Delta-Color Complement of a Graph

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***Abstract*—Let Ģ $= (V, E)$ be a finite, simple colored graph of order n and size m. In this paper, we define δ-color complement and δ'-color complement of graph as follows. For any two points h and i of Ģ with $d(h) = d(i)$, remove the edge between h and i in Ģ and add the edges of $\overline{Ģ}$ joining the vertices h and i. Resultant graph is called δ-color complement of Ģ. For any two points h and i of Ģ with $d(h) \neq d(i)$, delete the edge between h and i in Ģ and add corresponding edge of $\overline{Ģ}$ between h and i. The graph thus obtained is called δ'-color complement of Ģ. This paper presents different properties of δ-color and δ'-color complements, examining their connectivity, self-color complementary, and edge counts in specific graphs.**

***Index Terms*—δ-color complement, δ'-color complement, self-color complementary, degree sequence, proper coloring.**

I. Introduction

All the graphs considered here are finite, undirected, no loops and multiple edges. As usual $n = |V|$ and $m = |E|$, denote the number of vertices and edges in a graph G, respectively. To describe adjacency relations, consider two vertices h and i in G. If $hi \in E$, then h is adjacent to i, denoted as $h \sim i$. Otherwise, if h and i are not adjacent, we write $h \nsim i$. The complement of a graph G, denoted by $\overline{G}$, has the same vertex set as G, and two vertices h and i are adjacent in G if and only if they are non-adjacent in $\overline{G}$. A graph G is called self-complementary if it is isomorphic to $\overline{G}$. The degree of a vertex v in a graph, denoted by $d(v)$, represents the number of edges incident to v.

Graph Coloring is one of the most common optimization problems in the field of computer science and mathematics. There are various real life applications of graph coloring. Proper coloring is to color vertices of a graph with minimum color in such a way that no two vertices that share an edge are assigned the same color. The Chromatic number of graph G is the minimum number of colors required to properly color a graph and is represented by $\chi(G)$. For notation and graph theory terminology we generally follow [4] and [7].

Graph partitioning is a widely studied problem with applications in computing, engineering, and network science. It is widely used in clustering, route optimization, biological networks, and high-performance computing. A partition of G consists of disjoint subsets of $V(G)$. One notable type is the equal-degree partition, where all vertices of the same degree are grouped together. In 2022, Pai et. al. [6] introduced a variant of graph complements in which the complement takes place only among the vertices of the same degree. They called it as δ-complement graph G_δ of G.

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II. Definitions and Preliminaries

In this article, we denote Ģ $= (V, E)$ as a colored graph. The complement of Ģ, denoted by $\overline{Ģ}$, preserves the same vertex coloring and satisfies the following properties,

- If h and i are non-adjacent in Ģ and have different colors, then they become adjacent in $\overline{Ģ}$.
- If h and i are non-adjacent in Ģ and share the same color, they remain non-adjacent in $\overline{Ģ}$.
- All adjacent pairs in Ģ become non-adjacent in $\overline{Ģ}$.

Definition 1. *Let Ģ be a (n, m) colored graph. For any two points h and i in V(Ģ) such that $d(h) = d(i)$, delete the edge between h and i in Ģ and insert the corresponding edge from the $\overline{Ģ}$ between them. The graph thus obtained is called the delta-color complement or δ-color complement of Ģ and is denoted by $Ģ_{\delta_c}$.*

i. *A graph Ģ is δ-self color complementary (δ-s.c.c) if Ģ $\cong Ģ_{\delta_c}$.*

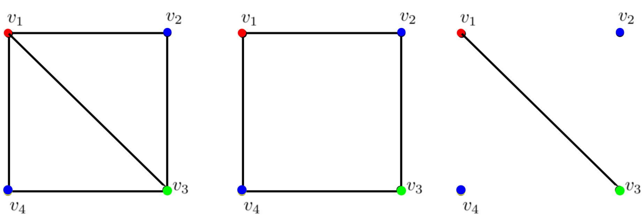
ii. *A graph Ģ is δ-co-self color complementary (δ-co.s.c.c) if $Ģ_{\delta_c} \cong \overline{Ģ}$.*

Definition 2. *For any two vertices h and i of Ģ with $d(h) \neq d(i)$, delete the edge between h and i in Ģ and add corresponding edge of $\overline{Ģ}$ between h and i. The graph thus obtained is called δ'-color complement of Ģ, is denoted by $Ģ_{\delta'_c}$.*

i. *A graph Ģ is δ'-self color complementary (δ'-s.c.c) if Ģ $\cong Ģ_{\delta'_c}$.*

ii. *A graph Ģ is δ'-co-self color complementary (δ'-co.s.c.c) if $Ģ_{\delta'_c} \cong \overline{Ģ}$.*

Example:



(a) Ģ (b) $Ģ_{\delta_c}$ (c) $Ģ_{\delta'_c}$

Fig. 1
$Ģ_{\delta_c}$ and $Ģ_{\delta'_c}$ with respect to equal degree partition of V(Ģ).

