General Degree Product Energy of Graphs

SANGAM MADABHAVI, K ARATHI BHAT*, and SHAHISTHA HANIF

***Abstract*—Let $\mathcal{G}$ be a simple connected graph with n vertices and m edges. In this article, we introduce the concept of the general degree product matrix and the general degree product energy of a graph $\mathcal{G}$ and obtain the bounds for the general degree product eigenvalues and general degree product energy of any connected graph $\mathcal{G}$. Further, we obtain the general degree product energy of a new family of graphs from an existing base graph, such as the duplication graph, $\mathcal{R}$-graph, and middle graph.**

***Index Terms*—Degree product matrix, duplication graph, $\mathcal{R}$-graph, middle graph.**

I. INTRODUCTION

GRAPHS considered here are simple, finite, undirected, and connected with the vertex set $V_{\mathcal{G}} = V(\mathcal{G})$ and edge set $E_{\mathcal{G}} = E(\mathcal{G})$. Motivated by work on the degree product energy of graphs [12], we define the general degree product matrix of $\mathcal{G}$ denoted by $\mathcal{D}_\alpha(\mathcal{G})$ as the square matrix of order n whose $(i,j)^{th}$ entry is given by

$$\mathcal{D}_\alpha(\mathcal{G})_{ij} = \begin{cases} (d_i d_j)^\alpha; & \text{if } i \neq j \\ 0; & \text{otherwise,} \end{cases}$$

for any real number α.

For a graph $\mathcal{G}$, the general degree product characteristic polynomial of $\mathcal{G}$ is $f(\mathcal{D}_\alpha(\mathcal{G}), x) = det(xI_n - \mathcal{D}_\alpha(\mathcal{G}))$ and its zeros are the general degree product eigenvalues of $\mathcal{G}$. $\mathcal{D}_\alpha\mathcal{E}(\mathcal{G})$ represents the general degree product energy, which is the sum of every eigenvalue of the general degree product matrix's absolute values.

The first Zagreb energy of graphs of various graph products are discussed in article [9]. The results on the spectral properties and the determinant of the ISI matrix are derived in [7]. Atom bond connectivity index for graph with self-loops and its application are discussed in [14].

The Laplacian and signless Laplacian adjacency spectra of the $\mathcal{R}$-edge neighborhood corona, the $\mathcal{R}$-vertex neighborhood corona, $\mathcal{R}$-edge corona, and $\mathcal{R}$-vertex corona for an arbitrary graph $\mathcal{H}$ and a regular graph $\mathcal{G}$ are discussed in 2015 [8], according to $\mathcal{G}$ and $\mathcal{H}$'s corresponding spectra (as well as some other quantities). The duplication edge corona, duplication neighborhood corona, and duplication corona of two graphs have been demonstrated, and their adjacency spectrum was examined by Adiga et al. [1] in 2018.

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Sangam Madabhavi is a research scholar in the department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India, 576104 (email:sangammadabhavi340@gmail.com).

K Arathi Bhat is an Associate Professor in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India, 576104 (*Corresponding author to provide email: arathi.bhat@manipal.edu; Phone: +91-9964282648).

Shahistha Hanif is an Assistant Professor in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India, 576104 (email: shahistha.hanif@manipal.edu).

Let A be the block-form description of the matrix of order n.

$$A = \begin{pmatrix} N_{11} & .... & N_{1l} \\ \vdots & \ddots & \vdots \\ N_{l1} & .... & N_{ll} \end{pmatrix}_{n\times n},$$

where for any $1 \leq i, j \leq l$ and $n = n_1 + \cdots + n_l$, the blocks N_{ij} are $n_i \times n_j$ matrices. A square matrix A has an equitable partition D if each diagonal block has square order and each block of the partitioned matrix has constant row sums. An equitable partition is represented by a quotient matrix B of a square matrix A, a matrix with constant row sums of the corresponding blocks of A as its entries. The theory of graph spectra depends on equitable partitions because of the following result.

Theorem I.1. *[5] Let A be a real symmetric matrix and B is its quotient matrix, then the characteristic polynomial of A is divided by the characteristic polynomial of B.*

The rest of the paper is organized as follows: In Section II, a few preliminary results on the bounds for the eigenvalues and energy of the general degree product matrix are discussed. Further, the expression for the general degree product energy of a k-regular graph, path, and complete bipartite graphs are obtained in the same section. The results on the general degree product energy of a new family of graphs, which are obtained by taking the base graph as the $\mathcal{R}$-graph, duplication graph, and middle graph, are presented in Section III.

II. PRELIMINARY RESULTS

In this section, we first obtain the expression for the general degree product energy of a few standard graphs. Further, we obtain some results related to the eigenvalues and energy of the general degree product matrix.

Theorem II.1. *Let $\mathcal{G}$ be a k-regular graph of order n. Then*

$$\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}) = 2k^{2\alpha}(n-1).$$

Proof: As $\mathcal{D}_\alpha(\mathcal{G}) = k^{2\alpha}A(K_n)$, the result follows by noting that

$$Spec(\mathcal{D}_\alpha(\mathcal{G})) = \begin{pmatrix} -k^{2\alpha} & k^{2\alpha}(n-1) \\ n-1 & 1 \end{pmatrix}.$$

■

Theorem II.2. *Suppose that P_n is a path graph on n vertices. Then*

$$\mathcal{D}_\alpha\mathcal{E}(P_n) = 1+4^\alpha(n-3)+\sqrt{4^{\alpha+1}(n-1)+(1+4^\alpha(n-3))^2}.$$

Proof: Observe that $\mathcal{D}_\alpha(P_n)$ can be expressed as,

$$\mathcal{D}_\alpha(P_n) = \begin{pmatrix} A(K_2) & 2^\alpha J_{2,n-2} \\ 2^\alpha J_{n-2,2} & 4^\alpha A(K_{n-2}) \end{pmatrix}.$$

By performing row operations on $det(xI_n - \mathcal{D}_\alpha(P_n))$ and solving quotient matrix of $\mathcal{D}_\alpha(P_n)$, we get

$$Spec(\mathcal{D}_\alpha(P_n)) = \begin{pmatrix} -1 & -4^\alpha & \zeta_1 & \zeta_2 \\ 1 & n-3 & 1 & 1 \end{pmatrix},$$

where $(\zeta_1, \zeta_2) = \frac{1+4^\alpha(n-3) \pm \sqrt{4^{\alpha+1}(n-1)+(1+4^\alpha(n-3))^2}}{2}$. ■

Theorem II.3. *Suppose that K_{n_1,n_2} is a complete bipartite graph on n vertices. Then*

$$\mathcal{D}_\alpha\mathcal{E}(K_{n_1,n_2}) = (n_2-1)n_1^{2\alpha} + (n_1-1)n_2^{2\alpha} + \sqrt{4(n_1n_2)^{2\alpha}(n_1+n_2-1) + ((n_2-1)n_1^{2\alpha} + (n_1-1)n_2^{2\alpha})^2}.$$

Proof: The general degree product matrix of K_{n_1,n_2} is expressed as,

$$\mathcal{D}_\alpha(K_{n_1,n_2}) = \begin{pmatrix} n_2^{2\alpha}A(K_{n_1}) & (n_1n_2)^\alpha J_{n_1,n_2} \\ (n_1n_2)^\alpha J_{n_2,n_1} & n_1^{2\alpha}A(K_{n_2}) \end{pmatrix}.$$

By performing row operations on $det(xI_n - \mathcal{D}_\alpha(K_{n_1,n_2}))$ and solving quotient matrix of $\mathcal{D}_\alpha(K_{n_1,n_2})$, we get

$$Spec(\mathcal{D}_\alpha(K_{n_1,n_2})) = \begin{pmatrix} -n_2^{2\alpha} & -n_1^{2\alpha} & \zeta_1 & \zeta_2 \\ n_1-1 & n_2-1 & 1 & 1 \end{pmatrix},$$

where $(\zeta_1, \zeta_2) = \frac{1}{2}((n_2-1)n_1^{2\alpha} + (n_1-1)n_2^{2\alpha} + \sqrt{4(n_1n_2)^{2\alpha}(n_1+n_2-1) + ((n_2-1)n_1^{2\alpha} + (n_1-1)n_2^{2\alpha})^2})$. ■

The following lemma is one of the basic properties of general degree product eigenvalues. We omit the proof because it is straightforward.

Lemma II.1. *Suppose $\zeta_1 \geq \zeta_2 \geq \cdots \geq \zeta_n$ are the general degree product eigenvalues of graph $\mathcal{G}$ with n vertices. Then*

(i) $\sum_{i=1}^{n} \zeta_i = 0$.

(ii) $\sum_{i=1}^{n} \zeta_i^2 = 2\mathcal{N}$, *where* $\mathcal{N} = \sum_{i<j} (d_id_j)^{2\alpha}$.

Lemma II.2 (Cauchy-Schwartz inequality). *If $(\alpha_1, \alpha_2, \ldots, \alpha_t)$ and $(\beta_1, \beta_2, \ldots, \beta_t)$ are real t-vectors then,*

$$\left(\sum_{k=1}^{t} \alpha_k\beta_k\right)^2 \leq \left(\sum_{k=1}^{t} \alpha_k^2\right)\left(\sum_{k=1}^{t} \beta_k^2\right).$$

In all the results discussed below, $\mathcal{N} = \sum_{i<j} (d_id_j)^{2\alpha}$. Using the Cauchy-Schwarz inequality, the bound for the largest general degree product eigenvalue is given in the following theorem.

