg]\Bigg]^2 +v\Bigg[-1+24\sum_{1}^{\infty}\Bigg[\frac{2rq^{6r}}{1-q^{6r}}-\frac{rq^{3r}}{1-q^{3r}}\Bigg]\Bigg]^2 =a(q)b^3(q). \tag{15}$$

$$(iv)\ \Big[1-4u-v\Big]+240\sum_{1}^{\infty}\Bigg[\left(\frac{1}{10}-\frac{2u}{5}\right)\frac{r^3q^r}{1-q^r} +\left(\frac{9}{10}-\frac{18u}{5}-\frac{v}{5}\right)\frac{r^3q^{3r}}{1-q^{3r}}-\left(\frac{4v}{5}\right)\frac{r^3q^{6r}}{1-q^{6r}}\Bigg] +u\Bigg[-2+24\sum_{1}^{\infty}\Bigg[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Bigg]\Bigg]^2 +v\Bigg[-1+24\sum_{1}^{\infty}\Bigg[\frac{2rq^{6r}}{1-q^{6r}}-\frac{rq^{3r}}{1-q^{3r}}\Bigg]\Bigg]^2 =\{3a(q^3)-2b(q)\}^2. \tag{16}$$

$$(v)\ \Big[1-4u-v\Big]+240\sum_{1}^{\infty}\Bigg[\left(\frac{1}{40}-\frac{2u}{5}\right)\frac{r^3q^r}{1-q^r} -\left(\frac{3}{20}\right)\frac{r^3q^{2r}}{1-q^{2r}}-\left(\frac{9}{40}+\frac{18u}{5}+\frac{v}{5}\right)\frac{r^3q^{3r}}{1-q^{3r}} +\left(\frac{27}{20}-\frac{4v}{5}\right)\frac{r^3q^{6r}}{1-q^{6r}}\Bigg]+u\Bigg[-2+24 \sum_{1}^{\infty}\Bigg[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Bigg]\Bigg]^2+v\Bigg[-1+24 \sum_{1}^{\infty}\Bigg[\frac{2rq^{6r}}{1-q^{6r}}-\frac{rq^{3r}}{1-q^{3r}}\Bigg]\Bigg]^2=a(q)a^3(q^2). \tag{17}$$

Proof:

$$C_1M(q)+C_2M(q^2)+C_3M(q^3)+C_4M(q^6) +C_5\{L(q)-3L(q^3)\}^2+C_6\{3L(q^3)-4L(q^4)\}^2 +C_7\{L(q^3)-2L(q^6)\}^2+C_8\{L(q^4)-3L(q^{12})\}^2 =a^3(q)a(q^2). \tag{18}$$

Subsequently upon (p,k) parametrization of the above expression using Lemma II.3 and by equating the coefficients of $k^4, pk^4, p^2k^4,\ p^3k^4, p^4k^4, p^5k^4, p^6k^4, p^7k^4$ and p^8k^4 on either sides, we obtain

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 4 & 1 & 1 & 4\\ 124 & 4 & 4 & 4 & 64 & 4 & 4 & 16\\ 964 & 64 & 4 & 4 & 400 & 4 & 4 & 16\\ 2788 & 178 & 28 & -2 & 1216 & 16 & 4 & -8\\ 3910 & 235 & 70 & -5 & 1816 & 28 & 10 & -14\\ 2788 & 178 & 28 & -2 & 1216 & -8 & 4 & 4\\ 964 & 64 & 4 & 4 & 400 & 64 & 4 & 4\\ 124 & 4 & 4 & 4 & 64 & -32 & 4 & -2\\ 1 & 1 & 1 & 1 & 4 & 4 & 1 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} C_1\\ C_2\\ C_3\\ C_4\\ C_5\\ C_6\\ C_7\\ C_8 \end{pmatrix} = \begin{pmatrix} 1\\ 13\\ 64\\ 151\\ 190\\ 151\\ 64\\ 13\\ 1 \end{pmatrix}.$$

We note that, the system results in a infinitely many solutions,

$$C_1=\left(\frac{3}{40}-\frac{2u}{5}\right),\ C_2=-\left(\frac{1}{5}\right), C_3=-\left(\frac{27}{40}+\frac{18u}{5}+\frac{v}{5}\right), C_4=\left(\frac{9}{5}-\frac{4v}{5}\right),\ C_5=u, C_6=0,\ C_7=v,\ C_8=0.$$

