Practical Exponential Stability of Uncertain Nonlinear Delayed Urban Traffic Systems

Zhuan Liu, Peng Gao and Chao Wei

Abstract—**Practical stability can describe qualitative behavior and quantitative properties of systems in comparison with traditional Lyapunov stability theory. In this paper, such stability problem is formulated for nonlinear delayed urban traffic systems described by uncertain differential equations which are a type of differential equations driven by Liu processes. First of all, we prove the existence and uniqueness of the solution. Then, we analyze the pth moment practically exponential stability and quasi surely globally practically uniformly exponential stability of the system by employing general Itô formula, Gronwall's inequality, Hölder inequality and Borel-Cantelli lemma. Moreover, an example is presented to verify the validity of our theoretical methods.**

Index Terms—**Nonlinear delayed urban traffic systems; uncertain differential equations; practical exponential stability; existence and uniqueness; Liu process**

I. Introduction

In recent years, the intensification of global climate change has led to frequent rainstorms, which have caused huge economic losses and casualties in many Chinese cities. Urban road traffic system, as an important part of urban infrastructure, is vital to ensure the daily travel of citizens and rapid recovery after disasters. Therefore, on the basis of accurately quantifying the existing urban road traffic system's ability to cope with rainstorm weather, exploring its improvement path has great theoretical and practical significance for improving the urban system's ability to cope with natural disasters. At the same time, with the rapid development of intelligent transportation systems, the field of transportation research has made remarkable progress in data acquisition. Many scholars have studied the urban traffic systems. For example, Gu et al. ([11]) analyzed the similarities and differences among the three concepts of reliability, vulnerability and resilience in traffic networks. Zhou et al. ([35]) believed that the transportation system toughness was mostly defined from two aspects: the ability of the system to maintain its own function in the face of disturbance and the time and resources required for the system to recover to the original state after disturbance. Goncalves and Ribeiro ([10]) divided the toughness of urban transportation system into static and dynamic resiliency. Akbari et al. ([3]) given the corresponding definition of toughness for road transportation system.

In practical problems, it is difficult to apply the general theory to build models because of some emergencies. Liu ([17]) created the uncertainty theory to address this uncertainty. Then, Liu ([18]) improved the uncertainty theory and put forward the Liu process. Liu process is the uncertain process which deals with dynamic systems in uncertain phenomenon. Meanwhile, the systems driven by Liu process have been studied by many scholars. For example, Abdar et al. ([1]) proposed different uncertainty quantification methods. Deng et al. ([8]) discussed a new distributed event-triggered observer for uncertain nonlinear multiagent systems. For uncertain systems, Dong et al. ([9]) investigated the cooperative output regulation problem. Kohler et al. ([14]) presented a tube-based framework for systems with uncertainty parameter. Liu et al. ([20]) used uncertain measurement to study the sliding mode control problem. Yang et al. ([30]) designed the MIMO uncertainty systems.

It is worth pointing out that the time delay is always unavoidable due to the uncertain communication environment. Therefore, it is required to be taken into consideration for stochastic systems. To deal with the Partial-Nodes-Based state estimation problem for time-varying delayed complex networks, Liu et al. ([19]) established a novel framework via the Lyapunov stability theory. Wang et al. ([26]) designed the stiffness nonlinearities, asymmetric smooth and discontinuous oscillator under time-delayed feedback control. Yu et al. ([32]) investigated the influence of bounded noise and time delay on the sub-threshold signal transmission in FitzHugh-Nagumo neuronal networks. Zhang and Fridman ([33]) proposed an improved time-delay method and extended it to L_2-gain analysis. The stability of fractional-order time-delayed systems was considered in ([21], [24], [34]).