Theorem II.4. *Let $\mathcal{G}$ be any graph with n vertices. Then*

$$\zeta_1 \leq \sqrt{\frac{2\mathcal{N}(n-1)}{n}}.$$

Proof. Choosing $\alpha_i = 1$, $\beta_i = \zeta_i$ for $i = 2, 3, \ldots, n$, in the Lemma II.2, we get

$$\left(\sum_{i=2}^{n} \zeta_i\right)^2 \leq (n-1)\sum_{i=2}^{n} \zeta_i^2. \tag{1}$$

From Lemma II.1, we know $\sum_{i=2}^{n} \zeta_i = -\zeta_1$ and $\sum_{i=2}^{n} \zeta_i^2 = -\zeta_1^2 + 2\mathcal{N}$.

Then Eq. (1) becomes

$$(-\zeta_1)^2 \leq (n-1)(2\mathcal{N} - \zeta_1^2)$$
$$\zeta_1 \leq \sqrt{\frac{2\mathcal{N}(n-1)}{n}}.$$

A few further bounds for the general degree product energy of graphs are discussed below.

Theorem II.5. *Let $\mathcal{G}$ be any graph with n vertices. Then*

$$\sqrt{2\mathcal{N}} \leq \mathcal{D}_\alpha\mathcal{E}(\mathcal{G}) \leq \sqrt{2n\mathcal{N}}.$$

Proof: Replacing $\alpha_i = 1$ and $\beta_i = |\zeta_i|$ in the Lemma II.2, we get

$$\left(\sum_{i=1}^{n} |\zeta_i|\right)^2 \leq n\sum_{i=1}^{n} \zeta_i^2$$
$$(\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}))^2 \leq 2n\mathcal{N}$$
$$\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}) \leq \sqrt{2n\mathcal{N}}$$

Now, $(\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}))^2 = \left(\sum_{i=1}^{n} |\zeta_i|\right)^2 \geq \sum_{i=1}^{n} |\zeta_i|^2 = 2\mathcal{N}$,

$$\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}) \geq \sqrt{2\mathcal{N}}.$$

■

Theorem II.6. *Let $\mathcal{G}$ be any graph with n vertices, and Δ be the absolute value of the determinant of the general degree product matrix $\mathcal{D}_\alpha(\mathcal{G})$. Then,*

$$\sqrt{2\mathcal{N} + n(n-1)\Delta^{\frac{2}{n}}} \leq \mathcal{D}_\alpha\mathcal{E}(\mathcal{G}) \leq \sqrt{2n\mathcal{N}}.$$

Proof: Let

$$(\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}))^2 = (\sum_{i=1}^{n} \zeta_i)^2 = 2\mathcal{N} + \sum_{i\neq j} |\zeta_i||\zeta_j|. \tag{2}$$

Since the geometric mean is smaller than the arithmetic mean for non-negative numbers,

$$\frac{1}{n(n-1)}\sum_{i\neq j} |\zeta_i||\zeta_j| \geq \left(\prod_{i\neq j} |\zeta_i||\zeta_j|\right)^{\frac{1}{n(n-1)}}$$
$$= \prod_{i=1}^{n} |\zeta_i|^{\frac{2}{n}}$$
$$= \Delta^{\frac{2}{n}}. \tag{3}$$

Consider,

$$T = \sum_{i=1}^{n}\sum_{j=1}^{n} (|\zeta_i| - |\zeta_j|)^2$$
$$= 4n\mathcal{N} - 2(\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}))^2$$

Since $T \geq 0$,

$$4n\mathcal{N} - 2(\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}))^2 \geq 0.$$

Hence,

$$\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}) \leq \sqrt{2n\mathcal{N}}. \tag{4}$$

Combining Equations (2), (3), and (4), we obtain the lower and upper bounds. ■

A few results that are useful for proving further results on the bounds of the general degree product energy are given below.

Theorem II.7. *[11] Suppose α_k and β_k, $1 \leq k \leq t$ are non-negative real numbers. Then*

$$\sum_{k=1}^{t} \alpha_k^2 \sum_{k=1}^{t} \beta_k^2 \leq \frac{1}{4}\left(\sqrt{\frac{P_1P_2}{Q_1Q_2}} + \sqrt{\frac{Q_1Q_2}{P_1P_2}}\right)^2 \left(\sum_{k=1}^{t} \alpha_k\beta_k\right)^2$$

where $P_1 = \max_{1\le k\le t}(\alpha_k)$*;* $P_2 = \max_{1\le k\le t}(\beta_k)$*;* $Q_1 = \min_{1\le k\le t}(\alpha_k)$ *and* $Q_2 = \min_{1\le k\le t}(\beta_k)$*.*

Theorem II.8. *[10] Let* α_k *and* β_k*,* $1 \le k \le t$ *are positive real numbers. Then*

$$\sum_{k=1}^{t}\alpha_k^2\sum_{k=1}^{t}\beta_k^2 - \left(\sum_{k=1}^{t}\alpha_k\beta_k\right)^2 \le \frac{t^2}{4}(P_1P_2 - Q_1Q_2)^2$$

where $P_1 = \max_{1\le k\le t}(\alpha_k)$*;* $P_2 = \max_{1\le k\le t}(\beta_k)$*;* $Q_1 = \min_{1\le k\le t}(\alpha_k)$ *and* $Q_2 = \min_{1\le k\le t}(v_k)$*.*

Theorem II.9. *[2] Let* α_k *and* β_k*,* $1 \le k \le t$ *are non-negative real numbers. Then*

$$\left|t\sum_{k=1}^{t}\alpha_k\beta_k - \sum_{k=1}^{t}\alpha_k\sum_{k=1}^{t}\beta_k\right| \le \chi(t)(Q-P)(S-R).$$

where P*,* Q*,* R *and* S *are real constants, that for each* k*,* $1 \le k \le t$*,* $P \le \alpha_k \le Q$ *and* $R \le \beta_k \le S$*. Further,* $\chi(t) = t\left\lceil\frac{t}{2}\right\rceil\left(1 - \frac{1}{t}\left\lceil\frac{t}{2}\right\rceil\right)$*.*

Theorem II.10. *[4] Let* α_k *and* β_k*,* $1 \le k \le t$ *be non-negative real numbers. Then*

$$\sum_{k=1}^{t}\beta_k^2 + KL\sum_{k=1}^{t}\alpha_k^2 \le (K+L)\left(\sum_{k=1}^{t}\alpha_k\beta_k\right),$$

where K *and* L *are real constants, so that for each* k*,* $1 \le k \le t$*, holds* $K\alpha_k \le \beta_k \le L\alpha_k$*.*

Theorem II.11. *For any graph* $\mathcal{G}$ *with* n *vertices. Then*

$$\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}) \ge \sqrt{2n\mathcal{N} - \frac{n^2}{4}(\zeta_1 - \zeta_{min})^2}$$

where $\zeta_1 = \zeta_{max} = \max_{1\le i\le n}|\zeta_i|$ *and* $\zeta_{min} = \min_{1\le i\le n}|\zeta_i|$*.*

Proof: For $\mathcal{D}_\alpha(\mathcal{G})$, let $\zeta_1, \zeta_2, \ldots, \zeta_n$ be its eigenvalues. According to Theorem II.8, we assume that $\alpha_i = 1$ and $\beta_i = |\zeta_i|$, this implies

$$\sum_{i=1}^{n}1^2\sum_{i=1}^{n}|\zeta_i|^2 - \left(\sum_{i=1}^{n}|\zeta_i|\right)^2 \le \frac{n^2}{4}(\zeta_1 - \zeta_{min})^2$$
$$2n\mathcal{N} - (\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}))^2 \le \frac{n^2}{4}(\zeta_1 - \zeta_{min})^2$$
$$\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}) \ge \sqrt{2n\mathcal{N} - \frac{n^2}{4}(\zeta_1 - \zeta_{min})^2}.$$

■

Theorem II.12. *If* $\mathcal{D}_\alpha(\mathcal{G})$ *does not have zero as an eigenvalue, then*

$$\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}) \ge \frac{2\sqrt{2n\mathcal{N}}\sqrt{\zeta_1\zeta_{min}}}{\zeta_1 + \zeta_{min}}$$

where $\zeta_1 = \zeta_{max} = \max_{1\le i\le n}|\zeta_i|$ *and* $\zeta_{min} = \min_{1\le i\le n}|\zeta_i|$*.*

Proof: Let $\mathcal{D}_\alpha(\mathcal{G})$ have eigenvalues $\zeta_1, \zeta_2, \ldots, \zeta_n$. Assuming $\alpha_i = |\zeta_i|$ and $\beta_i = 1$, we can infer from Theorem II.7

$$\sum_{i=1}^{n}|\zeta_i|^2\sum_{i=1}^{n}1^2 \le \frac{1}{4}\left(\sqrt{\frac{\zeta_1}{\zeta_{min}}} + \sqrt{\frac{\zeta_{min}}{\zeta_1}}\right)^2\left(\sum_{i=1}^{n}|\zeta_i|\right)^2$$
$$2n\mathcal{N} \le \frac{1}{4}\left(\frac{(\zeta_1 + \zeta_{min})^2}{\zeta_1\zeta_{min}}\right)(\mathcal{D}_\alpha\mathcal{E}(G))^2$$
$$\mathcal{D}_\alpha\mathcal{E}(G) \ge \frac{2\sqrt{\zeta_1\zeta_{min}}\sqrt{2n\mathcal{N}}}{\zeta_1 + \zeta_{min}}.$$

