On Borewein's Cubic Theta Functions and h-Functions

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Abstract—**On page 188 of his lost notebook, Ramanujan introduced distinct classes of exquisite infinite series, representing them in terms of Eisenstein series. The objective of this paper is to develop various differential identities related to Borwein's cubic theta functions and h-functions. Also, we present certain identities containing Eisenstein series of level 6 and h-functions that are used to establish relations involving class one infinite series and h-functions. Additionally, we present a straightforward approach to evaluate a discrete convolution sum by employing relationships involving Eisenstein series and h-functions.**

Index Terms—**Eisenstein series, Dedekind η-function, Cubic theta functions, Convolution Sum.**

I. Introduction

Convolution can be employed in any scientific discipline that necessitates the computation of mathematical data through multiplication followed by accumulation. Convolution is frequently used in the field of numerical analysis and numerical linear algebra, and in the design and implementation of finite impulse response filters in signal processing. Convolution may also be applicable in communication systems and forms the theoretical basis of Digital Signal processing. For the determination of convolution sums, mathematicians most frequently use Ramanujan's discriminant function, Gaussian hypergeometric series, quasimodular forms, Ramanujan-type Eisenstein series and many more. In this article, we employ Ramanujan-type Eisenstein series relations to evaluate a convolution sum $\sum_{2i+3j=l} \delta(i)\delta(j)$. Differential equations performs a critical role in applied mathematics. They are truly essential in the development of all clinical and engineering disciplines, along with physics and chemistry. In this direction, inspired by way of Ramanujan's work, we construct an ordinary differential equations with the aid of derived Eisenstein series relations of level 6. Further, we formulate certain.

Many research efforts have been devoted to ordinary differential equations satisfied by modular forms. In his notebook [2], Ramanujan devoted much attention to Eisenstein series, most notably to P, Q, and R and supplied a few interesting differential identities consisting of infinite series and theta functions. The main interest of this paper is to develop several differential identities that involve Borwein's cubic theta functions and an infinite product, namely h-functions. A detailed study of h-functions and numerous modular equations for h has been derived by M. S. M. Naika et.al [10], [11]. In their work, S. Cooper and D. Ye [6] provide a comprehensive study of the h-function and showcased the application of this parameter by mentioning several elegant formulas involving it. Also, S. Cooper [7] has documented certain relations among Eisenstein series and h-functions. Ramanujan[2] recorded certain differential equations concerning η-functions and B. C. Berndt [4] extensively enlightened the importance of developing differential equations that involve η-functions and Eisenstein series. H. C. Vidya and B. Ashwath Rao, in their work referenced as [5], developed elegant connections between Eisenstein series and theta functions. These connections were subsequently leveraged to derive and evaluate specific convolution identities. Moreover, H. C. Vidya and B. R. Srivatsa Kumar, as referenced in [13], derived differential formulas incorporating identities related to the η-function. They highlighted the significance of formulating these formulas in the context of generating incomplete integrals that feature η-functions. Further, on page 188 of his lost notebook [12], Ramanujan formulated certain formulas relating the class one infinite series $T_{2r}(q)$, $r = 1, 2, ..., 6$ with the Eisenstein series P, Q and R. The primary proof of six formulas for $T_{2r}(q)$ appeared in a research document by B. C. Berndt and A. J. Yee [3]. Another proof of these formulas appeared in Liu's paper [9, p. 9-12].

II. Preliminaries

Definition 2.1: For any complex r and s, Ramanujan [2, p.35] documented a general theta function,

$$f(r,s) := \sum_{n=-\infty}^{\infty} r^{n(n+1)/2} s^{n(n-1)/2}$$
$$:= (-r;rs)_\infty(-s;rs)_\infty(rs;rs)_\infty,$$

where

$$(r;q)_\infty := \prod_{n=0}^{\infty}(1-rq^n), \qquad |q|<1.$$

The special case of theta functions defined by Ramanujan [2, p.35] are given as follows:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty,$$

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$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^x q^{n(3n-1)/2}$$

$$= (q;q)_\infty = q^{-1/24}\eta(q), \qquad (1)$$

where $\eta(q)$ is known as the Dedekind η-function. We denote $f(-q^n) = f_n$.