Substituting these values in (18) yields, (13). The remaining results are obtained by applying Definition II.1. By modifying the right-hand side of equation (13) and then employing (18), we derive equations (i) through (iv) as presented below.

$$\begin{aligned}
(i)\quad & -\left(\frac{2u}{5}\right)M(q)-\left(\frac{1}{80}\right)M(q^2)-\left(\frac{18u}{5}+\frac{v}{5}\right)M(q^3)\\
&+\left(\frac{81}{80}-\frac{4v}{5}\right)M(q^6)+u\{L(q)-3L(q^3)\}^2\\
&+v\{L(q^3)-2L(q^6)\}^2=a(q^2)b^3(q^2).\\
(ii)\quad & -\left(\frac{1}{80}+\frac{2u}{5}\right)M(q)+\left(\frac{81}{80}-\frac{18u}{5}-\frac{v}{5}\right)M(q^3)\\
&-\left(\frac{4v}{5}\right)M(q^6)+u\{L(q)-3L(q^3)\}^2+v\{L(q^3)\\
&-2L(q^6)\}^2=a(q)b^3(q).\\
(iii)\quad & \left(\frac{3}{40}-\frac{2u}{5}\right)M(q)-\left(\frac{1}{5}\right)M(q^2)-\left(\frac{27}{40}+\frac{18u}{5}+\frac{v}{5}\right)\\
&M(q^3)+\left(\frac{9}{5}-\frac{4v}{5}\right)M(q^6)+u\{L(q)-3L(q^3)\}^2\\
&+v\{L(q^3)-2L(q^6)\}^2=a^3(q)a(q^2).\\
(iv)\quad & \left(\frac{1}{40}-\frac{2u}{5}\right)M(q)-\left(\frac{3}{20}\right)M(q^2)-\left(\frac{9}{40}+\frac{18u}{5}\right.\\
&\left.+\frac{v}{5}\right)M(q^3)+\left(\frac{27}{20}-\frac{4v}{5}\right)M(q^6)+u\{L(q)\\
&-3L(q^3)\}^2+v\{L(q^3)-2L(q^6)\}^2=a(q)a^3(q^2).
\end{aligned}$$

Upon simplification of the aforementioned equations, we arrive at equations (14) through (17). ■

Theorem III.4. *The relation amongst an infinite series and theta functions holds:*

$$\begin{aligned}
(i)\quad & \Big[1-4u\Big]+240\sum_{1}^{\infty}\Bigg[-\left(\frac{1}{80}\right)\frac{r^3q^r}{1-q^r}-\left(\frac{2u}{5}\right)\\
&\frac{r^3q^{2r}}{1-q^{2r}}+\left(\frac{81}{80}\right)\frac{r^3q^{3r}}{1-q^{3r}}-\left(\frac{18u}{5}\right)\frac{r^3q^{6r}}{1-q^{6r}}\\
&+u\Bigg[-2+24\sum_{1}^{\infty}\Bigg[\frac{3rq^{6r}}{1-q^{6r}}-\frac{rq^{2r}}{1-q^{2r}}\Bigg]\Bigg]^2\\
&=a(q)b^3(q).
\end{aligned}\tag{19}$$

$$\begin{aligned}
(ii)\quad & \Big[1-4u\Big]+240\sum_{1}^{\infty}\Bigg[-\left(\frac{1}{80}+\frac{2u}{5}\right)\frac{r^3q^{2r}}{1-q^{2r}}+\left(\frac{81}{80}\right.\\
&\left.-\frac{18u}{5}\right)\frac{r^3q^{6r}}{1-q^{6r}}+u\Bigg[-2+24\sum_{1}^{\infty}\Bigg[\frac{3rq^{6r}}{1-q^{6r}}\\
&-\frac{rq^{2r}}{1-q^{2r}}\Bigg]\Bigg]^2=a(q^2)b^3(q^2).
\end{aligned}\tag{20}$$