■

Theorem II.13. *Let* $\mathcal{G}$ *be a graph of order* n *and* $\zeta_1 \ge \zeta_2 \ge \cdots \ge \zeta_n$ *be the eigenvalues of* $\mathcal{D}_\alpha(\mathcal{G})$*. Then*

$$\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}) \ge \frac{2\mathcal{N} + n\zeta_1\zeta_{min}}{\zeta_1 + \zeta_{min}}$$

where $\zeta_1 = \zeta_{max} = \max_{1\le i\le n}|\zeta_i|$ *and* $\zeta_{min} = \min_{1\le i\le n}|\zeta_i|$*.*

Proof: Suppose $\beta_i = |\zeta_i|$, $u_i = 1$, $L = |\zeta_1|$, and $K = |\zeta_{min}|$. Theorem II.10 yields:

$$\sum_{i=1}^{n}|\zeta_i|^2 + \zeta_1\zeta_{min}\sum_{i=1}^{n}1^2 \le (\zeta_1 + \zeta_{min})\sum_{i=1}^{n}|\zeta_i|$$
$$2\mathcal{N} + n\zeta_1\zeta_{min} \le (\zeta_1 + \zeta_{min})\mathcal{D}_\alpha\mathcal{E}(G)$$
$$\mathcal{D}_\alpha\mathcal{E}(G) \ge \frac{2\mathcal{N} + n\zeta_1\zeta_{min}}{\zeta_1 + \zeta_{min}}.$$

■

Theorem II.14. *Let* $\mathcal{G}$ *be a graph of order* n *and* $\zeta_1 \ge \zeta_2 \ge \cdots \ge \zeta_n$ *be the eigenvalues of* $\mathcal{D}_\alpha(\mathcal{G})$*. Then*

$$\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}) \ge \sqrt{2n\mathcal{N} - \alpha(n)(\zeta_1 - \zeta_{min})^2},$$

where $\zeta_1 = \zeta_{max} = \max_{1\le i\le n}|\zeta_i|$ *and* $\zeta_{min} = \min_{1\le i\le n}|\zeta_i|$ *and* $\chi(n) = n\left\lceil\frac{n}{2}\right\rceil\left(1 - \frac{1}{n}\left\lceil\frac{n}{2}\right\rceil\right)$*.*

Proof: Assuming $\alpha_i = |\zeta_i| = \beta_i$, $Q \le |\zeta_i| \le S$, and $P \le |\zeta_n| \le R$, we obtain the following using Theorem (II.9):

$$\left|n\sum_{i=1}^{n}|\zeta_i|^2 - \left(\sum_{i=1}^{n}|\zeta_i|\right)^2\right| \le \chi(n)(\zeta_1 - \zeta_{min})^2$$
$$|2n\mathcal{N} - (\mathcal{D}_\alpha\mathcal{E}(G))^2| \le \chi(n)(\zeta_1 - \zeta_{min})^2$$
$$\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}) \ge \sqrt{2n\mathcal{N} - \chi(n)(\zeta_1 - \zeta_{min})^2}.$$

■

III. Graph Operations

In this section, we obtain the general degree product energy of a new family of graphs from an existing base graph such as duplication graph, $\mathcal{R}$-graph and middle graph.

A. Duplication graph

Definition III.1. [13] The duplication graph $D_u(\mathcal{G})$ is a bipartite graph with vertex partition sets $V_\mathcal{G} = \{v_1, \ldots, v_n\}$ and $U_\mathcal{G} = \{u_1, \ldots, u_n\}$, where v_iu_j is an edge if and only if v_iv_j is an edge in $\mathcal{G}$.

Definition III.2. [1] The duplication corona $\mathcal{G}_1 \oplus \mathcal{G}_2$ of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ is the graph obtained by taking one copy of $D_u(\mathcal{G}_1)$ and $|V_{\mathcal{G}_1}|$ copies of $\mathcal{G}_2$, and then joining vertex v_i of $D_u(\mathcal{G}_1)$ to every vertex in the ith copy of $\mathcal{G}_2$.

Definition III.3. [1] The duplication neighborhood corona $\mathcal{G}_1 \boxtimes \mathcal{G}_2$ of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ is the graph obtained by taking one copy of $D_u(\mathcal{G}_1)$ and $|V_{\mathcal{G}_1}|$ copies of $\mathcal{G}_2$ and then joining the neighbors of the vertex v_i of $D_u(\mathcal{G}_1)$ to every vertex in the ith copy of $\mathcal{G}_2$.

Definition III.4. [1] The duplication edge corona $\mathcal{G}_1 \boxplus \mathcal{G}_2$ of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ is the graph obtained by taking one copy of $D_u(\mathcal{G}_1)$ and $|E_{\mathcal{G}_1}|$ copies of $\mathcal{G}_2$ and then joining a pair of vertices v_i and v_j of $D_u(\mathcal{G}_1)$ to every vertex in the kth copy of $\mathcal{G}_2$ whenever $v_iv_j = e_k \in E_{\mathcal{G}_1}$.

Theorem III.1. *Suppose that k-regular graph $\mathcal{G}$ has order n. Then*

$$\mathcal{D}_\alpha\mathcal{E}(D_u(\mathcal{G})) = 2(2n-1)k^{2\alpha}.$$

Proof: As $\mathcal{D}_\alpha(D_u(\mathcal{G})) = k^{2\alpha}A(K_{2n})$, the result follows by noting that

$$Spec(\mathcal{D}_\alpha(D_u(\mathcal{G}))) = \begin{pmatrix} k^{2\alpha}(2n-1) & -k^{2\alpha} \\ 1 & (2n-1) \end{pmatrix}. \qquad \blacksquare$$

Theorem III.2. *Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ be k_1, k_2-regular graphs with n_1, n_2 vertices. Then $\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}_1 \oplus \mathcal{G}_2) = \sum_{i=1}^{3}|\zeta_i| + (n_1n_2-1)(k_2+1)^{2\alpha} + (n_1-1)\left(k_1^{2\alpha} + (k_1+n_2)^{2\alpha}\right)$, where $\zeta_1, \zeta_2, \zeta_3$ are the roots of the polynomial of $x^3 - ((k_2+1)^{2\alpha}(n_1n_2-1) + (k_1^{2\alpha} + (k_1+n_2)^{2\alpha})(n_1-1))x^2 - ((n_1n_2+n_1-1)(k_1^{2\alpha} + (k_1+n_2)^{2\alpha})(k_2+1)^{2\alpha} + (2n_1-1)(k_1(k_1+n_2))^{2\alpha})x - (n_1n_2+2n_1-1)(k_1(k_1+n_2)(k_2+1))^{2\alpha} = 0$.*

Proof: The general degree product matrix of duplication corona of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ be expressed as a block matrix as follows:

$$\mathcal{D}_\alpha(\mathcal{G}_1 \oplus \mathcal{G}_2) = \begin{pmatrix} (k_2+1)^{2\alpha}A(K_{n_1n_2}) & (k_1^\alpha(k_2+1)^\alpha J_{n_1n_2,n_1}) & ((k_1+n_2)^\alpha(k_2+1)^\alpha J_{n_1n_2,n_1}) \\ k_1^\alpha(k_2+1)^\alpha J_{n_1,n_1n_2} & k_1^{2\alpha}A(K_{n_1}) & k_1^\alpha(k_1+n_2)^\alpha J_{n_1} \\ ((k_1+n_2)^\alpha(k_2+1)^\alpha J_{n_1,n_1n_2}) & k_1^\alpha(k_1+n_2)^\alpha J_{n_1} & (k_1+n_2)^{2\alpha}A(K_{n_1}) \end{pmatrix}$$

The following is an expression for $\mathcal{D}_\alpha(\mathcal{G}_1 \oplus \mathcal{G}_2)$'s characteristic polynomial:

$$f(\mathcal{D}_\alpha(\mathcal{G}_1 \oplus \mathcal{G}_2), x) = \begin{vmatrix} (xI_{n_1n_2} - (k_2+1)^{2\alpha}A(K_{n_1n_2})) & -k_1^\alpha(k_2+1)^\alpha J_{n_1n_2,n_1} & -((k_1+n_2)^\alpha(k_2+1)^\alpha J_{n_1n_2,n_1}) \\ -(k_1^\alpha(k_2+1)^\alpha J_{n_1,n_1n_2}) & xI_{n_1} - k_1^{2\alpha}A(K_{n_1}) & -k_1^\alpha(k_1+n_2)^\alpha J_{n_1} \\ -((k_1+n_2)^\alpha(k_2+1)^\alpha J_{n_1,n_1n_2}) & -k_1^\alpha(k_1+n_2)^\alpha J_{n_1} & (xI_{n_1} - (k_1+n_2)^{2\alpha}A(K_{n_1})) \end{vmatrix}$$