Definition 2.2: [7] Let $y = exp(2\pi i/3)$. The Borwein's cubic theta functions are defined as follows:

$$a(q) := \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} q^{j^2+jk+k^2},$$

$$b(q) := \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} y^{j-k} q^{j^2+jk+k^2},$$

$$c(q) := \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} q^{(j+\frac{1}{3})^2+(j+\frac{1}{3})(k+\frac{1}{3})+(k+\frac{1}{3})^2}.$$

Definition 2.3: [7] For $|q| < 1$, the h-function defined by

$$h := h(q) := q \prod_{x=1}^{\infty} \frac{(1-q^{12x-1})(1-q^{12x-11})}{(1-q^{12x-5})(1-q^{12x-7})}.$$

The weight two modular form y_{12} in terms of h-function is defined by

$$y_{12} = q\frac{d}{dq} logh = 1 - \sum_{s=1}^{\infty} \chi_{12}(s) \frac{sq^s}{1-q^s},$$

where

$$\chi_{12}(s) = \begin{cases} 1 & \text{if s=1 or 11 (mod 12)}, \\ -1 & \text{if s=5 or 7 (mod 12)}, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.4: Ramanujan[12] documented the class one infinite series,

$$T_{2k}(q) := 1 + \sum_{x=1}^{\infty} (-1)^x \Big[(6x-1)^{2k} \{ q^{\frac{x(3x-1)}{2}} + (6x+1)^{2k} q^{\frac{x(3x+1)}{2}} \} \Big], \qquad (2)$$

and expressed $T_{2k}(q)$ for $k = 1, 2, ..., 6$ in terms of Ramnujan-type Eisenstein series:

$$P(q) := 1 - 24 \sum_{j=1}^{\infty} \frac{jq^j}{1-q^j} = 1 - 24 \sum_{j=1}^{\infty} \delta_1(j) q^j, \qquad (3)$$

$$= 1 + 24q\frac{d}{dq} \sum_{j=1}^{\infty} log(1-q^j), \qquad (4)$$

$$Q(q) := 1 + 240 \sum_{j=1}^{\infty} \frac{j^3 q^j}{1-q^j} = 1 + 240 \sum_{j=1}^{\infty} \delta_3(j) q^j, \qquad (5)$$

$$R(q) := 1 - 504 \sum_{j=1}^{\infty} \frac{j^5 q^j}{1-q^j} = 1 - 504 \sum_{j=1}^{\infty} \delta_5(j) q^j.$$

Throughout, we denote $P(q^n) = P_n$.

Further, B. C. Berndt [3] established an interesting relation

$$T_2(q) = (q;q)_\infty P(q). \qquad (6)$$

lemma 2.1: [7] The relation among an infinite series and h-functions hold:

$$\begin{pmatrix} P(q) \\ P(q^2) \\ P(q^3) \\ P(q^6) \end{pmatrix} = \begin{pmatrix} 6 & 2 & 3 & -6 & 2 & 0 \\ 3 & 2 & 0 & 0 & \frac{1}{2} & -\frac{3}{2} \\ 2 & -2 & \frac{5}{3} & \frac{2}{3} & 0 & \frac{2}{3} \\ 1 & 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{2} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} \frac{h(dy_{12})}{dh} \\ \frac{(1-h^2)}{(1+h^2)} y_{12} \\ \frac{(1-h^2)}{(1-h+h^2)} y_{12} \\ \frac{(1-h^2)}{(1-4h+h^2)} y_{12} \\ \frac{(1-h^2)}{(1-2h+h^2)} y_{12} \\ \frac{(1-h^2)}{(1+2h+h^2)} y_{12} \end{pmatrix}.$$

III. Formation of differential identities involving cubic theta functions and h-functions

In succeeding Theorems 3.1−3.4, we formulate differential equations in consequence of Eisenstein series of level 6 that associates h-functions and the derivative of weight two modular forms.

Theorem 3.1: If

$$S(q) := 2a(q^2) - a(q) = \frac{b^2(q)}{b(q^2)},$$

then the following differential identity holds:

$$q\frac{dS}{dq} + \frac{v}{4} \left[\frac{(2u^4+u^3-12u^2+16u-16)}{u(u-1)(u-4)(u^2-4)} y_{12} - 2\frac{w}{v} \right] S = 0,$$

where $h + \frac{1}{h} = u$, $-h + \frac{1}{h} = v$ and $h\frac{dy_{12}}{dh} = w$.