Definition 3. *[7] The star graph S_n of order n, is a tree on n vertices where one vertex has a degree of $n-1$, and all other vertices have degree 1.*

Definition 4. *[3] A unicyclic graph $\tau = S_n + e$ is formed by adding an edge between two pendant vertices of the star graph S_n of order n.*

Definition 5. *[3] A bicyclic graph* $\Upsilon = S_n + 2e$ *is obtained by taking the unicyclic graph* $\tau = S_n + e$ *and adding one more edge between a pendant vertex and a vertex of degree two in* τ.

Definition 6. *[5] A double star* $S(p,q)$ *is obtained by connecting the central vertices of two star graphs* S_p *and* S_q *with an additional edge.*

Definition 7. *[1] A wheel graph* W_n *is formed by joining a single universal vertex to every vertex of a cycle, resulting in a graph of order* n.

Definition 8. *[5] The friendship graph* F_p *is a planar graph with* $2p+1$ *vertices and* $3p$ *edges, constructed by joining* p *copies of the cycle* C_3 *at a single common vertex, which serves as the universal vertex.*

Definition 9. *[5] The p-book graph* B_p, *is defined as* $B_p = S_{p+1} \times P_2$, *where* S_{p+1} *is a star graph with* $p+1$ *vertices and* P_2 *is a path with two vertices. The graph has* $2p+2$ *vertices.*

Definition 10. *[5] The Triangular book graph* $B(3,p)$, *consists of* p *triangles sharing a single common edge. It is a planar graph with* $p+2$ *vertices and* $2p+1$ *edges.*

Definition 11. *[5] A graph amalgamation is a relationship between two graphs. Amalgamations can provide a way to reduce a graph to a simpler graph while keeping certain structure intact.*

Let $\{G_1, G_2, \dots, G_l\}$ *be a finite collection of graphs and each* G_i *has a fixed vertex* v_{0i} *called as a terminal vertex. The amalgamation* $Amal(v_{0i}, G_i)$ *is formed by taking all the* G_i*'s and identifying their terminals.*

Definition 12. *[2] The tadpole graph,* $T_{p,q}$ *consists of a cycle with* p *vertices* ($p \geq 3$) *joined by a bridge to a path with* q *vertices.*

III. Main Results

The following results illustrates the isomorphism between the δ-color complements of a graph.

Proposition 1. *For any graph* $\c{G}$,

i. $\overline{\c{G}_{\delta_c}} \cong (\overline{\c{G}})_{\delta_c}$.
ii. $\overline{\c{G}_{\delta'_c}} \cong (\overline{\c{G}})_{\delta'_c}$.

Proof: *Consider two vertices* x *and* y *in* $\c{G}$.

i. *Let* $x \sim y$ *in* $\overline{\c{G}_{\delta_c}}$
$\iff$ $x \nsim y$ *in* $\c{G}_{\delta_c}$
$\iff$ $d(x) = d(y)$, *and* $x \sim y$ *in* $\c{G}$ *if* $c(x) \neq c(y)$, *or* $d(x) \neq d(y)$, *and* $x \nsim y$ *in* $\c{G}$
$\iff$ $d(x) = d(y)$, *and* $x \nsim y$ *in* $\overline{\c{G}}$, *or* $d(x) \neq d(y)$, *and* $x \sim y$ *in* $\overline{\c{G}}$ *if* $c(x) \neq c(y)$
$\iff$ $x \sim y$ *in* $(\overline{\c{G}})_{\delta_c}$ *if* $c(x) \neq c(y)$.
ii. *Proof technique is same as (i).*

■

As a consequence of Proposition 1, we have

Corollary 1. *For any graph* $\c{G}$,

i. $\overline{\c{G}_{\delta'_c}} \cong \c{G}_{\delta_c}$.
ii. $\overline{\c{G}_{\delta_c}} \cong \c{G}_{\delta'_c}$.

Corollary 2. *For any graph* $\c{G}$,

i. $\c{G}_{\delta'_c} \cong \c{G} \iff \overline{\c{G}_{\delta'_c}} \cong \overline{\c{G}}$.
ii. $\c{G}_{\delta_c} \cong \c{G} \iff \overline{\c{G}_{\delta_c}} \cong \overline{\c{G}}$.

Corollary 3. *For any graph* $\c{G}$,

i. $\c{G}_{\delta_c} \cong \c{G} \iff \c{G}_{\delta'_c} \cong \overline{\c{G}}$.
ii. $\c{G}_{\delta'_c} \cong \c{G} \iff \c{G}_{\delta_c} \cong \overline{\c{G}}$.