By performing, $R_i \rightarrow R_i - R_{n_1n_2}; R_j \rightarrow R_j - R_{n_1n_2+n_1}; R_l \rightarrow R_l - R_{n_1n_2+2n_1}$, where $1 \le i \le (n_1n_2-1); (n_1n_2+1) \le j \le (n_1n_2+n_1-1); (n_1n_2+n_1+1) \le l \le (n_1n_2+2n_1-1)$ we get,

$$f(\mathcal{D}_\alpha(\mathcal{G}_1 \oplus \mathcal{G}_2), x) = (x+(k_2+1)^{2\alpha})^{n_1n_2-1}(x+k_1^{2\alpha})^{n_1-1}(x+(k_1+n_2)^{2\alpha})^{n_1-1}|M|,$$

where $|M|$ is the remaining quantity of the $f(\mathcal{D}_\alpha(\mathcal{G}_1 \oplus \mathcal{G}_2), x)$ after taking out common factors. Now we get all the eigenvalues except 3 of the matrix, and the remaining 3 eigenvalues are obtained from the quotient matrix of $\mathcal{D}_\alpha(\mathcal{G}_1 \oplus \mathcal{G}_2)$. Therefore, $Q(\mathcal{D}_\alpha(\mathcal{G}_1 \oplus \mathcal{G}_2)) = \begin{pmatrix} (k_2+1)^{2\alpha}(n_1n_2-1) & k_1^\alpha(k_2+1)^\alpha n_1 & ((k_1+n_2)^\alpha(k_2+1)^\alpha n_1) \\ k_1^\alpha(k_2+1)^\alpha n_1n_2 & k_1^{2\alpha}(n_1-1) & k_1^\alpha(k_1+n_2)^\alpha n_1 \\ ((k_1+n_2)^\alpha(k_2+1)^\alpha n_1n_2) & k_1^\alpha(k_1+n_2)^\alpha n_1 & (k_1+n_2)^{2\alpha}(n_1-1) \end{pmatrix}$
The characteristic equation of the matrix $Q(\mathcal{D}_\alpha(\mathcal{G}_1 \oplus \mathcal{G}_2))$ is expressed as $x^3 - ((k_2+1)^{2\alpha}(n_1n_2-1) + (k_1^{2\alpha} + (k_1+n_2)^{2\alpha})(n_1-1))x^2 - ((n_1n_2+n_1-1)(k_1^{2\alpha} + (k_1+n_2)^{2\alpha})(k_2+1)^{2\alpha} + (2n_1-1)k_1^{2\alpha}(k_1+n_2)^{2\alpha})x - (n_1n_2+2n_1-1)k_1^{2\alpha}(k_1+n_2)^{2\alpha}(k_2+1)^{2\alpha} = 0$.
By Theorem I.1, we conclude that the remaining 3 eigenvalues $\zeta_1, \zeta_2, \zeta_3$ are the roots of the above characteristic equation of the quotient matrix. Hence, the result follows by noting that

$$Spec(\mathcal{D}_\alpha(\mathcal{G}_1 \oplus \mathcal{G}_2)) = \begin{pmatrix} -(k_2+1)^{2\alpha} & -k_1^{2\alpha} & -(k_1+n_2)^{2\alpha} & \zeta_1 & \zeta_2 & \zeta_3 \\ n_1n_2-1 & n_1-1 & n_1-1 & 1 & 1 & 1 \end{pmatrix}. \qquad \blacksquare$$

Theorem III.3. *Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ be k_1, k_2-regular graphs with n_1, n_2 vertices. Then $\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}_1 \boxtimes \mathcal{G}_2) = \sum_{i=1}^{3}|\zeta_i| + (k_1+k_2)^{2\alpha}(n_1n_2-1) + k_1^{2\alpha}(n_1-1)\left(1+(n_2+1)^{2\alpha}\right)$, where $\zeta_1, \zeta_2, \zeta_3$ are the roots of the polynomial of $x^3 - ((k_1+k_2)^{2\alpha}(n_1n_2-1) + (1+(n_2+1)^{2\alpha})k_1^{2\alpha}(n_1-1))x^2 - ((n_1n_2+n_1-1)(1+(n_2+1)^{2\alpha})k_1^{2\alpha}(k_1+k_2)^{2\alpha} + (2n_1-1)k_1^{4\alpha}(n_2+1)^{2\alpha})x - (n_1n_2+2n_1-1)k_1^{4\alpha}(k_1+k_2)^{2\alpha}(n_2+1)^{2\alpha} = 0$.*

Proof: The general degree product matrix of duplication neighborhood corona of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ be expressed as a block matrix as follows:

$$\mathcal{D}_\alpha(\mathcal{G}_1 \boxtimes \mathcal{G}_2) = \begin{pmatrix} (k_1+k_2)^{2\alpha}A(K_{n_1n_2}) & (k_1^\alpha(k_1+k_2)^\alpha(n_2+1)^\alpha J_{n_1n_2,n_1}) & (k_1^\alpha(k_1+k_2)^\alpha J_{n_1n_2,n_1}) \\ (k_1^\alpha(k_1+k_2)^\alpha(n_2+1)^\alpha J_{n_1,n_1n_2}) & k_1^{2\alpha}(n_2+1)^{2\alpha}A(K_{n_1}) & k_1^{2\alpha}(n_2+1)^\alpha J_{n_1} \\ k_1^\alpha(k_1+k_2)^\alpha J_{n_1,n_1n_2} & k_1^{2\alpha}(n_2+1)^\alpha J_{n_1} & k_1^{2\alpha}A(K_{n_1}) \end{pmatrix}$$

By performing row operations on $det(xI_{n_1n_2+2n_1} - \mathcal{D}_\alpha(\mathcal{G}_1 \boxtimes \mathcal{G}_2))$ and solving quotient matrix of $\mathcal{D}_\alpha(\mathcal{G}_1 \boxtimes \mathcal{G}_2)$, we get $Spec(\mathcal{D}_\alpha(\mathcal{G}_1 \boxtimes \mathcal{G}_2)) = \begin{pmatrix} -(k_1+k_2)^{2\alpha} & -k_1^{2\alpha}(n_2+1)^{2\alpha} & -k_1^{2\alpha} & \zeta_1 & \zeta_2 & \zeta_3 \\ n_1n_2-1 & n_1-1 & n_1-1 & 1 & 1 & 1 \end{pmatrix}$, where the quotient matrix's characteristic equation has $\zeta_1, \zeta_2, \zeta_3$ as its roots. $\blacksquare$

Theorem III.4. *Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ be k_1, k_2-regular graphs with n_1, n_2 vertices. Then $\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}_1 \boxplus \mathcal{G}_2) = \sum_{i=1}^{3}|\zeta_i| + (k_2+2)^{2\alpha}(n_2m_1-1) + (n_1-1)(k_1^{2\alpha} + (k_1+k_1n_2)^{2\alpha})$, where $\zeta_1, \zeta_2, \zeta_3$ are the roots of the polynomial of $x^3 - ((k_2+2)^{2\alpha}(n_2m_1-1) + (k_1^{2\alpha} + (k_1+k_1n_2)^{2\alpha})(n_1-1))x^2 - ((n_2m_1+n_1-1)(k_1^{2\alpha} + (k_1+k_1n_2)^{2\alpha})(k_2+2)^{2\alpha} + (2n_1-1)k_1^{4\alpha}(n_2+1)^{2\alpha})x - (n_2m_1+2n_1-1)k_1^{4\alpha}(k_2+2)^{2\alpha}(n_2+1)^{2\alpha} = 0$.*

Proof: The general degree product matrix of duplication edge corona of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ be expressed as a block matrix as follows:

$$\mathcal{D}_\alpha(\mathcal{G}_1 \boxplus \mathcal{G}_2) = \begin{pmatrix} (k_2+2)^{2\alpha}A(K_{n_2m_1}) & (k_1^\alpha(k_2+2)^\alpha(n_2+1)^\alpha J_{n_2m_1,n_1}) & (k_1^\alpha(k_2+2)^\alpha J_{n_2m_1,n_1}) \\ (k_1^\alpha(k_2+2)^\alpha(n_2+1)^\alpha J_{n_1,n_2m_1}) & k_1^{2\alpha}(n_2+1)^{2\alpha}A(K_{n_1}) & k_1^{2\alpha}(n_2+1)^\alpha J_{n_1} \\ k_1^\alpha(k_2+2)^\alpha J_{n_1,n_2m_1} & k_1^{2\alpha}(n_2+1)^\alpha J_{n_1} & k_1^{2\alpha}A(K_{n_1}) \end{pmatrix}$$

By performing row operations on $det(xI_{n_2m_1+2n_1} - \mathcal{D}_\alpha(\mathcal{G}_1 \boxplus \mathcal{G}_2))$ and solving the quotient matrix of $\mathcal{D}_\alpha(\mathcal{G}_1 \boxplus \mathcal{G}_2)$, we get $Spec(\mathcal{D}_\alpha(\mathcal{G}_1 \boxplus \mathcal{G}_2) = \begin{pmatrix} -(k_2+2)^{2\alpha} & -(k_1+k_1n_2)^{2\alpha} & -k_1^{2\alpha} & \zeta_1 & \zeta_2 & \zeta_3 \\ n_2m_1-1 & n_1-1 & n_1-1 & 1 & 1 & 1 \end{pmatrix}$, where the quotient matrix's characteristic equation has $\zeta_1, \zeta_2, \zeta_3$ as its roots. $\blacksquare$

B. $\mathcal{R}$- Graph

Definition III.5. [3] The $\mathcal{R}$-graph of $\mathcal{G}$, denoted by $\mathcal{R}_\mathcal{G}$ is the graph obtained from $\mathcal{G}$ by adding a vertex u_e and joining u_e to the end vertices of e for each $e \in E_\mathcal{G}$.