$$\begin{aligned}
(iii)\quad & \Big[1-4u\Big]+240\sum_{1}^{\infty}\Bigg[\left(\frac{3}{40}\right)\frac{r^3q^r}{1-q^r}-\left(\frac{1}{5}\right.\\
&\left.+\frac{2u}{5}\right)\frac{r^3q^{2r}}{1-q^{2r}}-\left(\frac{27}{40}\right)\frac{r^3q^{3r}}{1-q^{3r}}+\left(\frac{9}{5}\right.\\
&\left.-\frac{18u}{5}\right)\frac{r^3q^{6r}}{1-q^{6r}}+u\Bigg[-2+24\sum_{1}^{\infty}\Bigg[\frac{3rq^{6r}}{1-q^{6r}}\\
&-\frac{rq^{2r}}{1-q^{2r}}\Bigg]\Bigg]^2=a^3(q)a(q^2).
\end{aligned}\tag{21}$$

$$\begin{aligned}
(iv)\quad & \Big[1-4u\Big]+240\sum_{1}^{\infty}\Bigg[\left(\frac{1}{10}\right)\frac{r^3q^r}{1-q^r}\\
&-\left(\frac{2u}{5}\right)\frac{r^3q^{2r}}{1-q^{2r}}+\left(\frac{9}{10}\right)\frac{r^3q^{3r}}{1-q^{3r}}\\
&-\left(\frac{18u}{5}\right)\frac{r^3q^{6r}}{1-q^{6r}}+u\Bigg[-2+24\sum_{1}^{\infty}\Bigg[\frac{3rq^{6r}}{1-q^{6r}}\\
&-\frac{rq^{2r}}{1-q^{2r}}\Bigg]\Bigg]^2=\{3a(q^3)-2b(q)\}^2.
\end{aligned}\tag{22}$$

$$\begin{aligned}
(v)\quad & \Big[1-4u\Big]+240\sum_{1}^{\infty}\Bigg[\left(\frac{1}{40}\right)\frac{r^3q^r}{1-q^r}-\left(\frac{3}{20}\right.\\
&\left.+\frac{2u}{5}\right)\frac{r^3q^{2r}}{1-q^{2r}}-\left(\frac{9}{40}\right)\frac{r^3q^{3r}}{1-q^{3r}}+\left(\frac{27}{20}\right.\\
&\left.-\frac{18u}{5}\right)\frac{r^3q^{6r}}{1-q^{6r}}+u\Bigg[-2+24\sum_{1}^{\infty}\Bigg[\frac{3rq^{6r}}{1-q^{6r}}\\
&-\frac{rq^{2r}}{1-q^{2r}}\Bigg]\Bigg]^2=a(q)a^3(q^2).
\end{aligned}\tag{23}$$

Proof: Let us persume that,

$$\begin{aligned}
&C_1M(q)+C_2M(q^2)+C_3M(q^3)+C_4M(q^6)+C_5M(q^{12})\\
&\quad+C_6\{L(q^2)-3L(q^6)\}^2+C_7\{3L(q^3)-4L(q^4)\}^2\\
&\quad+C_8\{L(q^4)-3L(q^{12})\}^2=a(q)b^3(q).
\end{aligned}\tag{24}$$

By employing the (p,k) parametrization described in Lemma II.3 on the preceding expression, and subsequently comparing the coefficients of k^4, pk^4, p^2k^4, p^3k^4, p^4k^4, p^5k^4, p^6k^4, p^7k^4, and p^8k^4 on both sides, we obtain

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 4 & 1 & 4\\
124 & 4 & 4 & 4 & 4 & 16 & 4 & 16\\
964 & 64 & 4 & 4 & 4 & 40 & 4 & 16\\
2788 & 178 & 28 & -2 & -2 & 64 & 16 & -8\\
3910 & 235 & 70 & -5 & -5 & 76 & 128 & -14\\
2788 & 178 & 28 & -2 & -2 & 64 & -8 & 4\\
964 & 64 & 4 & 4 & \frac{1}{4} & 40 & 64 & 4\\
124 & 4 & 4 & 4 & \frac{1}{4} & 16 & -32 & -2\\
1 & 1 & 1 & 1 & \frac{1}{16} & 4 & 4 & \frac{1}{4}
\end{pmatrix}
\begin{pmatrix}
C_1\\ C_2\\ C_3\\ C_4\\ C_5\\ C_6\\ C_7\\ C_8
\end{pmatrix}
=
\begin{pmatrix}
1\\ \frac{5}{2}\\ -8\\ -\frac{13}{2}\\ 22\\ -\frac{13}{2}\\ -8\\ -\frac{5}{2}\\ 1
\end{pmatrix}.$$