Proof: *Let* $\c{G}$ *be a graph.*

i. $\c{G}_{\delta_c} \cong \c{G} \iff \overline{\c{G}_{\delta_c}} \cong \overline{\c{G}} \iff \c{G}_{\delta'_c} \cong \overline{\c{G}}$ *(from Corollary 1).*
ii. $\c{G}_{\delta'_c} \cong \c{G} \iff \overline{\c{G}_{\delta'_c}} \cong \overline{\c{G}} \iff \c{G}_{\delta_c} \cong \overline{\c{G}}$ *(from Colollary 1).*

■

Proposition 2. *For any graph* $\c{G}$,

i. $(\overline{\c{G}})_{\delta_c} \cong \overline{\c{G}} \iff (\overline{\c{G}})_{\delta'_c} \cong \c{G}$.
ii. $(\overline{\c{G}})_{\delta'_c} \cong \overline{\c{G}} \iff (\overline{\c{G}})_{\delta_c} \cong \c{G}$.

Proof:

i. *Suppose* $(\overline{\c{G}})_{\delta_c} \cong \overline{\c{G}}$, *then* $\overline{(\overline{\c{G}})_{\delta_c}} \cong \overline{(\overline{\c{G}})} \cong \c{G}$.
Assume that $\c{H} = \overline{\c{G}}$. *Then* $\overline{(\c{H})_{\delta_c}} \cong \c{G}$.
From Corollary 1,
$\c{H}_{\delta'_c} \cong \c{G} \Rightarrow (\overline{\c{G}})_{\delta'_c} \cong \c{G}$.
Conversely, $(\overline{\c{G}})_{\delta'_c} \cong \c{G}$
$\overline{(\overline{\c{G}})_{\delta'_c}} \cong \overline{\c{G}} \Rightarrow \overline{(\c{H})_{\delta'_c}} \cong \overline{\c{G}}$
From Corollary 1,
$\c{H}_{\delta_c} \cong \overline{\c{G}} \Rightarrow (\overline{\c{G}})_{\delta_c} \cong \overline{\c{G}}$.
Similarly we can prove (ii).

■

Proposition 3. *For any graph* $\c{G}$ *of order* n *and partition of its vertex set into* k *parts, then*

i. $\c{G}_{\delta_c} \cong \c{G}$ *and* $\c{G}_{\delta'_c} \cong \overline{\c{G}}$, *for* $k = n$.
ii. $\c{G}_{\delta_c} \cong \overline{\c{G}}$ *and* $\c{G}_{\delta'_c} \cong \c{G}$, *for* $k = 1$.

Proposition 4. *Let* $\c{G}$ *be a complete multipartite graph* $K_{n_1,n_2,\dots,n_k}$, *with a vertex partition* $\{V_1, V_2, \dots, V_k\}$, *where* $|V_i| \neq |V_j|$ *for all* $i \neq j$. *Then,* $\c{G}$ *is a* δ*-self color complementary and* δ'*-co-self color complementary graph.*

Proof: *Let* $\c{G}$ *be a complete multipartite graph with a vertex partition* $\{V_1, V_2, \dots, V_k\}$. *Note that all vertices in each* V_i *share the same color, forming a distinct color class for each partite. Since a vertex partition by color classes leads to the* δ*-color complement of* $\c{G}$ *being isomorphic to* $\c{G}$ *and* δ'*-color complement of* $\c{G}$ *being isomorphic to* $\overline{\c{G}}$, *it follows that* $\c{G}$ *is a* δ*-self color complementary and* δ'*-co-self color complementary graph.* ■

Remark 1.

i. *All regular graphs are* δ'*-s.c.c and* δ*-co-s.c.c.*
ii. *For a graph* $\c{G} \cong K_n$, $\chi(\c{G}_{\delta'_c}) = n$ *and* $\chi(\c{G}_{\delta_c}) = 1$.

Theorem 1. *Let* $\c{G}$ *be a connected graph of order* n. *The graph* $\c{G}_{\delta_c}$ *is disconnected under any of the following conditions:*

i. $\c{G} \cong K_n$.
ii. $\c{G}$ *is a* $(n-2)$*-regular graph.*
iii. $\c{G}$ *contains a vertex* v *of degree* l *such that all of its neighbors also have degree* l.

Proof: *Let* $\c{G}$ *be a connected graph.*

i. *If* $\c{G}$ *is a complete graph, every pair of vertices are adjacent, then* $\c{G}_{\delta_c}$ *is completely disconnected by the definition of* δ*-color complement graph.*

ii. *Suppose $Ç$ is a connected $(n-2)$-regular graph. Then n must be even. Since the non-adjacent vertices shares same color, $Ç_{\delta_c}$ is disconnected.*
iii. *Suppose a vertex $v \in Ç$ is only adjacent to l number of vertices of degree l. Then v is non-adjacent to any of the vertices in $Ç_{\delta_c}$, resulting in a disconnected graph.*

■

Theorem 2. *Let $Ç$ be a graph with n vertices. Then, $Ç_{\delta'_c}$ is disconnected if*

i. *Let $d_1, d_2, \ldots, d_m$ be all the distinct values of the degrees of the vertices in $Ç$. Partition $V(Ç)$ into non-empty sets $V_{d_1}, V_{d_2}, \ldots, V_{d_m}$ such that there exist at least one V_{d_i} whose all the vertices are adjacent to every vertex of $V - V_{d_i}$.*
ii. *$Ç$ is a disconnected regular graph.*
iii. *There exists a vertex $v \in V(Ç)$ such that all of its neighbors have degrees different from that of v.*