Proof: We note from [7, pp. 189] that, $S(q)$ may be reformulated in terms of η-functions given by

$$S(q) = \frac{\varphi^3(-q)}{\varphi(-q^3)} = \frac{\eta_1^6 \eta_6}{\eta_2^3 \eta_3^2}.$$

From equation 1, we have

$$S(q) = \frac{f_1^6 f_6}{f_2^3 f_3^2}.$$

By utilizing Definition 2.1 and logarithmically differentiating $S(q)$ with respect to q, we arrive at

$$\frac{1}{S}\frac{dS}{dq} = \frac{1}{q} \left[-6 \sum_{r=1}^{\infty} \frac{rq^r}{1-q^r} + 6 \sum_{r=1}^{\infty} \frac{rq^{2r}}{1-q^{2r}} + 6 \sum_{r=1}^{\infty} \frac{rq^{3r}}{1-q^{3r}} - 6 \sum_{r=1}^{\infty} \frac{rq^{6r}}{1-q^{6r}} \right].$$

Using (4), we get

$$\frac{q}{S}\frac{dS}{dq} = \frac{1}{4}[P_1 - P_2 - P_3 + P_6]. \qquad (7)$$

Incorporating Lemma 2.1 and simplifying, we obtain

$$P_1 - P_2 - P_3 + P_6 = 2h\frac{dy_{12}}{dh} + \frac{(1-h^2)}{(1+h^2)}y_{12}*$$

$$\left[\frac{(-4-2h+8h^2-38h^3+56h^4-38h^5+8h^6-2h^7-4h^8)}{(1-h+h^2)(1-4h+h^2)(1-2h+h^2)(1+2h+h^2)}\right].$$

Further, denoting $h+\frac{1}{h}=u$, $-h+\frac{1}{h}=v$ and $h\frac{dy_{12}}{dh}=w$ and simplifying, we arrive at

$$\begin{aligned} &P_1 - P_2 - P_3 + P_6 \\ &= -v\left[\frac{(2u^4+u^3-12u^2+16u-16)}{u(u-1)(u-4)(u^2-4)}y_{12} - 2\frac{w}{v}\right]. \end{aligned} \tag{8}$$

The claimed result follows by substituting (8) in (7).

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Theorem 3.2: If

$$S(q) = \frac{a(q)+a(q^2)}{2} = \frac{b^2(q^2)}{b(q)},$$

then the following differential identity holds:

$$q\frac{dS}{dq} - \frac{v}{2}\left[\frac{(7u^3-24u^2+24u-16)}{u(u-1)(u-4)(u^2-4)}y_{12} + \frac{w}{v}\right]S = 0,$$

where $h+\frac{1}{h}=u$, $-h+\frac{1}{h}=v$ and $h\frac{dy_{12}}{dh}=w$.

Proof: We note from [7, pp. 189] that

$$S(q) = \frac{\psi^3(q)}{\psi(q^3)} = \frac{\eta_2^6\eta_3}{\eta_1^3\eta_6^2}.$$

Using the definition of theta functions, $S(q)$ may be reformulated as

$$S(q) = \frac{f_2^6 f_3}{f_1^3 f_6^2}.$$

Employing Definition 2.1 and logarithmically differentiating, we arrive at

$$\begin{aligned} \frac{1}{S}\frac{dS}{dq} = \frac{1}{q}\Bigg[&3\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r} - 12\sum_{r=1}^{\infty}\frac{rq^{2r}}{1-q^{2r}} \\ &-3\sum_{r=1}^{\infty}\frac{rq^{3r}}{1-q^{3r}} + 12\sum_{r=1}^{\infty}\frac{rq^{6r}}{1-q^{6r}}\Bigg]. \end{aligned}$$