Because nonlinear characteristic of the systems make the property of the systems more complex, it is difficult to analyze the stability of systems. Therefore, many scholars studied the stability of the systems ([5], [6], [12], [16], [27]–[29]). Sucec ([25]) introduced the practical stability in 1987. Then, the practical stability has been widely used in fields of control. Compared with traditional Lyapunov stability theory, practical stability could describe quantitative properties and behavior. Many scholars studied the practical stability theory ([2], [13], [22]). For instance, Ben Makhlouf ([4]) investigated the stability of nonlinear fractional systems. Deghat et al. ([7]) analyzed the singularly perturbed nonlinear systems which have boundary-layer solutions. Li and Zhao ([15]) considered the practical stability for delayed positive systems. Platonov ([23]) considered the nonlinear stochastic systems which satisfies sector constraints. Yao et al. ([31]) discussed practical partially stability for delayed

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systems by using Lyapunov-Razumikhin method.

However, lots of systems considered in the literature were driven by Brownian motion. Since uncertain systems model the time evolution of a dynamic system with uncertain influences and have been widely used in the fields of financial markets, it is of great important to explore the stability of uncertain systems. Motivated by the above considerations, in the present paper, we deals with the practical exponential stability of uncertain nonlinear delayed systems. The main contributions and novelties of this article are summarized as follows:

(1) Different from existing literature about practical exponential stability for Itô stochastic systems, this paper considers uncertain influences and study the practical exponential stability for uncertain nonlinear time-delay systems.

(2) This paper derive the existence and uniqueness of the solution under linear growth condition and local Lipschitz condition by employing general Itô formula, Cauchy-Schwarz inequality and stochastic analysis.

(3) Combining the uncertain theory to the practical stability theory, this paper provides some sufficient conditions which guarantee pth moment practically exponential stability and quasi surely globally practically uniformly exponential stability of the system. At last, an example is presented to verify the validity of our theoretical methods.

The rest of this paper is organized as follows. The uncertain nonlinear delayed systems and some definitions are given in Section 2. The existence and uniqueness, pth moment practically exponential stability and quasi surely globally practically uniformly exponential stability of the system are studied In Section 3. An example is provided In Section 4. The conclusion is given in Section 5.

II. Problem Formulation and Preliminaries

Firstly, we give some definitions about uncertain variables and Liu process.

Definition 1: ([17], [18]) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}:\mathcal{L}\to[0,1]$ is called an uncertain measure if it satisfies the following axioms:
Axiom 1: (Normality Axiom) $\mathcal{M}(\Gamma)=1$ for the universal set Γ.
Axiom 2: (Duality Axiom) $\mathcal{M}(\Lambda)+\mathcal{M}(\Lambda^c)=1$ for any event Λ.
Axiom 3: (Subadditivity Axiom) For every countable sequence of events Λ_1, Λ_2, $\cdots$,

$$\mathcal{M}\{\bigcup_{i=1}^{\infty}\Lambda_i\}\le\sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

Axiom 4: (Product Axiom) Let $(\Gamma_k,\mathcal{L}_k,\mathcal{M}_k)$ be uncertainty spaces for $k=1,2,\cdots$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\{\Pi_{k=1}^{\infty}\Lambda_k\}=\min_{k\ge1}\mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k=1,2,\cdots$.

An uncertain variable ξ is a measurable function from the uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to the set of real numbers.

Definition 2: ([17]) For any real number x, let ξ be an uncertain variable and its uncertainty distribution is defined by

$$\Phi(x)=\mathcal{M}(\xi\le x).$$

In particular, an uncertain variable ξ is called normal if it has an uncertainty distribution

$$\Phi(x)=(1+\exp(\frac{\pi(\mu-x)}{\sqrt{3}\sigma}))^{-1},x\in\Re,$$

denoted by $\mathcal{N}(\mu,\sigma)$. If $\mu=0$, $\sigma=1$, ξ is called a standard normal uncertain variable.