Proof:

i. *Suppose every vertex with degree d_i is connected all the vertices of degree other than d_i, then in $Ç_{\delta'_c}$, the vertices of degree d_i will have no connections to those of different degrees, resulting in a disconnected graph.*
ii. *Let $Ç$ be a disconnected regular graph. It is known that for a regular graph $Ç$, $Ç \cong Ç_{\delta'_c}$, and consequently, $Ç_{\delta'_c}$ is disconnected.*
iii. *Let v be a vertex in $V(Ç)$ such that its neighborhood consists of vertices $u \in V(Ç)$ with a different degree than v, $d(u) \neq d(v)$. In this case, v becomes an isolated vertex in $Ç_{\delta'_c}$, resulting a disconnected graph.*

■

Proposition 5. *Let $Ç$ be a Complete graph K_n. Then, $m(Ç_{\delta'_c}) = \frac{n(n-1)}{2}$ and $m(Ç_{\delta_c}) = \phi$.*

Proposition 6. *Let $Ç$ be a Complete multipartite graph $K_{n_1,n_2,\ldots,n_k}$, where $n_1 \neq n_2 \neq \ldots \neq n_k$. Then, $m(Ç_{\delta'_c}) = \phi$ and $m(Ç_{\delta_c}) = \sum_{1 \le i < j \le k} n_i n_j$.*

Theorem 3. *Let $Ç$ be a Path $P_n, n \geq 2$. Then,*

$$m(Ç_{\delta'_c}) = \begin{cases} 2n-7, & \text{if } n \text{ is even,} \\ 2n-6, & \text{if } n \text{ is odd} \end{cases} \quad \text{and}$$

$$m(Ç_{\delta_c}) = \begin{cases} \frac{n^2-8n+28}{4}, & \text{if } n \text{ is even,} \\ \frac{n^2-8n+23}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Let $Ç = P_n$ be the path on n vertices, where $n \geq 2$ and $v_1, v_2, \ldots, v_n$ be the vertices of P_n. The vertices v_1 and v_n are of degree 1, while all other vertices have degree 2. Thus, there exists a partition of V into two disjoint subsets V_1 and V_2, where $V_1 = \{v_1, v_n\}$ and $V_2 = V \setminus V_1$.

Case i. *When n is even.*

The subgraph induced by V_2 forms a path P_{n-2}, which contains $n-3$ edges, as a path with n vertices has $n-1$ edges. In $(Ç)_{\delta'_c}$, the two edges between V_1 to V_2, namely (v_1, v_2) and (v_{n-1}, v_n), are removed. Additionally, $n-4$ new edges are introduced. As a result, the total number of edges in $Ç_{\delta'_c}$ is given by $(n-3) + (n-4) = 2n-7$.

In $Ç_{\delta_c}$, the two edges connecting V_1 to V_2 remain unchanged. Since the 2 vertices in $\langle V_1 \rangle$ have different colors, an edge is introduced between them. Within $\langle V_2 \rangle$, taking the color complement, we know for a path graph P_n of even order, the number of edges is given by $(\frac{n}{2} - 1)^2$. Since V_2 contains $n-2$ vertices, applying the same formula to P_{n-2} gives, $(\frac{n-2}{2} - 1)^2$. Thus, the total number of edges in $Ç_{\delta_c}$ is $\frac{n^2-8n+28}{4}$.

Case ii. *When n is odd.*

Similarly, in $Ç_{\delta'_c}$, the subgraph induced by V_2 forms a path P_{n-2}, which contains $n-3$ edges. The two edges between V_1 and V_2, namely (v_1, v_2) and (v_{n-1}, v_n) are removed, and instead, $n-3$ new edges are added. Thus, the total number of edges in $Ç_{\delta'_c}$ is $(n-3) + (n-3) = 2n-6$.

In $Ç_{\delta_c}$, the two edges connecting V_1 and V_2 remain unchanged. Since all vertices in V_1 share the same color, no adjacency appears between them. Within $\langle V_2 \rangle$, taking the color complement, for a path graph P_n of odd order, the number of edges is $\frac{n^2-4n+3}{4}$. Since V_2 contains $n-2$ vertices, applying the same formula to P_{n-2} modifies the expression to, $\frac{(n-2)^2-4(n-2)+3}{4}$. Therefore, the total number of edges in $Ç_{\delta_c}$ is $\frac{n^2-8n+23}{4}$. ■

Theorem 4. *Let $Ç$ be a Cycle C_n with $n \geq 3$ vertices. Then, $m(Ç_{\delta'_c}) = n$ and*

$$m(G_{\delta_c}) = \begin{cases} \frac{n(n-4)}{4}, & n \text{ is even,} \\ \frac{n^2-2n-3}{4}, & n \text{ is odd.} \end{cases}$$

Proof: Let $Ç$ be a Cycle C_n consisting of n vertices. Since every vertex has degree 2, by the Proposition 3, $Ç$ is both δ'-self color complement and δ-co-self color complement. Thus, the δ'-color complement of $Ç$ contains n edges.