Here, consider $I_\mathcal{G}$ to be the set of newly added vertices, i.e., $I_\mathcal{G} = V_{\mathcal{R}_\mathcal{G}} \backslash V_\mathcal{G}$. and let us consider $\mathcal{G}_1$ and $\mathcal{G}_2$ be two vertex-disjoint graphs.

Definition III.6. [8] The $\mathcal{R}$-vertex corona of $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted by $\mathcal{G}_1 \odot \mathcal{G}_2$, is the graph obtained from vertex-disjoint

$\mathcal{R}_{\mathcal{G}_1}$ and $|V_{\mathcal{G}_1}|$ copies of $\mathcal{G}_2$ by joining the i-th vertex of $V_{\mathcal{G}_1}$ to every vertex in the i-th copy of $\mathcal{G}_2$.

Definition III.7. [8] The $\mathcal{R}$-edge corona of $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted by $\mathcal{G}_1 \ominus \mathcal{G}_2$, is the graph obtained from the vertex-disjoint $\mathcal{R}_{\mathcal{G}_1}$ and $|I_{\mathcal{G}_1}|$ copies of $\mathcal{G}_2$ by joining the i-th vertex of $I_{\mathcal{G}_1}$ to every vertex in the i-th copy of $\mathcal{G}_2$.

Definition III.8. [8] The $\mathcal{R}$-vertex neighborhood corona of $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted by $\mathcal{G}_1 \boxdot \mathcal{G}_2$, is the graph obtained from vertex-disjoint $\mathcal{R}_{\mathcal{G}_1}$ and $|V_{\mathcal{G}_1}|$ copies of $\mathcal{G}_2$ by joining the neighbors of the i-th vertex of $\mathcal{G}_1$ in $\mathcal{R}_{\mathcal{G}_1}$ to each vertex in the i-th copy of $\mathcal{G}_2$.

Definition III.9. [8] The $\mathcal{R}$-edge neighborhood corona of $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted by $\mathcal{G}_1 \boxminus \mathcal{G}_2$, is the graph obtained from vertex-disjoint $\mathcal{R}_{\mathcal{G}_1}$ and $|I_{\mathcal{G}_1}|$ copies of $\mathcal{G}_2$ by joining neighbors of the i-th vertex of $I_{\mathcal{G}_1}$ in $\mathcal{R}_{\mathcal{G}_1}$ to every vertex in the i-th copy of $\mathcal{G}_2$.

Theorem III.5. *Suppose that k-regular graph $\mathcal{G}$ has order n and size m. Then* $\mathcal{D}_\alpha\mathcal{E}(\mathcal{R}_\mathcal{G}) = 2^{2\alpha}\left(k^{2\alpha}(n-1) + (m-1) + \sqrt{(k^{2\alpha}(n-1)+(m-1))^2 + 4k^{2\alpha}(m+n-1)}\right)$.

Proof: The general degree product matrix of the $\mathcal{R}$-graph is expressed as a block matrix as follows:

$$\mathcal{D}_\alpha(\mathcal{R}_\mathcal{G}) = \begin{pmatrix} (2k)^{2\alpha}A(K_n) & (4k)^\alpha J_{n,m} \\ (4k)^\alpha J_{m,n} & 2^{2\alpha}A(K_m) \end{pmatrix}$$

By performing row operations on $det(xI_{n+m} - \mathcal{D}_\alpha(\mathcal{R}_\mathcal{G})$ and solving quotient matrix of $\mathcal{D}_\alpha(\mathcal{R}_\mathcal{G})$, we have the following result noting that

$$Spec(\mathcal{D}_\alpha(\mathcal{R}_\mathcal{G})) = \begin{pmatrix} -(2k)^{2\alpha} & -2^{2\alpha} & \zeta_1 & \zeta_2 \\ n-1 & m-1 & 1 & 1 \end{pmatrix},$$

where $(\zeta_1, \zeta_2) = 2^{(2\alpha-1)}(k^{2\alpha}(n-1) + (m-1)) \pm \sqrt{(k^{2\alpha}(n-1)+(m-1))^2 + 4k^{2\alpha}(m+n-1)}$. ■

Theorem III.6. *Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are k_1, k_2-regular graphs with m_1, m_2 edges and n_1, n_2 vertices. Then* $\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}_1 \odot \mathcal{G}_2) = \sum_{i=1}^{3}|\zeta_i| + (k_2+1)^{2\alpha}(n_1n_2-1) + 2^{2\alpha}(m_1-1) + (2k_1+n_2)^{2\alpha}(n_1-1)$,
where $\zeta_1, \zeta_2, \zeta_3$ are the roots of the polynomial of $x^3 - ((k_2+1)^{2\alpha}(n_1n_2-1) + 2^{2\alpha}(m_1-1) + (2k_1+n_2)^{2\alpha}(n_1-1))x^2 - ((n_1n_2+m_1-1)2^{2\alpha}(k_2+1)^{2\alpha} + (n_1n_2+n_1-1)(2k_1+n_2)^{2\alpha}(k_2+1)^{2\alpha} + (n_1+m_1-1)2^{2\alpha}(2k_1+n_2)^{2\alpha})x - (n_1n_2+n_1+m_1-1)2^{2\alpha}(k_2+1)^{2\alpha}(2k_1+n_2)^{2\alpha} = 0$.

Proof: The general degree product matrix of $\mathcal{R}$-vertex corona of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ be expressed as a block matrix as follows:

$$\mathcal{D}_\alpha(\mathcal{G}_1 \odot \mathcal{G}_2) = \begin{pmatrix} (k_2+1)^{2\alpha}A(K_{n_1n_2}) & ((k_2+1)^\alpha(2k_1+n_2)^\alpha J_{n_1n_2,n_1}) & (2^\alpha(k_2+1)^\alpha J_{n_1n_2,m_1}) \\ ((k_2+1)^\alpha(2k_1+n_2)^\alpha J_{n_1,n_1n_2}) & (2k_1+n_2)^{2\alpha}A(K_{n_1}) & (2^\alpha(2k_1+n_2)^\alpha J_{n_1,m_1}) \\ (2^\alpha(k_2+1)^\alpha J_{m_1,n_1n_2}) & 2^\alpha(2k_1+n_2)^\alpha J_{m_1,n_1} & 2^{2\alpha}A(K_{m_1}) \end{pmatrix}$$

By performing row operations on $det(xI_{n_1n_2+n_1+m_1} - \mathcal{D}_\alpha(\mathcal{G}_1 \odot \mathcal{G}_2))$ and solving quotient matrix of $\mathcal{D}_\alpha(\mathcal{G}_1 \odot \mathcal{G}_2)$, we get $Spec(\mathcal{D}_\alpha(\mathcal{G}_1 \odot \mathcal{G}_2) = \begin{pmatrix} -(k_2+1)^{2\alpha} & -2^{2\alpha} & -(2k_1+n_2)^{2\alpha} & \zeta_1 & \zeta_2 & \zeta_3 \\ n_1n_2-1 & m_1-1 & n_1-1 & 1 & 1 & 1 \end{pmatrix}$,
where the quotient matrix's characteristic equation has $\zeta_1, \zeta_2, \zeta_3$ as its roots. ■

Theorem III.7. *Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are k_1, k_2-regular graphs with m_1, m_2 edges and n_1, n_2 vertices. Then* $\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}_1 \ominus \mathcal{G}_2) = \sum_{i=1}^{3}|\zeta_i| + (k_2+1)^{2\alpha}(n_2m_1-1) + (2k_1)^{2\alpha}(n_1-1) + (2+n_2)^{2\alpha}(m_1-1)$,
where $\zeta_1, \zeta_2, \zeta_3$ are the roots of the polynomial of $x^3 - ((k_2+1)^{2\alpha}(n_2m_1-1) + 2^{2\alpha}k_1^{2\alpha}(n_1-1) + (2+n_2)^{2\alpha}(m_1-1))x^2 - ((n_2m_1+n_1-1)2^{2\alpha}k_1^{2\alpha}(k_2+1)^{2\alpha} + (n_2m_1+m_1-1)(2+n_2)^{2\alpha}(k_2+1)^{2\alpha} + (n_1+m_1-1)2^{2\alpha}k_1^{2\alpha}(2+n_2)^{2\alpha})x - (n_2m_1+n_1+m_1-1)2^{2\alpha}k_1^{2\alpha}(k_2+1)^{2\alpha}(2+n_2)^{2\alpha} = 0$.