Definition 3: ([18]) An uncertain process C_t is called a Liu process if
(i) $C_0=0$ and almost all sample paths are Lipschitz continuous, (ii) C_t has stationary and independent increments, (iii) the increment $C_{s+t}-C_s$ has a normal uncertainty distribution

$$\Phi_t(x)=(1+\exp(\frac{-\pi x}{\sqrt{3}t}))^{-1},x\in\Re.$$

Considering the uncertain nonlinear delayed systems as follows:

$$\begin{aligned} dv(t) &= h_1(v(t),v(t-\tau(t)),t)dt\\ &\quad+h_2(v(t),v(t-\tau(t)),t)dC(t), \end{aligned}\qquad(1)$$

where $\tau(t)\in[0,\tau]$, $\varsigma(t)_{-\tau\le t\le0}=\varsigma\in\mathcal{C}([-\tau,0];\mathbb{R}^n)$ is the nonrandom initial data, $C(t)$ is a one dimensional Liu process. $h_1:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+\to\mathbb{R}^n$, $h_2:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+\to\mathbb{R}^n$.

Let $V(v,v_1,t)\ge0$ be the real-valued functions and $V(v,v_1,t)\in\mathcal{C}^{1,2}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+;\mathbb{R}_+)$. Define the operator $\mathcal{L}V$ as follows:

$$\begin{aligned} &\mathcal{L}V(v,v_1,t)\\ =\;&V_t(v,v_1,t)+V_v(v,v_1,t)h_1(v,v_1,t)\\ &+\frac{1}{2}h_2^T(v,v_1,t)V_{vv}(v,v_1,t)h_2(v,v_1,t), \end{aligned}$$

$$\begin{aligned} dV(v,v_1,t) &= \mathcal{L}V(v,v_1,t)dt\\ &\quad+V_v(v,v_1,t)h_1(v,v_1,t)dC(t). \end{aligned}$$

Assumption 1: $\forall t\ge0$, $|v|\vee|v^{'}|\vee|v_1|\vee|v_1^{'}|\le n$,

$$\begin{aligned} &|h_1(v,v_1,t)-h_1(v^{'},v_1^{'},t)|\vee|h_2(v,v_1,t)-h_2(v^{'},v_1^{'},t)|\\ &\le K_n(|v-v^{'}|+|v_1-v_1^{'}|), \end{aligned}$$

where $K_n>0$ is a constant and $n\ge1$.

Assumption 2:

$$|h_1(v,v_1,t)|\vee|h_2(v,v_1,t)|\le K(1+|v|+|v_1|),$$

where $K>0$, $t\ge0$ and $v,v_1\in\mathbb{R}^n$.

Assumption 3:

$$h_1(0,t)\equiv0,\quad h_2(0,t)\equiv0.$$

Assumption 4:

$$\lim_{|v|\to\infty}\inf_{t\ge0}V(v,v_1,t)=\infty,\quad\mathcal{L}V(v,v_1,t)\le-c_1V(v,v_1,t),$$

where $c_1>0$.

Definition 4: ([36]) If there exist constants $C>0$, $\lambda>0$ and $\alpha>0$ satisfy

$$\mathbb{E}|v(t,t_0,v_0)|^p\le C|v_0|^pe^{-\lambda(t-t_0)}+\alpha,\quad t\ge t_0,$$

for all $v_0\in\mathbb{R}^n$, the system (1) is pth moment practically exponentially stable. When $p=2$, it is practically exponentially stable in mean square.

Definition 5: ([36]) If there exists a constant $\beta > 0$ satisfy $|v(t,t_0,v_0)| - \beta > 0$ and

$$\lim_{t\to\infty}\sup \frac{1}{t}\log(|v(t,t_0,v_0)| - \beta) < 0,$$

for all $v_0 \in \mathbb{R}^n$, the system (1) is quasi surely globally practically uniformly exponentially stable.

III. Main Results and Proofs

Theorem 1: Under the conditions 1-4, the global solution of system (1) is existent and unique.