Using (4), we get

$$\frac{q}{S}\frac{dS}{dq} = \frac{1}{8}[-P_1+4P_2+P_3-4P_6]. \tag{9}$$

Incorporating Lemma 2.1 and simplifying, we obtain

$$-P_1+4P_2+P_3-4P_6 = 2h\frac{dy_{12}}{dh} + \frac{(1-h^2)}{(1+h^2)}y_{12}$$

$$*\left[\frac{(14h-48h^2+90h^3-128h^4+90h^5-48h^6+14h^7)}{(1-h+h^2)(1-4h+h^2)(1-2h+h^2)(1+2h+h^2)}\right].$$

Further denoting $h+\frac{1}{h}=u$, $-h+\frac{1}{h}=v$ and $h\frac{dy_{12}}{dh}=w$ and then simplifying, we deduce

$$\begin{aligned} &-P_1+4P_2+P_3-4P_6 \\ &= 4v\left[\frac{(7u^3-24u^2+24u-16)}{u(u-1)(u-4)(u^2-4)}y_{12} + \frac{w}{v}\right]. \end{aligned} \tag{10}$$

The claimed result follows immediately by using (10) in (9).

Theorem 3.3: If

$$S(q) = \frac{a(q)+2a(q^2)}{3} = \frac{c^2(q)}{3c(q^2)},$$

then the following differential identity holds:

$$q\frac{dS}{dq} - \frac{v}{2}\left[\frac{(5u^4-18u^3-18u^2+88u-48)}{u(u-1)(u-4)(u^2-4)}y_{12} + \frac{w}{v}\right]S = 0,$$

where $h+\frac{1}{h}=u$, $-h+\frac{1}{h}=v$ and $h\frac{dy_{12}}{dh}=w$.

Proof: We note from [7, pp. 189] that

$$S(q) = \frac{\varphi^3(-q^3)}{\varphi(-q)} = \frac{\eta_2\eta_3^6}{\eta_1^2\eta_6^3}.$$

Utilizing the definition of theta functions, we can reformulate $S(q)$ as

$$S(q) = \frac{f_2 f_3^6}{f_1^2 f_6^3}.$$

Now applying Definition 2.1 and logarithmically differentiating, we arrive at

$$\begin{aligned} \frac{1}{S}\frac{dS}{dq} = \frac{1}{q}\Bigg[&2\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r} - 2\sum_{r=1}^{\infty}\frac{rq^{2r}}{1-q^{2r}} \\ &-18\sum_{r=1}^{\infty}\frac{rq^{3r}}{1-q^{3r}} + 18\sum_{r=1}^{\infty}\frac{rq^{6r}}{1-q^{6r}}\Bigg]. \end{aligned}$$

Using (4), we get

$$\frac{q}{S}\frac{dS}{dq} = \frac{1}{12}[-P_1+P_2+9P_3-9P_6]. \tag{11}$$

Incorporating Lemma 2.1 and simplifying, we obtain

$$-P_1+P_2+9P_3-9P_6 = 2h\frac{dy_{12}}{dh} + \frac{(1-h^2)}{(1+h^2)}y_{12}*$$

$$\left[\frac{(10-36h+4h^2+68h^3-108h^4+68h^5+4h^6-36h^7+10h^8)}{(1-h+h^2)(1-4h+h^2)(1-2h+h^2)(1+2h+h^2)}\right],$$

Further denoting $h+\frac{1}{h}=u$, $-h+\frac{1}{h}=v$ and $h\frac{dy_{12}}{dh}=w$ and subsequently simplifying, we deduce

$$\begin{aligned} &-P_1+P_2+9P_3-9P_6 \\ &= 6v\left[\frac{(5u^4-18u^3-18u^2+88u-48)}{u(u-1)(u-4)(u^2-4)}y_{12} + \frac{w}{v}\right]. \end{aligned} \tag{12}$$

The claimed result follows immediately by using (12) in (11).

■

Theorem 3.4: If

$$S(q) = \frac{a(q)-a(q^2)}{6} = \frac{c^2(q^2)}{3c(q)},$$

then the following differential identity holds:

$$q\frac{dS}{dq} + \frac{v}{8}\left[\frac{(3u^4-14u^3+20u^2+32u-32)}{u(u-1)(u-4)(u^2-4)}y_{12} + \frac{w}{v}\right]S = 0,$$

where $h+\frac{1}{h}=u$, $-h+\frac{1}{h}=v$ and $h\frac{dy_{12}}{dh}=w$.