In the δ-color complement, when n is even, the vertices are divided into two color classes, each containing $\frac{n}{2}$ vertices. Each vertex then connects to $\frac{n-4}{2}$ vertices. Since color class containing $\frac{n}{2}$ vertices, the number of edges in $Ç_{\delta_c}$ is $\frac{n(n-4)}{4}$.

When n is odd, the chromatic number is 3, meaning one vertex has a unique color, while the remaining $n-1$ vertices are evenly split between two color classes, each containing $\frac{n-1}{2}$ vertices. Uniquely colored vertex connects to $n-3$ other vertices, while the remaining vertices form $\frac{(n-3)^2}{4}$ edges. Therefore, the total number of edges in $Ç_{\delta_c}$ is $\frac{n^2-2n-3}{4}$. ■

Theorem 5. *Let $Ç$ be a Wheel graph $W_n, n > 4$ and $m = 2(n-1)$. Then, $m(Ç_{\delta'_c}) = n-1$ and*

$$m(Ç_{\delta_c}) = \begin{cases} \frac{(n-1)(n+1)-3}{4}, & n \text{ is even,} \\ \frac{(n-1)^2}{4}, & n \text{ is odd.} \end{cases}$$

Proof: Let $Ç$ be a Wheel graph W_n with $n > 4$. The central vertex has a degree of $n-1$, while all other vertices have a degree 3. Let V_1, V_2 be 2 partite set, where V_1 contains the central vertex, and $V_2 = V \setminus V_1$.

In $Ç_{\delta'_c}$, the edges connecting V_1 and V_2 are removed. As a result, $\langle V_1 \rangle$ consists of an isolated vertex and $\langle V_2 \rangle$ forms a cycle C_{n-1}. Therefore, the graph $Ç_{\delta'_c}$ is the disjoint union of K_1 and C_{n-1}, meaning the total number of edges is $n-1$.

In $Ç_{\delta_c}$, edges between V_1 and V_2 remain intact, contributing $n-1$ edges. When n is even, $\langle V_2 \rangle$ forms a cycle C_{n-1}, which has an odd number of vertices. From Theorem 4, for a cycle of odd order, the number of edges in $Ç_{\delta_c}$ is $\frac{(n-1)^2-2(n-1)-3}{4}$. Thus, the total number of edges in the δ-color complement is $(n-1) + \frac{(n-1)^2-2(n-1)-3}{4} = \frac{(n-1)(n+1)-3}{4}$.

When n is odd, $\langle V_2\rangle$ forms a cycle C_{n-1}, which has an even number of vertices. From Theorem 4, for a cycle of even order, the number of edges in $Ç_{\delta_c}$ is $\frac{(n-1)(n-5)}{4}$. Therefore, the total number of edges in the δ-color complement is $(n-1)+\frac{(n-1)(n-5)}{4}=\frac{(n-1)^2}{4}$. ■

Theorem 6. *Let $Ç$ be a Star graph $S_n, n\geq 2$ and $m=n-1$. Then, $m(Ç_{\delta'_c})=\phi$ and $m(Ç_{\delta_c})=n-1$.*

Proof: *Let $Ç$ be a Star graph $S_n, n\geq 2$. The central vertex has a degree of $n-1$, with the remaining $n-1$ vertices being pendant. Thus, we get two partites V_1 and V_2, where V_1 contains only the central vertex, and $V_2=V\setminus V_1$ consists of the remaining vertices. $\langle V_2\rangle$ is a totally disconnected graph, as all its vertices belong to the same color class.*
In $Ç_{\delta'_c}$, all $n-1$ edges between V_1 and V_2 are removed. Consequently, $Ç_{\delta'_c}$ is a null graph with no edges. In $Ç_{\delta_c}$, the $n-1$ edges between V_1 to V_2 are remain unchanged. Since $\langle V_2\rangle$ is completely disconnected where all vertices share the same color, no additional edges are introduced. Thus, $Ç_{\delta_c}\cong Ç$, and the number of edges in $Ç_{\delta_c}$ remains $n-1$. ■

Theorem 7. *Let $Ç$ be a Double star graph $S(p,q)$. Then,*

$$m(Ç_{\delta'_c})=\begin{cases}\phi, & \text{if } p\neq q,\\ 1, & \text{if } p=q\end{cases}\quad \text{and}$$
$$m(Ç_{\delta_c})=\begin{cases}pq, & \text{if } p\neq q,\\ p^2-1, & \text{if } p=q.\end{cases}$$

Proof: *Let $Ç$ be a double star graph $S(p,q)$ with $p+q$ vertices and $p+q-1$ edges.*

When $p=q$, the two central vertices each have a degree p, while the remaining vertices are pendant vertices. We get 2 partites, where V_1 consists of the two central vertices and V_2 contains the $p+q-2$ pendant vertices. In $Ç_{\delta'_c}$, the $2(p-1)$ edges connecting V_1 and V_2 are removed. $\langle V_1\rangle$ forms K_2, while $\langle V_2\rangle$ having no edges, forms a null graph. Thus, $Ç_{\delta'_c}$ is the disjoint union of K_2 and a null graph of order $2(p-1)$, resulting in a total of one edge.
In $Ç_{\delta_c}$, $2(p-1)$ edges between V_1 and V_2 are remain unaltered. Since the $2(p-1)$ pendant vertices are distributed into two color classes, each containing $p-1$ vertices, every pendant vertex in one class becomes adjacent to all $p-1$ pendant vertices in the other class. Consequently, the total number of edges in $Ç_{\delta_c}$ is $2(p-1)+(p-1)^2=p^2-1$.