It is important to observe that the system yields an infinite number of solutions,

$$C_1 = -\frac{1}{80},\ C_2 = \frac{2u}{5},\ C_3 = \frac{81}{80},\ C_4 = -\frac{18u}{5},$$
$$C_5 = 0,\ C_6 = u,\ C_7 = 0,\ C_8 = 0.$$

Substituting these values in (24) yields, (19). The remaining outcomes are derived through the application of Definition II.1. By altering the right-hand side of equation (19) and utilizing (24), we derive equations (i) to (iv).

$$(i)\ -\left(\frac{1}{80}+\frac{2u}{5}\right)M(q^2)+\left(\frac{81}{80}-\frac{18u}{5}\right)M(q^6) + u\{L(q^2)-3L(q^6)\}^2 = a(q^2)b^3(q^2).$$

$$(ii)\ \left(\frac{3}{40}\right)M(q)-\left(\frac{1}{5}+\frac{2u}{5}\right)M(q^2)-\left(\frac{27}{40}\right)M(q^3) + \left(\frac{9}{5}-\frac{18u}{5}\right)M(q^6)+u\{L(q^2)-3L(q^6)\}^2 = a^3(q)a(q^2).$$

$$(iii)\ \left(\frac{1}{10}\right)M(q)-\left(\frac{2u}{5}\right)M(q^2)+\left(\frac{9}{10}\right)M(q^3) - \left(\frac{18u}{5}\right)M(q^6)+u\{L(q^2)-3L(q^6)\}^2 + = \{3a(q^3)-2b(q)\}^2.$$

$$(iv)\ \left(\frac{1}{40}\right)M(q)-\left(\frac{3}{20}+\frac{2u}{5}\right)M(q^2)-\left(\frac{9}{40}\right)M(q^3) + \left(\frac{27}{20}-\frac{18u}{5}\right)M(q^6)+u\{L(q^2)-3L(q^6)\}^2 = a(q)a^3(q^2).$$

After simplification of the aforementioned equations, equations (20) through (23) are obtained. ■

Theorem III.5. *The relation amongst an infinite series and theta functions holds:*

$$(i)\ \left(\frac{1}{2}-2u\right)+240\sum_{1}^{\infty}\Bigg[\left(-\frac{2}{5}+\frac{8u}{5}\right)\frac{r^3q^{2r}}{1-q^{2r}} + \left(\frac{9}{10}-\frac{18u}{5}\right)\frac{r^3q^{3r}}{1-q^{3r}}-\left(\frac{1}{2}-2u\right)\Bigg[-1 + 24\sum_{1}^{\infty}\left[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\right]\Bigg]^2+u\Bigg[-2 + 24\sum_{1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\right]\Bigg]^2 = \{3a(q^3)-2b(q)\}^2. \tag{25}$$

$$(ii)\ \left(\frac{17}{16}-2u\right)+240\sum_{1}^{\infty}\Bigg[\left(\frac{1}{20}+\frac{8u}{5}\right)\frac{r^3q^{2r}}{1-q^{2r}} + \left(\frac{81}{80}-\frac{18u}{5}\right)\frac{r^3q^{3r}}{1-q^{3r}}-\left(\frac{1}{16}-2u\right)\Bigg[-1 + 24\sum_{1}^{\infty}\left[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\right]\Bigg]^2+u\Bigg[-2 + 24\sum_{1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\right]\Bigg]^2 = a(q)b^3(q). \tag{26}$$

Proof: Let us persume that,

$$C_1M(q^2)+C_2M(q^3)+C_3\{L(q)-2L(q^2)\}^2 + C_4\{L(q)-3L(q^3)\}^2+C_5\{L(q)-4L(q^4)\}^2 + C_6\{3L(q^3)-4L(q^4)\}^2+C_7\{L(q)-6L(q^6)\}^2 + C_8\{L(q)-12L(q^{12})\}^2 = \{3a(q^3)-2b(q)\}^2. \tag{27}$$