Proof: We note from [7, pp. 189] that

$$S(q) = q\frac{\psi^3(q^3)}{\psi(q)} = q\frac{\eta_1\eta_6^6}{\eta_2^2\eta_3^3}.$$

Utilizing the definition of theta functions, we can reformulate $S(q)$ as

$$S(q) = q\frac{f_1 f_6^6}{f_2^2 f_3^3}.$$

Employing Definition 2.1 and logarithmically differentiating, we arrive at

$$\frac{1}{S}\frac{dS}{dq}=\frac{1}{q}-\frac{1}{q}\left[\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}-4\sum_{r=1}^{\infty}\frac{rq^{2r}}{1-q^{2r}}\right.$$
$$\left.-9\sum_{r=1}^{\infty}\frac{rq^{3r}}{1-q^{3r}}+36\sum_{r=1}^{\infty}\frac{rq^{6r}}{1-q^{6r}}\right].$$

Using (4), we have

$$\frac{q}{S}\frac{dS}{dq}=\frac{1}{24}[P_1-4P_2-9P_3+36P_6]. \tag{13}$$

Incorporating Lemma 2.1 and further denoting $h+\frac{1}{h}=u$, $-h+\frac{1}{h}=v$ and $h\frac{dy_{12}}{dh}=w$ and further simplifying, we get

$$P_1-4P_2-9P_3+36P_6$$
$$=-3v\left[\frac{(3u^4-14u^3+20u^2+32u-32)}{u(u-1)(u-4)(u^2-4)}y_{12}+\frac{w}{v}\right]. \tag{14}$$

The claimed result follows by using (14) in (13).

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IV. Relations among class one infinite series and h-functions

Inspired by the research conducted by B. C. Berndt [3], we develop a representation for Eisenstein series by leveraging classical class one infinite series. Moreover, utilizing this expression, we derive intriguing equations that establish connections between class one infinite series and h-functions.

Theorem 4.1: For every $n\geq 2$, the subsequent relationship among the two distinct series holds:

$$P(q^n)=1+nq^{n-1}\left[\frac{T_2(q^n)+1}{(q^n;q^n)_\infty}-1\right]. \tag{15}$$

Proof: First we prove the result for $n=2$. Replacing q to q^2 in (4), we obtain

$$P(q^2)=1+24q^2\frac{d}{dq}log(q^2;q^2)$$
$$=1+24q^2\frac{1}{(q^2;q^2)_\infty}\frac{d}{dq}(q^2;q^2)_\infty.$$

On simplifying, we arrive at

$$(q^2;q^2)_\infty P(q^2)=(q^2;q^2)_\infty$$
$$+24q^2\frac{d}{dq}\left[1+\sum_{x=1}^{\infty}(-1)^x\{q^{x(3x-1)}+q^{x(3x+1)}\}\right]$$
$$=(q^2;q^2)_\infty+24q$$
$$*\sum_{x=1}^{\infty}(-1)^x\left[x(3x-1)q^{x(3x-1)}+x(3x+1)q^{x(3x+1)}\right]$$

$$=(q^2;q^2)_\infty-2q(q^2;q^2)_\infty+2q$$
$$*\sum_{x=1}^{\infty}(-1)^x\ [((6x-1)^2-1)q^{x(3x-1)}$$
$$((6x+1)^2+1)q^{x(3x+1)}]$$
$$=(q^2;q^2)_\infty+2qT_2(q^2)-2q(q^2;q^2)_\infty+2q.$$

Dividing throughout by $(q^2;q^2)_\infty$ and further rearranging the terms, we deduce the result for $n=2$. Similarly, the proof of $n>2$ follows by replacing q to q^n in (4) and using the series (2).