Proof: The general degree product matrix of $\mathcal{R}$-edge corona of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ be expressed as a block matrix as follows:

$$\mathcal{D}_\alpha(\mathcal{G}_1 \ominus \mathcal{G}_2) = \begin{pmatrix} (k_2+1)^{2\alpha}A(K_{n_2m_1}) & ((2k_1)^\alpha(k_2+1)^\alpha J_{n_2m_1,n_1}) & ((2+n_2)^\alpha(k_2+1)^\alpha J_{n_2m_1,m_1}) \\ ((2k_1)^\alpha(k_2+1)^\alpha J_{n_1,n_2m_1}) & (2k_1)^{2\alpha}A(K_{n_1}) & ((2k_1)^\alpha(2+n_2)^\alpha J_{n_1,m_1}) \\ ((2+n_2)^\alpha(k_2+1)^\alpha J_{m_1,n_2m_1}) & ((2k_1)^\alpha(2+n_2)^\alpha J_{m_1,n_1}) & (2+n_2)^{2\alpha}A(K_{m_1}) \end{pmatrix}$$

By performing row operations on $det(xI_{n_2m_1+n_1+m_1} - \mathcal{D}_\alpha(\mathcal{G}_1 \ominus \mathcal{G}_2))$ and solving quotient matrix of $\mathcal{D}_\alpha(\mathcal{G}_1 \ominus \mathcal{G}_2)$, we get $Spec(\mathcal{D}_\alpha(\mathcal{G}_1 \ominus \mathcal{G}_2) = \begin{pmatrix} -(k_2+1)^{2\alpha} & -2^{2\alpha}k_1^{2\alpha} & -(2+n_2)^{2\alpha} & \zeta_1 & \zeta_2 & \zeta_3 \\ n_2m_1-1 & n_1-1 & m_1-1 & 1 & 1 & 1 \end{pmatrix}$,
where the quotient matrix's characteristic equation has $\zeta_1, \zeta_2, \zeta_3$ as its roots. ■

Theorem III.8. *Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are k_1, k_2-regular graphs with m_1, m_2 edges and n_1, n_2 vertices. Then* $\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}_1 \boxdot \mathcal{G}_2) = \sum_{i=1}^{3}|\zeta_i| + (k_2+2k_1)^{2\alpha}(n_1n_2-1) + k_1^{2\alpha}(2+n_2)^{2\alpha}(n_1-1) + 2^{2\alpha}(n_2+1)^{2\alpha}(m_1-1)$,
where $\zeta_1, \zeta_2, \zeta_3$ are the roots of the polynomial of $x^3 - ((k_2+2k_1)^{2\alpha}(n_1n_2-1) + k_1^{2\alpha}(2+n_2)^{2\alpha}(n_1-1) + 2^{2\alpha}(n_2+1)^{2\alpha}(m_1-1))x^2 - (k_1^{2\alpha}(2+n_2)^{2\alpha}(k_2+2k_1)^{2\alpha}(n_1n_2+n_1-1) + 2^{2\alpha}(k_2+2k_1)^{2\alpha}(n_2+1)^{2\alpha}(n_1n_2+m_1-1) + 2^{2\alpha}k_1^{2\alpha}(n_2+1)^{2\alpha}(2+n_2)^{2\alpha}(n_1+m_1-1))x - 2^{2\alpha}k_1^{2\alpha}(k_2+2k_1)^{2\alpha}(2+n_2)^{2\alpha}(n_2+1)^{2\alpha}(n_1n_2+n_1+m_1-1) = 0$.

Proof: The general degree product matrix of $\mathcal{R}$-vertex neighborhood corona of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ be expressed as a block matrix as follows:

$$\mathcal{D}_\alpha(\mathcal{G}_1 \boxdot \mathcal{G}_2) = \begin{pmatrix} (k_2+2k_1)^{2\alpha}A(K_{n_1n_2}) & ((k_1(k_2+2k_1)(2+n_2))^\alpha J_{n_1n_2,n_1}) & ((2(n_2+1)(k_2+2k_1))^\alpha J_{n_1n_2,m_1}) \\ (k_1(k_2+2k_1)(2+n_2))^\alpha J_{n_1,n_1n_2}) & (k_1(2+n_2))^{2\alpha}A(K_{n_1}) & ((2k_1(n_2+1)(2+n_2))^\alpha J_{n_1,m_1}) \\ ((2(n_2+1)(k_2+2k_1))^\alpha J_{m_1,n_1n_2}) & ((2k_1(n_2+1)(2+n_2))^\alpha J_{m_1,n_1}) & (2(n_2+1))^{2\alpha}A(K_{m_1}) \end{pmatrix}$$

By performing row operations on $det(xI_{n_1n_2+n_1+m_1} - \mathcal{D}_\alpha(\mathcal{G}_1 \boxdot \mathcal{G}_2))$ and solving quotient matrix of $\mathcal{D}_\alpha(\mathcal{G}_1 \boxdot \mathcal{G}_2)$, we get $Spec(\mathcal{D}_\alpha(\mathcal{G}_1 \boxdot \mathcal{G}_2) = \begin{pmatrix} -(k_2+2k_1)^{2\alpha} & -(k_1(2+n_2))^{2\alpha} & -(2(n_2+1))^{2\alpha} & \zeta_1 & \zeta_2 & \zeta_3 \\ n_1n_2-1 & n_1-1 & m_1-1 & 1 & 1 & 1 \end{pmatrix}$,
where the quotient matrix's characteristic equation has $\zeta_1, \zeta_2, \zeta_3$ as its roots. ■

Theorem III.9. *Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are k_1, k_2-regular graphs with m_1, m_2 edges and n_1, n_2 vertices. Then* $\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}_1 \boxminus \mathcal{G}_2) = \sum_{i=1}^{3}|\zeta_i| + (k_2+2)^{2\alpha}(n_2m_1-1) + (2k_1+k_1n_2)^{2\alpha}(n_1-1) + 2^{2\alpha}(m_1-1)$,
where $\zeta_1, \zeta_2, \zeta_3$ are the roots of the polynomial of $x^3 - ((k_2+2)^{2\alpha}(m_1n_2-1) + k_1^{2\alpha}(2+n_2)^{2\alpha}(n_1-1) + 2^{2\alpha}(m_1-1))x^2 - (k_1^{2\alpha}(2+n_2)^{2\alpha}(k_2+2)^{2\alpha}(m_1n_2+n_1-1) + 2^{2\alpha}(k_2+2)^{2\alpha}(m_1n_2+m_1-1) + 2^{2\alpha}k_1^{2\alpha}(2+n_2)^{2\alpha}(n_1+m_1-1))x - 2^{2\alpha}k_1^{2\alpha}(k_2+2)^{2\alpha}(n_2+1)^{2\alpha}(m_1n_2+n_1+m_1-1) = 0$.

Proof: The general degree product matrix of $\mathcal{R}$-edge neighborhood corona of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ be expressed as a block matrix as follows:

$$\mathcal{D}_\alpha(\mathcal{G}_1 \boxminus \mathcal{G}_2) = \begin{pmatrix} (k_2+2)^{2\alpha} A(K_{n_2 m_1}) & ((k_2+2)^\alpha (2k_1+k_1 n_2)^\alpha J_{n_2 m_1, n_1}) & (2^\alpha (k_2+2)^\alpha J_{n_2 m_1, m_1}) \\ ((k_2+2)^\alpha (2k_1+k_1 n_2)^\alpha J_{n_1, n_2 m_1}) & (2k_1+k_1 n_2)^{2\alpha} A(K_{n_1}) & (2^\alpha (2k_1+k_1 n_2)^\alpha J_{n_1, m_1}) \\ 2^\alpha (k_2+2)^\alpha J_{m_1, n_2 m_1} & 2^\alpha (2k_1+k_1 n_2)^\alpha J_{m_1, n_1} & 2^{2\alpha} A(K_{m_1}) \end{pmatrix}$$

By performing row operations on $det(xI_{n_2 m_1 + n_1 + m_1} - \mathcal{D}_\alpha(\mathcal{G}_1 \boxminus \mathcal{G}_2))$ and solving quotient matrix of $\mathcal{D}_\alpha(\mathcal{G}_1 \boxminus \mathcal{G}_2)$, we have the following result noting that $Spec(\mathcal{D}_\alpha(\mathcal{G}_1 \boxminus \mathcal{G}_2) = \begin{pmatrix} -(k_2+2)^{2\alpha} & -(2k_1+k_1 n_2)^{2\alpha} & -2^{2\alpha} & \zeta_1 & \zeta_2 & \zeta_3 \\ n_2 m_1 - 1 & n_1 - 1 & m_1 - 1 & 1 & 1 & 1 \end{pmatrix}$, where the quotient matrix's characteristic equation has $\zeta_1, \zeta_2, \zeta_3$ as its roots. ■

C. Middle graph

Definition III.10. [6] The middle graph $M_\mathcal{G}$ of a graph $\mathcal{G}$ is the graph in which the vertex set is $V_\mathcal{G} \cup E_\mathcal{G}$ and two vertices are adjacent if and only if either they are adjacent edges of $\mathcal{G}$ or one is a vertex of $\mathcal{G}$ and the other is an edge incident with it.

Motivated by the corona operations based on the $\mathcal{R}$ -graph and duplicate graph, we define four new corona operations based on the middle graph as follows.

Definition III.11. The middle vertex corona of $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted by $\mathcal{G}_1 \circledcirc \mathcal{G}_2$, is the graph obtained from vertex-disjoint $M_{\mathcal{G}_1}$ and $|V_{\mathcal{G}_1}|$ copies of $\mathcal{G}_2$ by joining the i-th vertex of $V_{\mathcal{G}_1}$ to every vertex in the i-th copy of $\mathcal{G}_2$.