Proof: Let $|v_0| \le \xi$. For $n \ge \xi$, $n \in \mathbb{N}$, define the truncation function

$$h_{1_n}(v,v_1,t) = \begin{cases} h_1(v,v_1,t) & if|v|,|v_1| \le n, \\ h_1(\frac{nv}{|v|},\frac{nv_1}{|v_1|},t) & if|v|,|v_1| > n, \end{cases} \quad (2)$$

$$h_{2_n}(v,v_1,t) = \begin{cases} h_2(v,v_1,t) & if|v|,|v_1| \le n, \\ h_2(\frac{nv}{|v|},\frac{nv_1}{|v_1|},t) & if|v|,|v_1| > n, \end{cases} \quad (3)$$

Then, the following system satisfies the linear growth condition and local Lipschitz condition:

$$\begin{aligned} dv_n(t) &= h_{1_n}(v_n(t),v_n(t-\tau(t)),t)dt \\ &+h_{2_n}(v_n(t),v_n(t-\tau(t)),t)dC(t). \end{aligned} \quad (4)$$

Hence, the global solution of system (4) is existent and unique.

Let

$$\varpi_n = \inf\{t \ge 0 : |v_n(t)| \ge n\} \quad n \in \mathbb{N}, \quad (5)$$

where $\inf \varpi = \infty$.

When $0 \le t \le \varpi_n$, $v_n(t) = v_{n+1}$. This implies that $\{\varpi_n\}$ is increasing. Then, $\exists \varpi$ satisfies

$$\lim_{n\to\infty} \varpi_n = \varpi. \quad (6)$$

Define

$$\lim_{n\to\infty} v_n(t) = v(t), \quad -\tau \le t < \varpi. \quad (7)$$

It is easy to check that $v(t)$ is the unique solution of system (1).

For $t \ge 0$, by applying general Itô formula, it follows that

$$\begin{aligned} &V(v_n(t\wedge\varpi_n),v_n(t\wedge\varpi_n-\tau(t\wedge\varpi_n)),t\wedge\varpi_n)) \\ &= V(\varsigma(0),v_n(-\tau(0)),0) \\ &+\int_0^{t\wedge\varpi_n} \mathcal{L}_nV(v_n(s),v_n(s-\tau(s)),s)ds \end{aligned}$$

where $\mathcal{L}_nV(v_n(s),v_n(s-\tau(s)),s) = \mathcal{L}V(v_n(s),v_n(s-\tau(s)),s)$, $0 \le s \le t\wedge\varpi_n$.

Hence,

$$\begin{aligned} &\mathbb{E}[V(v_n(t\wedge\varpi_n),v_n(t\wedge\varpi_n-\tau(t\wedge\varpi_n)),t\wedge\varpi_n)] \\ &\le \mathbb{E}[V(\varsigma(0),v_n(-\tau(0)),0)] \\ &+\mathbb{E}[\int_0^{t\wedge\varpi_n} \mathcal{L}_nV(v_n(s),v_n(s-\tau(s)),s)ds] \\ &\le \mathbb{E}[V(\varsigma(0),v_n(-\tau(0)),0)] \\ &+\int_0^{t\wedge\varpi_n} \mathbb{E}[V(v_n(s),v_n(s-\tau(s)),s)]ds. \end{aligned}$$

Then,

$$\begin{aligned} &\mathbb{E}[V(v_n(t\wedge\varpi_n),v_n(t\wedge\varpi_n-\tau(t\wedge\varpi_n)),t\wedge\varpi_n)] \\ &\le \mathbb{E}[V(\varsigma(0),v_n(-\tau(0)),0)]e^{(t\wedge\varpi_n)}. \end{aligned} \quad (8)$$