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Theorem 4.2: The following series expansion among class one infinite series and h-functions hold:

$$i)\frac{T_2(q)}{f_1}-2q\frac{T_2(q^2)}{f_2}-3q^2\frac{T_2(q^3)}{f_3}+6q^5\frac{T_2(q^6)}{f_6}$$
$$+2q\left(1-\frac{1}{f_2}\right)+3q^2\left(1-\frac{1}{f_3}\right)-6q^5\left(1-\frac{1}{f_6}\right)$$
$$+v\left[\frac{(2u^4+u^3-12u^2+16u-16)}{u(u-1)(u-4)(u^2-4)}y_{12}-2\frac{w}{v}\right]-1=0,$$

$$ii)\frac{T_2(q)}{f_1}-8q\frac{T_2(q^2)}{f_2}-3q^2\frac{T_2(q^3)}{f_3}+24q^5\frac{T_2(q^6)}{f_6}$$
$$+8q\left(1-\frac{1}{f_2}\right)+3q^2\left(1-\frac{1}{f_3}\right)-24q^5\left(1-\frac{1}{f_6}\right)$$
$$+4v\left[\frac{(7u^3-24u^2+24u-16)}{u(u-1)(u-4)(u^2-4)}y_{12}+\frac{w}{v}\right]-1=0,$$

$$iii)\frac{T_2(q)}{f_1}-2q\frac{T_2(q^2)}{f_2}-27q^2\frac{T_2(q^3)}{f_3}+54q^5\frac{T_2(q^6)}{f_6}$$
$$+2q\left(1-\frac{1}{f_2}\right)+27q^2\left(1-\frac{1}{f_3}\right)-54q^5\left(1-\frac{1}{f_6}\right)$$
$$+6v\left[\frac{(5u^4-18u^3-18u^2+88u-48)}{u(u-1)(u-4)(u^2-4)}y_{12}+\frac{w}{v}\right]-1=0,$$

$$iv)\frac{T_2(q)}{f_1}-8q\frac{T_2(q^2)}{f_2}-27q^2\frac{T_2(q^3)}{f_3}+216q^5\frac{T_2(q^6)}{f_6}$$
$$+8q\left(1-\frac{1}{f_2}\right)+27q^2\left(1-\frac{1}{f_3}\right)-216q^5\left(1-\frac{1}{f_6}\right)$$
$$+3v\left[\frac{(3u^4-14u^3+20u^2+32u-32)}{u(u-1)(u-4)(u^2-4)}y_{12}+\frac{w}{v}\right]+23=0,$$

where $h+\frac{1}{h}=u$, $-h+\frac{1}{h}=v$ and $h\frac{dy_{12}}{dh}=w$.

Proof: Replacing $n=2,3$ and 6 in (15) and using (6), we deduce equalities for $P(q^n)$ in terms of $T_2(q^n)$. Further substituting these in (8), (10), (12) and (14), we arrive at the required result.

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V. Convolution Sum

In this section, we present a straightforward approach for computing a discrete convolution sum.
For $k,n\in\mathbb{N}$, we set

$$\delta_k(n)=\sum_{d/n}d^k,$$

where d runs through the positive integers divisors of n. If $n\notin\mathbb{N}$, we set $\delta_k(n)=0$. For $i,j\in\mathbb{N}$ with $i\leq j$, the convolution sum

$$W_{i,j}(n):=\sum_{il+jm=n}\delta(l)\delta(m).$$

For all n, the convolution $\sum\limits_{il+jm=n}\delta(l)\delta(m)$ has been evaluated explicitly for various values of i an j, by A. Alaca,

S. Alaca, K. S. Williams. For wonderful introduction one can see [1], [14]. The representations of the convolution sums $\sum_{i+6j=n} \sigma(i)\sigma(j)$ and $\sum_{i+12j=n} \sigma(i)\sigma(j)$ are found in the article by E. X. W. Xia and O. X. M. Yao [15].
Central to our proof are the assertions made by J. W. L. Glaisher [8],

$$P^2(q) = 1 + \sum_{l=1}^{\infty}(240\sigma_3(l) - 288l\sigma(l))q^l. \quad (16)$$

Theorem 5.1: For any $l \in \mathbb{N}$, we have

$$\sum_{2i+3j=l} \delta(i)\delta(j) = \frac{1}{24}\delta_1(l/2) + \frac{1}{24}\delta_1(l/3) + \frac{5}{36}\delta_3(l/2)$$
$$-\frac{1}{12}l\delta_1(l/2) - \frac{1}{8}l\delta_1(l/3) + \frac{5}{16}l\delta_3(l/3) - \frac{A(l)}{6912},$$

where

$$\sum_{l=1}^{\infty} A(l)q^l = \left[\frac{(u^4 - 4u^3 + 48u^2 - 160u + 160)v}{u(u-1)(u-4)(u^2-4)}\right]^2 y_{12}^2.$$