Following the (p,k) parametrization of the expression above using Lemma II.3, and by equating the coefficients of$k^4, pk^4, p^2k^4,\ p^3k^4, p^4k^4, p^5k^4, p^6k^4, p^7k^4$ and p^8k^4 on both sides, we acquire

$$\begin{pmatrix} 1 & 1 & 1 & 4 & 9 & 1 & 25 & 121 \\ 4 & 4 & 28 & 64 & 108 & 4 & 220 & 748 \\ 64 & 4 & 244 & 400 & 540 & 4 & 844 & 1948 \\ 178 & 28 & 700 & 1216 & 1440 & 16 & 1804 & 2800 \\ 235 & 70 & 970 & 1816 & 2160 & 28 & 2314 & 2428 \\ 178 & 28 & 700 & 1216 & 1728 & -8 & 1804 & 1288 \\ 64 & 4 & 244 & 400 & 576 & 64 & 844 & 400 \\ 4 & 4 & 28 & 64 & 0 & -32 & 220 & 64 \\ 1 & 1 & 1 & 4 & 0 & 4 & 25 & 4 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{pmatrix} = \begin{pmatrix} 1 \\ 16 \\ 100 \\ 304 \\ 454 \\ 304 \\ 100 \\ 16 \\ 1 \end{pmatrix}.$$

It's worth noting that the system yields infinitely many solutions,

$$C_1 = -\left(\frac{2}{5}-\frac{8u}{5}\right),\ C_2 = \left(\frac{9}{10}-\frac{18u}{5}\right),$$
$$C_3 = \left(\frac{1}{2}-2u\right),\ C_4 = u,\ C_5 = 0,\ C_6 = 0,$$
$$C_7 = 0,\ C_8 = 0.$$

Substituting these values into (27) results in (25). By adjusting the right-hand side of (25) and applying (27), we obtain equation (i).

$$(i)\ \left(\frac{1}{20}+\frac{8u}{5}\right)M(q^2)+\left(\frac{81}{80}-\frac{18u}{5}\right)M(q^3)-\left(\frac{1}{16} + 2u\right)\{L(q)-2L(q^2)\}^2+u\{L(q)-3L(q^3)\}^2 = a(q)b^3(q).$$

After performing simplification operations on the equation mentioned above, a simpler form is achieved, identified as equation (26). ■

Theorem III.6. *The connection between an infinite series and theta functions persists:*

$$(i)\ \Big(1-4u\Big)+240\sum_{1}^{\infty}\Big[\Big(-\frac{1}{80}-\frac{2u}{5}\Big)\frac{r^3q^r}{1-q^r}+\Big(\frac{81}{80}-\frac{18u}{5}\Big)\frac{r^3q^{3r}}{1-q^{3r}}+u\Big[-2+24\sum_{1}^{\infty}\Big[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Big]\Big]^2=a(q)b^3(q). \tag{28}$$

$$(ii)\ \Big(1-4u\Big)+240\sum_{1}^{\infty}\Big[\Big(\frac{1}{10}-\frac{2u}{5}\Big)\frac{r^3q^r}{1-q^r}+\Big(\frac{9}{10}-\frac{18u}{5}\Big)\frac{r^3q^{3r}}{1-q^{3r}}+u\Big[-2+24\sum_{1}^{\infty}\Big[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Big]\Big]^2=\{3a(q^3)-2b(q)\}^2. \tag{29}$$

Proof: Let us persume that,

$$C_1M(q)+C_2M(q^3)+C_3\{L(q)-2L(q^2)\}^2+C_4\{L(q)-3L(q^3)\}^2+C_5\{L(q)-4L(q^4)\}^2+C_6\{3L(q^3)-4L(q^4)\}^2+C_7\{L(q)-6L(q^6)\}^2+C_8\{L(q)-12L(q^{12})\}^2=a(q)b^3(q). \tag{30}$$