Since

$$\begin{aligned} &\mathbb{P}\{\varpi_n \le t\} \inf_{|v|\ge n,|v_1|\ge n,t\ge 0} V(v,v_1,t) \\ &\le \int_{\varpi_n\le t} V(v_n(t\wedge\varpi_n), \\ &v_n(t\wedge\varpi_n-\tau(t\wedge\varpi_n)),t\wedge\varpi_n)dP \\ &\le \mathbb{E}V(v_n(t\wedge\varpi_n),v_n(t\wedge\varpi_n-\tau(t\wedge\varpi_n)),t\wedge\varpi_n), \end{aligned}$$

we have

$$\mathbb{P}\{\varpi_n \le t\} \le \frac{\mathbb{E}[V(\varsigma(0),v_n(-\tau(0)),0)]e^{(t\wedge\varpi_n)}}{\inf_{|v|\ge n,|v_1|\ge n,t\ge 0} V(v,v_1,t)}. \quad (9)$$

$$\mathbb{P}\{\varpi \le t\} = 0, \quad t\to\infty. \quad (10)$$

Therefore,

$$\mathbb{P}\{\varpi = \infty\} = 1. \quad (11)$$

■

The proof is complete.

Theorem 2: For $\forall(v,v_1,t) \in (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+)$,

$$a_1|v|^p \le V(v,v_1,t) \le a_2|v|^p + a_3, \quad (12)$$

$$\mathcal{L}V(v,v_1,t) \le -c_1V(v,v_1,t), \quad (13)$$

where a_1, a_2, a_3, c_1 are positive constants, the system (1) is pth moment practically exponentially stable.

Proof: Fix $\forall v_0 \in \mathbb{R}^n$, for each $n \ge |v_0|$ and $t > 0$, by using general Itô formula, we have

$$\begin{aligned} &e^{c_1(t\wedge\varpi_n)}V(v(t\wedge\varpi_n),v(t\wedge\varpi_n-\tau(t\wedge\varpi_n)),t\wedge\varpi_n) \\ &= V(v_0,v(-\tau(0),0) \\ &+\int_0^{t\wedge\varpi_n} c_1e^{c_1s}V(v(s),v(s-\tau(s)),s)ds \\ &+\int_0^{t\wedge\varpi_n} e^{c_1s}\mathcal{L}V(v(s),v(s-\tau(s)),s)ds \\ &+\int_0^{t\wedge\varpi_n} e^{c_1s}V_v(v(s),v(s-\tau(s)) \\ &\times h_2(v(s),v(s-\tau(s))dC(s). \end{aligned} \quad (14)$$

Since

$$\begin{aligned} &\mathbb{E}[\int_0^{t\wedge\varpi_n} e^{c_1s}V_v(v(s),v(s-\tau(s)) \\ &\times h_2(v(s),v(s-\tau(s))dC(s)] = 0, \end{aligned} \quad (15)$$

we have

$$\begin{aligned} &\mathbb{E}[e^{c_1(t\wedge\varpi_n)} \\ &\times V(v(t\wedge\varpi_n),v(t\wedge\varpi_n-\tau(t\wedge\varpi_n)),t\wedge\varpi_n)] \\ &= V(v_0,v(-\tau(0),0) \\ &+\mathbb{E}\int_0^{t\wedge\varpi_n} e^{c_1s}[c_1V(v(s),v(s-\tau(s)),s) \\ &+\mathcal{L}V(v(s),v(s-\tau(s)),s)]ds. \end{aligned} \quad (16)$$

Hence, we obtain that

$$\begin{aligned} &\mathbb{E}[V(v(t\wedge\varpi_n),v(t\wedge\varpi_n-\tau(t\wedge\varpi_n)),t\wedge\varpi_n)] \\ &\le e^{-c_1(t\wedge\varpi_n)}V(v_0,v(-\tau(0),0) \\ &\le e^{-c_1(t\wedge\varpi_n)}a_2|v_0|^p + a_3. \end{aligned} \quad (17)$$

Then,

$$a_1\mathbb{E}|v(t\wedge\varpi_n)|^p \leq e^{-c_1(t\wedge\varpi_n)}a_2|v_0|^p + a_3. \quad (18)$$

Letting $n\to\infty$ yields that

$$a_1\mathbb{E}|v(t)|^p \leq e^{-c_1 t}a_2|v_0|^p + a_3. \quad (19)$$

Hence,

$$\mathbb{E}|v(t)|^p \leq \frac{a_2}{a_1}e^{-c_1 t}|v_0|^p + \frac{a_3}{a_1}. \quad (20)$$

■

The proof is complete.