When $p\neq q$, the two central vertices have different degrees, resulting in 3 partites. The partites V_1 and V_2 each contain one central vertex, while V_3 consists of the remaining $p+q-2$ pendant vertices. The graph $Ç_{\delta'_c}$ is a null graph because the vertex in V_1 and the $q-1$ vertices connected to V_2 share the same color. Likewise, the vertex in V_2 and the $p-1$ vertices connected to V_1 share the same color.
In $Ç_{\delta_c}$, all $p+q-1$ edges are remain unchanged. In $\langle V_3\rangle$, each of the $p-1$ vertices is connected to the $q-1$ vertices, as they belong to different color classes. Therefore, the total number of edges in $Ç_{\delta_c}$ is $p+(q-1)+(p-1)(q-1)=pq$. ■

Theorem 8. *Let $Ç=S_n+e$ is a unicyclic graph of order n. Then, $m(Ç_{\delta'_c})=n-2$ and $m(Ç_{\delta_c})=n-1$.*

Proof: *Let $Ç=S_n+e$ is a unicyclic graph of order n with $\chi=3$. Since an edge is added between two pendant vertices, we get the partites V_1, V_2 and V_3, where V_1 contains the central vertex of degree $n-1$, V_2 consists of $n-3$ pendant vertices, and V_3 includes the two vertices of degree 2.*
In $Ç_{\delta'_c}$, the $n-1$ edges connecting V_1 to V_2 and V_3 are removed. Since $\langle V_3\rangle$ forms K_2 and one of the vertices in V_3 has a unique color, the $n-3$ vertices in V_2 are connected to this uniquely colored vertex, forming S_{n-1}. As a result, $Ç_{\delta'_c}$ is the disjoint union of K_1 and S_{n-1}, yielding $n-2$ edges.
In $Ç_{\delta_c}$, since all vertices in V_2 share the same color, no additional edges are formed within $\langle V_2\rangle$. The $n-1$ edges connecting V_1 to V_2 and V_3 remain unaltered. Thus, $Ç_{\delta_c}\cong S_n$, that is, it has $n-1$ edges. ■

Theorem 9. *Let $Ç=S_n+2e$ is a bicyclic graph of order n. Then, $m(Ç_{\delta'_c})=n-4$ and $m(Ç_{\delta_c})=n+1$.*

Proof: *Let $Ç=S_n+2e$ be a bicyclic graph of order n obtained by adding an edge between pendant vertex and a vertex of degree two in the graph S_n+e with chromatic number $\chi=3$. Since the partition is with respect to equal degree, adding two edges to S_n results in four distinct vertex degrees, giving $\{V_1,V_2,V_3,V_4\}$, where V_1 contains the central vertex of degree $n-1$, V_2 consists of $n-4$ pendant vertices, V_3 includes the two vertices of degree 2, and V_4 contains a uniquely colored vertex with degree 3.*
In $Ç_{\delta'_c}$, the $n-1$ edges connecting V_1 to V_2, V_3 and V_4 are removed, isolating V_1 as K_1. Since the vertices in V_2 and V_3 share the same color, no edges are added between them. However, the uniquely colored vertex in V_4 connects to all $n-4$ vertices in V_2, forming S_{n-3}. As a result, $Ç_{\delta'_c}$ is the disjoint union of $3K_1$ and S_{n-3}, containing $n-4$ edges.
For $Ç_{\delta_c}$, since all vertices in V_2 and V_3 share the same color, no additional edges are introduced within $\langle V_2\rangle$ and $\langle V_3\rangle$. The $n-1$ edges connecting V_1 to V_2, V_3, and V_4 remain unchanged. Thus, $Ç_{\delta_c}\cong Ç$, meaning it retains $n+1$ edges. ■

Theorem 10. *Let $Ç$ be a Friendship graph F_p, $p\geq 2$ has $2p+1$ vertices and $3p$ edges. Then, $m(Ç_{\delta'_c})=p$ and $m(Ç_{\delta_c})=p(p+1)$.*