Following the (p,k) parametrization of the expression above using Lemma II.3, and by equating the coefficients of$k^4,pk^4,p^2k^4,\ p^3k^4,p^4k^4,p^5k^4,p^6k^4,p^7k^4$ and p^8k^4 on both sides, we acquire

$$\begin{pmatrix} 1 & 1 & 1 & 4 & 9 & 1 & 25 & 121\\ 124 & 4 & 28 & 64 & 108 & 4 & 220 & 748\\ 964 & 4 & 244 & 400 & 540 & 4 & 844 & 1948\\ 2788 & 28 & 700 & 1216 & 1440 & 16 & 1804 & 2800\\ 3910 & 70 & 970 & 1816 & 2160 & 28 & 2314 & 2428\\ 2788 & 28 & 700 & 1216 & 1728 & -8 & 1804 & 1288\\ 964 & 4 & 244 & 400 & 576 & 64 & 844 & 400\\ 124 & 4 & 28 & 64 & 0 & -32 & 220 & 64\\ 1 & 1 & 1 & 4 & 0 & 4 & 25 & 4 \end{pmatrix}\begin{pmatrix} C_1\\ C_2\\ C_3\\ C_4\\ C_5\\ C_6\\ C_7\\ C_8 \end{pmatrix}=\begin{pmatrix} 1\\ \frac{5}{2}\\ -8\\ -\frac{13}{2}\\ 22\\ -\frac{13}{2}\\ -8\\ -\frac{5}{2}\\ 1 \end{pmatrix}.$$

We observe that the system yields an infinite number of solutions,

$$C_1=-\Big(\frac{1}{80}+\frac{2u}{5}\Big),\ C_2=\Big(\frac{81}{80}-\frac{18u}{5}\Big),\ C_3=0,\ C_4=u,\ C_5=0,\ C_6=0,\ C_7=0,\ C_8=0.$$

Upon substituting these values into (30), we derive (28). The subsequent results are derived by applying Definition II.1. By adjusting the right-hand side of (28) and utilizing (30), we arrive at the following equation.

$$(i)\ \Big(\frac{1}{10}-\frac{2u}{5}\Big)M(q)+\Big(\frac{9}{10}-\frac{18u}{5}\Big)M(q^3)+u\{L(q)-3L(q^3)\}^2=\{3a(q^3)-2b(q)\}^2.$$

On simplyfing the above equation, we get equation (29). ∎

Theorem III.7. *The relation amongst an infinite series and theta functions holds:*

$$(i)\ -\Big(\frac{1}{8}+\frac{u}{5}\Big)+240\sum_{1}^{\infty}\Big[\Big(-\frac{1}{8}-\frac{u}{5}\Big)\frac{r^3q^r}{1-q^r}-\Big(\frac{4u}{5}\Big)\frac{r^3q^{2r}}{1-q^{2r}}+u\Big[-1+24\sum_{1}^{\infty}\Big[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\Big]\Big]^2+\frac{9}{32}\Big[-2+24\sum_{1}^{\infty}\Big[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Big]\Big]^2=a(q)b^3(q). \tag{31}$$

$$(ii)\ -\Big(\frac{1}{5}+\frac{4u}{5}\Big)+240\sum_{1}^{\infty}\Big[\Big(-\frac{1}{5}\Big)\frac{r^3q^r}{1-q^r}-\Big(\frac{4u}{5}\Big)\frac{r^3q^{2r}}{1-q^{2r}}+u\Big[-1+24\sum_{1}^{\infty}\Big[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\Big]\Big]^2+\frac{1}{4}\Big[-2+24\sum_{1}^{\infty}\Big[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Big]\Big]^2=\{3a(q^3)-2b(q)\}^2. \tag{32}$$

Proof: Let us persume that,

$$C_1M(q)+C_2M(q^2)+C_3\{L(q)-2L(q^2)\}^2+C_4\{L(q)-3L(q^3)\}^2+C_5\{L(q)-4L(q^4)\}^2+C_6\{3L(q^3)-4L(q^4)\}^2+C_7\{L(q)-6L(q^6)\}^2+C_8\{L(q)-12L(q^{12})\}^2=\{3a(q^3)-2b(q)\}^2. \tag{33}$$