Proof: Incorporating Lemma 2.1, we note that

$$2P_2 - 3P_3 = \frac{(1-h^2)}{(1+h^2)} *$$
$$\frac{(1-4h+52h^2-172h^3+262h^4-172h^5+52h^6-4h^7+h^8)}{(1-h+h^2)(1-2h+h^2)(1+2h+h^2)(1-4h+h^2)} y_{12}.$$

Denoting $h + \frac{1}{h} = u$, $-h + \frac{1}{h} = v$ and Squaring the above relation, we deduce

$$4P_2^2 + 9P_3^2 - 12P_2P_3 = \sum_{l=1}^{\infty} A(l)q^l,$$

where

$$\sum_{l=1}^{\infty} A(l)q^l = \left[\frac{(u^4 - 4u^3 + 48u^2 - 160u + 160)v}{u(u-1)(u-4)(u^2-4)}\right]^2 y_{12}^2.$$

Employing(16) and the definition of Eisenstein series and subsequently comparing the coefficient of q^l on either sides, we arrive at the desired result. ∎

REFERENCES

[1] A. Alaca, S. Alaca and K. S. Williams, Evaluation of the convolution sums $\sum_{l+12m=n} \sigma(l)\sigma(m)$ and $\sum_{3l+4m=n} \sigma(l)\sigma(m)$, *Advances in Theoretical and Applied Mathematics,* vol.1, , no. 1, pp. 27-48, 2006.

[2] Berndt BC, Ramanujan's Notebooks, Part III, Springer, New York, 1991.

[3] B. C. Berndt, A. J. Yee, A page on Eisenstein series in Ramanujan's lost notebook, *Glasgow Math. J.*, vol.45, pp. 123-129, 2003.

[4] B. C. Berndt, H. H. Chan, S. S. Haung, Incomplete elliptic integrals in Ramanujan's lost notebook, *Contemporary Mathematics,* vol. 254, pp. 79-124, 2004.

[5] Harekala Chandrashekara Vidya and Badanidiyoor Ashwath Rao, Intriguing Relationships among Eisenstein Series, Borewein's Cubic Theta Functions, and the Class One Infinite Series, *IAENG International Journal of Computer Science,* vol. 50, no. 4, pp. 1166-1173, 2023.

[6] S. Cooper, D. Ye., The level 12 analogue of Ramanujan's function k, *J. Aust. Math. Soc.,* vol. 101, pp. 29-53, 2016.

[7] S. Cooper, Ramanujan's Theta Functions, Springer, 2017.

[8] J. W. L. Glaisher, On the square of the series in which the coeff's are the sum of the divisors of the exponents, *Mess. Math.,* vol. 14, pp. 156-163, 1885.

[9] Z. -G. Liu, A three-term theta function identity and its applications, *Adv. in Math.*, vol. 195, pp. 1-23, 2005.

[10] M. S. M. Naika, B. N. Dharmendra, K. Shivashankara, A continued fraction of order twelve, *Central European Journal of Mathematics*, vol. 6, no.3, pp. 393-404, 2008.

[11] M. S. M. Naika, S. Chandan Kumar and K. Sushan Bairy, Some new identities for a continued fraction of order 12, *Souh East Asian J. Math. Math. Sci.,* vol. 10, pp. 129-140, 2012.

[12] Ramanujan S, The Lost Notebook and Other Unpublished Papers, Narosa: New Delhi, 1988.

[13] H. C. Vidya, B. R. Srivatsa Kumar, Some studies on Eisenstein series and its applications, *Notes on Number Theory and Discrete Mathematics,* vol. 25, no.4, pp. 30-43, 2019.

[14] K. S. Williams, *The convolution sum* $\sum_{m<n/8} \sigma(l)\sigma(n-8m)$, Pacific Journal of Mathematics, vol. 228, no. 2, pp. 387-396, 2006.

[15] E. X. W. Xia and O. X. M. Yao, Eisenstein series identities involving the Borwein's cubic theta functions, *Journal of Applied Mathematics,* Article 181264, 2012.