Proof: *Let $Ç$ be a Friendship graph F_p with $p\geq 2$, where the central vertex has a degree of $2p$ while all other vertices have a degree 2. We get 2 distinct vertex degrees, yielding the partition $\{V_1,V_2\}$, where V_1 contains only the central vertex, and $V_2=V\setminus V_1$, consists of all remaining vertices.*
In $Ç_{\delta'_c}$, all edges connecting V_1 and V_2 are removed. This results in $\langle V_1\rangle$ being an isolated vertex, while $\langle V_2\rangle$ consists of p disjoint edges, forming pK_2. Thus, the graph $Ç_{\delta'_c}$ is the disjoint union of K_1 and pK_2, with a total of p edges.
In $Ç_{\delta_c}$, the edges connecting V_1 and V_2 remain unchanged, contributing $2p$ edges. Within $\langle V_2\rangle$, the vertices are divided into two color classes, each containing p vertices. Since each vertex in one class connects to all $(p-1)$ vertices in the other class, an additional $p(p-1)$ edges are formed. Therefore, the total number of edges in $Ç_{\delta_c}$ is $2p+p(p-1)=p(p+1)$. ■

Theorem 11. *Let $Ç$ be the Book graph $B_p=S_{p+1}\times P_2$, of order $2p+2$. Then, $m(Ç_{\delta'_c})=p+1$ and $m(Ç_{\delta_c})=p(p+1)$.*

Proof: *Let $Ç$ be the Book graph $B_p=S_{p+1}\times P_2$ which has an order of $2p+2$ and a chromatic number $\chi=2$. The two central vertices each have a degree $p+1$, while all remaining vertices have a degree 2. Thus, two distinct vertex*

degrees are partitioned into $\{V_1, V_2\}$, where V_1 contains the two central vertices $\langle V_1\rangle = K_2$, and $V_2 = V \setminus V_1$, consists of all remaining vertices, each of degree 2.
In $Ç_{\delta_c'}$, all edges connecting V_1 to V_2 are removed. As a result, $\langle V_1\rangle$ remains as K_2, while $\langle V_2\rangle$ consists of p disjoint edges, forming pK_2. Therefore, the graph $Ç_{\delta_c'}$ is the disjoint union of $(p+1)K_2$, containing a total of $p+1$ edges.
In $Ç_{\delta_c}$, the edges between V_1 and V_2 remain unchanged, contributing $2p$ edges. Within $\langle V_2\rangle$, the vertices are grouped into two color classes, each consisting of p vertices. Each vertex in one class is linked to all $(p-1)$ vertices in the other class, resulting in an additional $p(p-1)$ edges. Therefore, the total number of edges in $Ç_{\delta_c}$ is $2p+p(p-1) = p(p+1)$. ■

Theorem 12. *Let Ç be a Triangular book graph $B(3,p)$ of order $p+2$ and $2p+1$ edges. Then, $m(Ç_{\delta_c'}) = 1$ and $m(Ç_{\delta_c}) = 2p$.*

Proof: Let Ç be a Triangular book graph $B(3,p)$ with $\chi = 3$. The graph consists of two vertices with a degree of $p+1$, while the remaining p vertices are of degree 2, such that V_1 contains the two vertices of degree $p+1$, forming $\langle V_1\rangle = K_2$, and V_2 consists of the remaining p vertices of degree 2.
In $Ç_{\delta_c'}$, all edges connecting V_1 to V_2 are removed. As a result, $\langle V_1\rangle$ remains as K_2, while $\langle V_2\rangle$ consists of p isolated vertices, forming pK_1. Therefore, the resulting graph $Ç_{\delta_c'}$ is the disjoint union of K_2 and pK_1, containing a single edge.
In $Ç_{\delta_c}$, the $2p$ edges between V_1 and V_2 remain unchanged. Since all vertices in $\langle V_2\rangle$ share the same color, no additional edges are introduced within V_2. However, the edge within $\langle V_1\rangle$ is removed. As a result, $Ç_{\delta_c}$ is isomorphic to the complete bipartite graph, $K_{(2,p)}$, containing a total of $2p$ edges. ■

Theorem 13. *Let Ç be the graph obtained by the amalgamation of l copies of K_p, $Amal(l, K_p)$, for $l \geq 2$ and of order $l(p-1)+1$. Then, $m(Ç_{\delta_c'}) = l\binom{p-1}{2}$.*

Proof: Let Ç be the graph formed by the amalgamation of l copies of K_p, denoted as $Amal(l, K_p)$, for $l \geq 2$. In this graph, the central vertex has a degree of $l(p-1)$, while all other vertices have a degree of $p-1$. We define 2 partites V_1 and V_2, where V_1 contains only the central vertex, and $V_2 = V \setminus V_1$, contains all the remaining vertices.
In $Ç_{\delta_c'}$, all edges between V_1 to V_2 are removed. As a result, $\langle V_1\rangle$ becomes an isolated vertex, while $\langle V_2\rangle$ consists of l disjoint copies of K_{p-1}. Therefore, the graph $Ç_{\delta_c'}$ is the disjoint union of K_1 and $l(K_{p-1})$, containing a total of $l\binom{p-1}{2}$ edges. ■