Theorem 3: Under the conditions in Theorem 3.2, the system (1) is quasi surely globally practically uniformly exponentially stable.

Proof: According to Theorem 2, when $p=2$,

$$\mathbb{E}|v(t)|^2 \leq \frac{a_2}{a_1}e^{-c_1 t}|v_0|^2 + \frac{a_3}{a_1}. \quad (21)$$

Then, we have

$$\begin{aligned}
&|v(t)|^2\\
&= |v_0 + \int_0^t h_1(v(s), v(s-\tau(s)), s)ds\\
&+\int_0^t h_2(v(s), v(s-\tau(s)), s)dC(s)|^2\\
&\leq 3|v_0|^2 + 3|\int_0^t h_1(v(s), v(s-\tau(s)), s)ds|^2\\
&+3|\int_0^t h_2(v(s), v(s-\tau(s)), s)dC(s)|^2.
\end{aligned}$$

For positive integer κ_0, let $\kappa=\kappa_0, \kappa_0+1, \kappa_0+2, \cdots$ and $\forall m>0$ satisfies $m^2K^2(2(\kappa+1)K^2(\gamma)+1) < \frac{1}{12}$ where $K(\gamma)$ is the Lipschitz constant of the sample path $v(\gamma)$. Then, we have

$$\begin{aligned}
&\mathbb{E}[\sup_{\kappa m\leq t\leq(\kappa+1)m}|v(t)|^2]\\
&\leq 3\mathbb{E}[|v(\kappa m)|^2]\\
&+3\mathbb{E}(\int_{\kappa m}^{(\kappa+1)m}|h_1(v(s), v(s-\tau(s)), s)|ds)^2\\
&+3\mathbb{E}|\int_0^t h_2(v(s), v(s-\tau(s)), s)dC(s)|^2\\
&\leq 3\mathbb{E}[|v(\kappa m)|^2]\\
&+3\mathbb{E}(m\sup_{\kappa m\leq s\leq(\kappa+1)m}|h_1(v(s), v(s-\tau(s)), s)|)^2\\
&+3\mathbb{E}[\sup_{\kappa m\leq s\leq(\kappa+1)m}\\
&|\int_{\kappa m}^{(\kappa+1)m} h_2(v(s), v(s-\tau(s)), s)dC(s)|^2]\\
&\leq 3\mathbb{E}[|v(\kappa m)|^2] + 6m^2K^2\\
&+12m^2K^2\mathbb{E}[\sup_{\kappa m\leq s\leq(\kappa+1)m}|v(s)|^2]\\
&+12K^2K^2(\gamma)(\kappa+1)m^2\\
&+24K^2K^2(\gamma)(\kappa+1)m^2\mathbb{E}[\sup_{\kappa m\leq s\leq(\kappa+1)m}|v(s)|^2]\\
&\leq 3\frac{a_2}{a_1}|v_0|^2e^{-c_1\kappa m} + \frac{a_3}{a_1}\\
&+6m^2K^2(2(\kappa+1)K^2(\gamma)+1)\\
&+12m^2K^2(2(\kappa+1)K^2(\gamma)+1)\\
&\mathbb{E}[\sup_{\kappa m\leq s\leq(\kappa+1)m}|v(s)|^2].
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E}[\sup_{\kappa m\leq t\leq(\kappa+1)m}|v(t)|^2]\\
&\leq \frac{3\frac{a_2}{a_1}|v_0|^2e^{-c_1\kappa m} + \frac{a_3}{a_1} + 6m^2K^2(2(\kappa+1)K^2(\gamma)+1)}{1-12m^2K^2(2(\kappa+1)K^2(\gamma)+1)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{P}(\omega: \sup_{\kappa m\leq t\leq(\kappa+1)m}|v(t)|\\
&-\frac{\frac{a_3}{a_1} + 6m^2K^2(2(\kappa+1)K^2(\gamma)+1)}{1-12m^2K^2(2(\kappa+1)K^2(\gamma)+1)} > 3\frac{a_2}{a_1}|v_0|^2e^{\frac{-c_1\kappa m}{2}})\\
&\leq e^{\frac{-c_1\kappa m}{2}}.
\end{aligned}$$