Following the (p,k) parametrization of the expression above using Lemma II.3, and by equating the coefficients of$k^4,pk^4,p^2k^4,\ p^3k^4,p^4k^4,p^5k^4,p^6k^4,p^7k^4$ and p^8k^4 on both sides, we acquire

$$\begin{pmatrix} 1 & 1 & 1 & 4 & 9 & 1 & 25 & 121 \\ 124 & 4 & 28 & 64 & 108 & 4 & 220 & 748 \\ 964 & 64 & 244 & 400 & 540 & 4 & 844 & 1948 \\ 2788 & 178 & 700 & 1216 & 1440 & 16 & 1804 & 2800 \\ 3910 & 235 & 970 & 1816 & 2160 & 28 & 2314 & 2428 \\ 2788 & 178 & 700 & 1216 & 1728 & -8 & 1804 & 1288 \\ 964 & 64 & 244 & 400 & 576 & 64 & 844 & 400 \\ 124 & 4 & 28 & 64 & 0 & -32 & 220 & 64 \\ 1 & 1 & 1 & 4 & 0 & 4 & 25 & 4 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{5}{2} \\ -8 \\ -\frac{13}{2} \\ 22 \\ -\frac{13}{2} \\ -8 \\ -\frac{5}{2} \\ 1 \end{pmatrix}.$$

It's worth noting that the system yields infinitely many solutions.

$$C_1 = -\left(\frac{1}{8} + \frac{u}{5}\right),\ C_2 = -\left(\frac{4u}{5}\right),\ C_3 = u,$$
$$C_4 = \frac{9}{32},\ C_5 = 0,\ C_6 = 0,\ C_7 = 0,\ C_8 = 0.$$

Upon substituting these values into (33), we obtain (31). The rest of the outcomes follow from the application of Definition II.1. By adjusting the right-hand side of (31) and utilizing (33), we derive the following equation.

$$(i)\ \left(-\frac{u}{5}\right)M(q) - \left(\frac{4u}{5}\right)M(q^2) + u\{L(q) - 2L(q^2)\}^2 + \frac{1}{4}\{L(q) - 3L(q^3)\}^2 = \{3a(q^3) - 2b(q)\}^2.$$

Simplifying the preceding equation yields (32). ■

REFERENCES

[1] A. Alaca, S. Alaca, and K. S. Williams, “On the two-dimensional theta functions of the Borweins”, Acta Arithmetica, vol. 124, no. 2, pp177-195, 2006

[2] B. C. Berndt, Ramanujan‘s Notebooks, Part III, Springer, New York, 1991

[3] J.M. Borwein and P.B. Borwein, “A cubic counterpart of Jacobi‘s identity and the AGM”, Transactions of the American Mathematical Society, vol. 323, no. 2, pp691-701, 1991

[4] J.M. Borwein, P.B. Browein, F.G. Garvan, “Some cubic modular identities of Ramanujan.” Transactions of the American Mathematical Society, vol. 343, no. 1, pp35-47, 1994

[5] Harekala Chandrashekara Vidya, and Badanidiyoor Ashwath Rao,“Intriguing Relationships among Eisenstein Series, Borewein’s Cubic Theta Functions, and the Class One Infinite Series,” IAENG International Journal of Computer Science, vol. 50, no.4, pp1166-1173, 2023

[6] S. Ramanujan, Notebooks, Vols. 1, 2, Tata Institute of Fundamental Research, Mumbai, India, 1957

[7] Shruti and B. R. Srivatsa Kumar, “Some new Eisenstein series containing the Borweins’ cubic theta functions and convolution sum $\sum_{i+4j=n} \sigma(i)\sigma(j)$”, Afrika Matematika, vol. 33, pp971-982, 2020

[8] H. C. Vidya and B Ashwath Rao, “Few more relation connecting Ramanujan type Eisenstein series and cubic theta functions of Borwein”, Proceedings of the Jangjeon Mathematical society, vol. 25, no. 2, pp253-263, 2022

[9] Vidya Harekala Chandrashekara and Smitha Ganesh Bhat, “Formulating Few Key Identities Relating Infinite Series of Eisenstein and Borweins’ Cubic Theta Functions”, Engineering Letters, vol. 31, no. 4, pp1383–1394, 2023

[10] E. X. W. Xia and O. X. M. Yao, “Eisenstein series identities involving the Borweins’ cubic theta functions”, Journal of Applied Mathematics, 2012, Article ID 181264.