Definition III.12. The middle edge corona of $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted by $\mathcal{G}_1 \ominus \mathcal{G}_2$, is the graph obtained from vertex-disjoint $M_{\mathcal{G}_1}$ and $|I_{\mathcal{G}_1}|$ copies of $\mathcal{G}_2$ by joining the i-th vertex of $I_{\mathcal{G}_1}$ to every vertex in the i-th copy of $\mathcal{G}_2$.

Definition III.13. The middle vertex neighborhood corona of $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted by $\mathcal{G}_1 \otimes \mathcal{G}_2$, is the graph obtained from vertex-disjoint $M_{\mathcal{G}_1}$ and $|V_{\mathcal{G}_1}|$ copies of $\mathcal{G}_2$ by joining the neighbors of the i-th vertex of $\mathcal{G}_1$ in $M_{\mathcal{G}_1}$ to every vertex in the i-th copy of $\mathcal{G}_2$.

Definition III.14. The middle edge neighborhood corona of $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted by $\mathcal{G}_1 \circledast \mathcal{G}_2$, is the graph obtained from vertex-disjoint $M_{\mathcal{G}_1}$ and $|I_{\mathcal{G}_1}|$ copies of $\mathcal{G}_2$ by joining neighbors of the i-th vertex of $I_{\mathcal{G}_1}$ in $M_{\mathcal{G}_1}$ to every vertex in the i-th copy of $\mathcal{G}_2$.

Note that, $\mathcal{G}_1$ and $\mathcal{G}_2$ have n_1, n_2 vertices and m_1, m_2 edges respectively. Then, the graph $\mathcal{G}_1 \circledcirc \mathcal{G}_2$ has $n_1 + m_1 + n_1 n_2$, $\mathcal{G}_1 \ominus \mathcal{G}_2$ has $n_1 + m_1 + m_1 n_2$, $\mathcal{G}_1 \otimes \mathcal{G}_2$ contains $n_1 + m_1 + n_1 n_2$, and $\mathcal{G}_1 \circledast \mathcal{G}_2$ includes $n_1 + m_1 + m_1 n_2$ vertices. The graphs C_3, C_4 and M_{C_4} are given in Figures 1-3. The middle vertex corona, middle edge corona, middle vertex neighborhood corona and middle edge neighborhood corona operations of C_3 and C_4 have been depicted pictorially in Figures 4- 7.

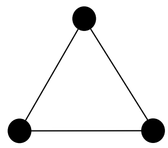

Figure 1: C_3

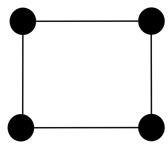

Figure 2: C_4

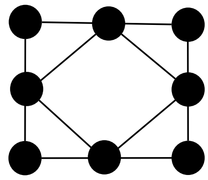

Figure 3: M_{C_4}

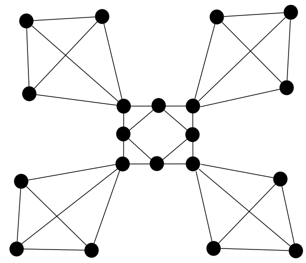

Figure 4: $C_4 \circledcirc C_3$

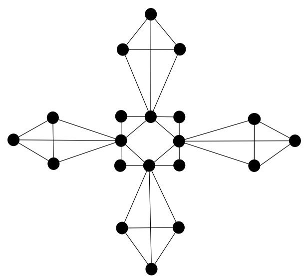

Figure 5: $C_4 \ominus C_3$

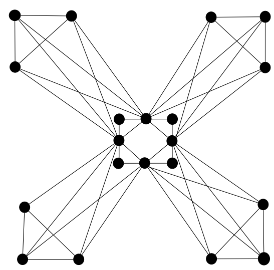

Figure 6: $C_4 \otimes C_3$

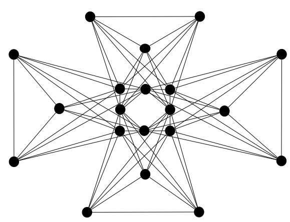

Figure 7: $C_4 \circledast C_3$

Theorem III.10. *Let $\mathcal{G}$ be a k-regular graph of order n and* *size m. Then*

$$\mathcal{D}_\alpha\mathcal{E}(M_G) = k^{2\alpha}\Bigg((n-1) + 2^{2\alpha}(m-1) + \sqrt{((n-1) + 2^{2\alpha}(m-1))^2 + 2^{2\alpha+2}(n+m-1)}\Bigg).$$

Proof: The middle graph's general degree product matrix can be represented as a block matrix in this way:

$$\mathcal{D}_\alpha(M_\mathcal{G}) = \begin{pmatrix} k^{2\alpha}A(K_n) & 2^\alpha k^{2\alpha}J_{n,m} \\ 2^\alpha k^{2\alpha}J_{m,n} & (2k)^{2\alpha}A(K_m) \end{pmatrix}.$$

By performing row operations on $det(xI_{n+m} - \mathcal{D}_\alpha(M_\mathcal{G})$ and solving quotient matrix of $\mathcal{D}_\alpha(M_\mathcal{G})$, we have

$$Spec(\mathcal{D}_\alpha(M_\mathcal{G})) = \begin{pmatrix} -k^{2\alpha} & -(2k)^{2\alpha} & \lambda_1 & \lambda_2 \\ (n-1) & (m-1) & 1 & 1 \end{pmatrix},$$

where $(\lambda_1, \lambda_2) = \dfrac{k^{2\alpha}}{2}\Bigg((n-1) + 2^{2\alpha}(m-1) \pm \sqrt{((n-1) + 2^{2\alpha}(m-1))^2 + 2^{2\alpha+2}(n+m-1)}\Bigg)$.

Hence, the proof follows. ■

Next, we obtain the general degree product energy of $\mathcal{G}_1 \circledcirc \mathcal{G}_2$, $\mathcal{G}_1 \ominus \mathcal{G}_2$, $\mathcal{G}_1 \otimes \mathcal{G}_2$ and $\mathcal{G}_1 \circledast \mathcal{G}_2$, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are regular graphs.

Theorem III.11. *Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are k_1, k_2-regular graphs with m_1, m_2 edges and n_1, n_2 vertices. Then* $\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}_1 \circledcirc \mathcal{G}_2) = \sum_{i=1}^{3}|\zeta_i| + (k_2+1)^{2\alpha}(n_1n_2-1) + (k_1+n_2)^{2\alpha}(n_1-1) + (2k_1)^{2\alpha}(m_1-1)$,
where $\zeta_1, \zeta_2, \zeta_3$ are the roots of the polynomial of $x^3 - ((k_2+1)^{2\alpha}(n_1n_2-1) + (k_1+n_2)^{2\alpha}(n_1-1) + (2k_1)^{2\alpha}(m_1-1))x^2 - ((n_1n_2+m_1-1)(2k_1)^{2\alpha}(k_2+1)^{2\alpha} + (n_1n_2+n_1-1)(k_1+n_2)^{2\alpha}(k_2+1)^{2\alpha} + (n_1+m_1-1)(2k_1)^{2\alpha}(k_1+n_2)^{2\alpha})x - (n_1n_2+n_1+m_1-1)(2k_1)^{2\alpha}(k_2+1)^{2\alpha}(k_1+n_2)^{2\alpha} = 0$.

Proof: The general degree product matrix of the middle vertex corona of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ be expressed as a block matrix as follows:

$$\mathcal{D}_\alpha(\mathcal{G}_1 \circledcirc \mathcal{G}_2) = \begin{pmatrix} (k_2+1)^{2\alpha}A(K_{n_1n_2}) & ((k_2+1)^\alpha(k_1+n_2)^\alpha J_{n_1n_2,n_1}) & ((2k_1)^\alpha(k_2+1)^\alpha J_{n_1n_2,m_1}) \\ ((k_2+1)^\alpha(k_1+n_2)^\alpha J_{n_1,n_1n_2}) & (k_1+n_2)^{2\alpha}A(K_{n_1}) & ((2k_1)^\alpha(k_1+n_2)^\alpha J_{n_1,m_1}) \\ (2k_1)^\alpha(k_2+1)^\alpha J_{m_1,n_1n_2} & ((2k_1)^\alpha(k_1+n_2)^\alpha J_{m_1,n_1}) & (2k_1)^{2\alpha}A(K_{m_1}) \end{pmatrix}$$

By performing row operations on $det(xI_{n_1n_2+n_1+m_1} - \mathcal{D}_\alpha(\mathcal{G}_1 \circledcirc \mathcal{G}_2))$ and solving quotient matrix of $\mathcal{D}_\alpha(\mathcal{G}_1 \circledcirc \mathcal{G}_2)$, we have $Spec(\mathcal{D}_\alpha(\mathcal{G}_1 \circledcirc \mathcal{G}_2) = \begin{pmatrix} -(k_2+1)^{2\alpha} & -(2k_1)^{2\alpha} & -(k_1+n_2)^{2\alpha} & \zeta_1 & \zeta_2 & \zeta_3 \\ n_1n_2-1 & m_1-1 & n_1-1 & 1 & 1 & 1 \end{pmatrix}$,
where the quotient matrix's characteristic equation has $\zeta_1, \zeta_2, \zeta_3$ as its roots. ■

Theorem III.12. *Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are k_1, k_2-regular graphs with m_1, m_2 edges and n_1, n_2 vertices. Then* $\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}_1 \ominus \mathcal{G}_2) = \sum_{i=1}^{3}|\zeta_i| + (k_2+1)^{2\alpha}(n_2m_1-1) + k_1^{2\alpha}(n_1-1) + (2k_1+n_2)^{2\alpha}(m_1-1)$,
where $\zeta_1, \zeta_2, \zeta_3$ are the roots of the polynomial of $x^3 - ((k_2+1)^{2\alpha}(n_2m_1-1) + k_1^{2\alpha}(n_1-1) + (2k_1+n_2)^{2\alpha}(m_1-1))x^2 - ((n_2m_1+n_1-1)k_1^{2\alpha}(k_2+1)^{2\alpha} + (n_2m_1+m_1-1)(2k_1+n_2)^{2\alpha}(k_2+1)^{2\alpha} + (n_1+m_1-1)k_1^{2\alpha}(2k_1+n_2)^{2\alpha})x - (n_2m_1+n_1+m_1-1)k_1^{2\alpha}(k_2+1)^{2\alpha}(2k_1+n_2)^{2\alpha} = 0$.