From the Borel-Cantelli lemma,

$$\begin{aligned}
&\sup_{\kappa m\leq t\leq(\kappa+1)m}|v(t)| - \frac{\frac{a_3}{a_1} + 6m^2K^2(2(\kappa+1)K^2(\gamma)+1)}{1-12m^2K^2(2(\kappa+1)K^2(\gamma)+1)}\\
&\leq 3\frac{a_2}{a_1}|v_0|^2e^{\frac{-c_1\kappa m}{2}}.
\end{aligned}$$

Thus, for $\kappa m\leq t\leq(\kappa+1)m$, we have

$$\begin{aligned}
&\lim_{t\to\infty}\sup\frac{1}{t}\log(|v(t)|\\
&-\frac{\frac{a_3}{a_1} + 6m^2K^2(2(\kappa+1)K^2(\gamma)+1)}{1-12m^2K^2(2(\kappa+1)K^2(\gamma)+1)})\\
&< -\frac{c_1}{2}\\
&< 0.
\end{aligned}$$

The proof is complete. ■

IV. Example

Consider the scalar uncertain nonlinear delayed systems as follows:

$$\begin{aligned}
dv(t) &= h_1(v(t), v(t-\tau(t)), t)dt\\
&+h_2(v(t), v(t-\tau(t)), t)dC(t),
\end{aligned}$$

where

$$h_1(v(t), v(t-\tau(t)), t) = -8v(t) + v(t-\tau(t))sin(t),$$

$$h_2(v(t), v(t-\tau(t)), t) = 3v(t)sin(t)$$

$$\tau(t) = 1 + 0.3cos(t).$$

Hence, $\tau=1.3$. Let $V(v, v_1, t) = v^2$, $p=2$. Therefore,

$$\mathcal{L}V(v, v_1, t) \leq -3v^2.$$

Then, the systems are pth moment practically exponentially stable and quasi surely globally practically uniformly exponentially stable.

Remark 1: If the time delay in systems is given, we can consider the following example:

$$\begin{aligned}
dv(t) &= h_1(v(t), v(t-1), t)dt\\
&+h_2(v(t), v(t-1), t)dC(t),
\end{aligned}$$

where

$$h_1(v(t), v(t-1), t) = -\frac{1}{1+t^2}v(t) + \frac{1}{1+t^4}v(t-1),$$

$$h_2(v(t), v(t-1, t) = \frac{1}{1+t^6}v(t),$$

$$\tau(t) = 1.$$

Let $V(v, v_1, t) = v^2$, $p = 2$. Therefore,

$$\mathcal{L}V(v, v_1, t) \leq -2v^2.$$

Then, the systems are practically exponentially stable in mean and quasi surely globally practically uniformly exponentially stable.

V. Conclusion

This paper has investigated the practical exponential stability of uncertain delayed nonlinear systems. The global solution has been proved existent and unique. The pth moment practically exponential stability and quasi surely globally practically uniformly exponential stability of the system have been derived by applying general Itô formula, Gronwall's inequality, Hölder inequality, Chebyshev inequality and Borel-Cantelli lemma. In future works, we will consider the practical partial stability of uncertain systems.

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