Proof: The general degree product matrix of the middle edge corona of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ be expressed as a block matrix as follows:

$$\mathcal{D}_\alpha(\mathcal{G}_1 \ominus \mathcal{G}_2) = \begin{pmatrix} (k_2+1)^{2\alpha}A(K_{n_2m_1}) & k_1^\alpha(k_2+1)^\alpha J_{n_2m_1,n_1} & ((2k_1+n_2)^\alpha(k_2+1)^\alpha J_{n_2m_1,m_1}) \\ k_1^\alpha(k_2+1)^\alpha J_{n_1,n_2m_1} & k_1^{2\alpha}A(K_{n_1}) & (k_1^\alpha(2k_1+n_2)^\alpha J_{n_1,m_1}) \\ ((2k_1+n_2)^\alpha(k_2+1)^\alpha J_{m_1,n_2m_1}) & k_1^\alpha(2k_1+n_2)^\alpha J_{m_1,n_1} & (2k_1+n_2)^{2\alpha}A(K_{m_1}) \end{pmatrix}$$

By performing row operations on $det(xI_{n_2m_1+n_1+m_1} - \mathcal{D}_\alpha(\mathcal{G}_1 \ominus \mathcal{G}_2))$ and solving quotient matrix of $\mathcal{D}_\alpha(\mathcal{G}_1 \ominus \mathcal{G}_2)$, we have $Spec(\mathcal{D}_\alpha(\mathcal{G}_1 \ominus \mathcal{G}_2) = \begin{pmatrix} -(k_2+1)^{2\alpha} & -k_1^{2\alpha} & -(2k_1+n_2)^{2\alpha} & \zeta_1 & \zeta_2 & \zeta_3 \\ n_2m_1-1 & n_1-1 & m_1-1 & 1 & 1 & 1 \end{pmatrix}$,
where the quotient matrix's characteristic equation has $\zeta_1, \zeta_2, \zeta_3$ as its roots. ■

Theorem III.13. *Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are k_1, k_2-regular graphs with m_1, m_2 edges and n_1, n_2 vertices. Then* $\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}_1 \otimes \mathcal{G}_2) = \sum_{i=1}^{3}|\zeta_i| + (k_1+k_2)^{2\alpha}(n_1n_2-1) + k_1^{2\alpha}(n_1-1) + (2(k_1+n_2))^{2\alpha}(m_1-1)$,
where $\zeta_1, \zeta_2, \zeta_3$ are the roots of the polynomial of $x^3 - ((k_1+k_2)^{2\alpha}(n_1n_2-1) + k_1^{2\alpha}(n_1-1) + (2(k_1+n_2))^{2\alpha}(m_1-1))x^2 - ((k_1(k_1+k_2))^{2\alpha}(n_1n_2+n_1-1) + (k_1+k_2)^{2\alpha}(2(k_1+n_2))^{2\alpha}(n_1n_2+m_1-1) + (2k_1(k_1+n_2))^{2\alpha}(n_1+m_1-1))x - (2k_1(k_1+k_2)(k_1+n_2))^{2\alpha}(n_1n_2+n_1+m_1-1) = 0$.

Proof: The general degree product matrix of the middle vertex neighborhood corona of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ be expressed as a block matrix as follows:

$$\mathcal{D}_\alpha(\mathcal{G}_1 \otimes \mathcal{G}_2) = \begin{pmatrix} (k_1+k_2)^{2\alpha}A(K_{n_1n_2}) & ((k_1(k_1+k_2))^\alpha J_{n_1n_2,n_1}) & ((2(k_1+k_2)(k_1+n_2))^\alpha J_{n_1n_2,m_1}) \\ (k_1(k_1+k_2))^\alpha J_{n_1,n_1n_2} & k_1^{2\alpha}A(K_{n_1}) & (2k_1(k_1+n_2))^\alpha J_{n_1,m_1} \\ ((2(k_1+k_2)(k_1+n_2))^\alpha J_{m_1,n_1n_2}) & ((2k_1(k_1+n_2))^\alpha J_{m_1,n_1}) & (2(k_1+n_2))^{2\alpha}A(K_{m_1}) \end{pmatrix}$$

By performing row operations on $det(xI_{n_1n_2+n_1+m_1} - \mathcal{D}_\alpha(\mathcal{G}_1 \otimes \mathcal{G}_2))$ and solving quotient matrix of $\mathcal{D}_\alpha(\mathcal{G}_1 \otimes \mathcal{G}_2)$, we have $Spec(\mathcal{D}_\alpha(\mathcal{G}_1 \otimes \mathcal{G}_2) = \begin{pmatrix} -(k_1+k_2)^{2\alpha} & -k_1^{2\alpha} & -(2(k_1+n_2))^{2\alpha} & \zeta_1 & \zeta_2 & \zeta_3 \\ n_1n_2-1 & n_1-1 & m_1-1 & 1 & 1 & 1 \end{pmatrix}$,
where the quotient matrix's characteristic equation has $\zeta_1, \zeta_2, \zeta_3$ as its roots. ■

Theorem III.14. *Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are k_1, k_2-regular graphs with m_1, m_2 edges and n_1, n_2 vertices. Then* $\mathcal{D}_\alpha\mathcal{E}(\mathcal{G}_1 \circledast \mathcal{G}_2) = \sum_{i=1}^{3}|\zeta_i| + (k_2+2k_1)^{2\alpha}(n_2m_1-1) + (k_1+k_1n_2)^{2\alpha}(n_1-1) + (2k_1)^{2\alpha}(m_1-1)$,
where $\zeta_1, \zeta_2, \zeta_3$ are the roots of the polynomial of $x^3 - ((k_2+2k_1)^{2\alpha}(m_1n_2-1) + (k_1+k_1n_2)^{2\alpha}(n_1-1) + (2k_1)^{2\alpha}(m_1-1))x^2 - ((k_1(1+n_2)(k_2+2k_1))^{2\alpha}(m_1n_2+n_1-1) + (2k_1(k_2+2k_1))^{2\alpha}(m_1n_2+m_1-1) + (2k_1^2(n_2+1))^{2\alpha}(n_1+m_1-1))x - (2k_1^2(k_2+2k_1)(n_2+1))^{2\alpha}(m_1n_2+n_1+m_1-1) = 0$.

Proof: The general degree product matrix of the middle edge neighborhood corona of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ be expressed as a block matrix as follows:

$$\mathcal{D}_\alpha(\mathcal{G}_1 \circledast \mathcal{G}_2) = \begin{pmatrix} (k_2+2k_1)^{2\alpha}A(K_{n_2m_1}) & ((k_1(k_2+2k_1)(1+n_2))^\alpha J_{n_2m_1,n_1}) & ((2k_1(k_2+2k_1))^\alpha J_{n_2m_1,m_1}) \\ ((k_1(k_2+2k_1)(1+n_2))^\alpha J_{n_1,n_2m_1}) & (k_1(1+n_2))^{2\alpha}A(K_{n_1}) & ((2k_1^2(1+n_2))^\alpha J_{n_1,m_1}) \\ ((2k_1(k_2+2k_1)^\alpha) J_{m_1,n_2m_1}) & (2k_1^2(1+n_2))^\alpha J_{m_1,n_1} & (2k_1)^{2\alpha}A(K_{m_1}) \end{pmatrix}$$

By performing row operations on $det(xI_{n_2m_1+n_1+m_1} - \mathcal{D}_\alpha(\mathcal{G}_1 \circledast \mathcal{G}_2))$ and solving quotient matrix of

$\mathcal{D}_\alpha(\mathcal{G}_1 \circledast \mathcal{G}_2)$, we have $Spec(\mathcal{D}_\alpha(\mathcal{G}_1 \circledast \mathcal{G}_2) = \begin{pmatrix} -(k_2+2k_1)^{2\alpha} & -(k_1(1+n_2))^{2\alpha} & -(2k_1)^{2\alpha} & \zeta_1 & \zeta_2 & \zeta_3 \\ n_2m_1-1 & n_1-1 & m_1-1 & 1 & 1 & 1 \end{pmatrix}$, where the quotient matrix's characteristic equation has $\zeta_1, \zeta_2, \zeta_3$ as its roots. ■

IV. Conclusion

In this article, the general degree product energy of a new family of graphs from an existing base graph such as the duplication graph, $\mathcal{R}$-graph, and middle graph are discussed. One can try to obtain the general degree product energy of a new family of graphs by taking the base graph as the central graph.

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