Theorem 14. *Let Ç be a Tadpole graph $T_{p,q}$, consisting of $p+q$ vertices and edges. The path P_q is attached to the cycle C_p at vertex v. Let $c(v)$ denotes the color of the vertex $v \in C_p$. Then,*
Case i. *When the path P_q is attached to a vertex v with $c(v) = c_1$ or $c(v) = c_2$,*

$$m(Ç_{\delta_c'}) = \begin{cases} 2(p+q)-7, & \text{if } p \text{ is odd and } q \text{ is even,} \\ 2(p+q)-8, & \text{if both } p \text{ and } q \text{ are} \\ & \text{either even or odd,} \\ 2(p+q)-9, & \text{if } p \text{ is even and } q \text{ is odd.} \end{cases}$$

Case ii. *When the path P_q is attached to a vertex v with $c(v) = c_3$,*

$$m(Ç_{\delta_c'}) = \begin{cases} \frac{5(p+q)-20}{2}, & \text{if } q \text{ is odd,} \\ \frac{5(p+q)-19}{2}, & \text{if } q \text{ is even.} \end{cases}$$

Proof: Let Ç be a Tadpole graph $T_{p,q}$, where the path P_q is attached to a single vertex of the cycle C_p. In this graph, one vertex of P_q has degree 1, while the vertex in C_p that is attached to a vertex of P_q has degree 3, and all other vertices have degree 2. Thus, there exists 3 partites V_1, V_2, V_3, where V_1 contains the vertex of degree 1, V_2 contains the vertex of degree 3, and V_3 consists of the remaining $p+q-2$ vertices, each having degree 2. The cycle C_p is colored with colors c_1, c_2 and if C_p is of odd order, it is colored with c_1, c_2 and c_3. The path P_q is colored using colors c_1 and c_2.

Case i. *When the path P_q is attached to a vertex with $c(v) = c_1$ or $c(v) = c_2, v \in C_p$.*
If p is odd and q is even, then $\chi = 3$. Since the vertices in V_1 and V_2 share the same color, no additional edges are introduced between them. The subgraph induced by V_3 contains $p+q-4$ edges, while $p+q-3$ edges connect V_3 to V_1 and V_2. Therefore, the total number of edges in $Ç_{\delta_c'}$ is $2(p+q)-7$.
When both p and q are odd, the chromatic number remains 3. In this case, the vertices in V_1 and V_2 are assigned different colors, leading to the addition of an edge between them. We have $m(\langle V_1\rangle) = \phi$, $m(\langle V_2\rangle) = \phi$ and $m(\langle V_3\rangle) = p+q-4$. Additionally, there are $p+q-5$ edges connect V_3 to V_1 and V_2. Thus, the total edge count in $Ç_{\delta_c'}$ has $2(p+q)-8$.
When both p and q are even, the chromatic number reduces to $\chi = 2$. Since the vertices in V_1 and V_2 share the same color, no edges are introduced between them. The subgraph induced by V_3 contains $p+q-4$ edges, with an additional $p+q-4$ edges connecting V_3 to V_1 and V_2, leading to a total edge count of $2(p+q)-8$ in $Ç_{\delta_c'}$.
For the case, where p is even and q is odd, $\chi = 2$. The vertices in V_1 and V_2 are assigned different colors, introducing an edge between them. The induced subgraph $\langle V_3\rangle$ consists of $p+q-4$ edges, while $p+q-6$ edges are added from V_3 to V_1 and V_2. Consequently, the total number of edges in $Ç_{\delta_c'}$ is $2(p+q)-9$.

Case ii. *When the path P_q is attached to a vertex with $c(v) = c_3, v \in C_p$.*
When p is odd, the value of q can be either odd or even, leading to two possible cases. In both situations, the chromatic number remains 3, and since the vertices in V_1 and V_2 have different colors, an edge is introduced between them. In the graph $Ç_{\delta_c'}$, $m(\langle V_3\rangle) = p+q-4$. Additionally, $\frac{p-1}{2}$ edges are formed between the vertices of $C_p \in V_3$ and V_1. The number of edges connecting the vertices of $P_q \in V_3$ to V_1 depends on whether q is odd or even. If q is odd, there are $\frac{q-3}{2}$ such edges, whereas if q is even, there are $\frac{q-2}{2}$ such edges. Moreover, from V_2, there are $q-2$ edges linked to the vertices of $P_q \in V_3$, while $p-3$ edges connecting it to the vertices of $C_p \in V_3$. Hence, the total number of edges in $Ç_{\delta_c'}$ is $\frac{5(p+q)-20}{2}$, when q is odd and $\frac{5(p+q)-19}{2}$, when q is even. ■

IV. Conclusion

In this paper, we introduced the concepts of δ-color complement and δ'-color complement of a finite, simple colored graph and explored their structural properties. We examined their connectivity, self-color complementary nature, and edge modifications in specific graphs. These transformations provide new insights into graph coloring and complement operations, offering potential applications in network theory and combinatorial optimization.

References

[1] C. Adiga, E. Sampathkumar, M. A. Sriraj, A. S. Shrikanth, "Color energy of a graph," *Proceedings of the Jangjeon Mathematical Society*, vol. 16, no. 3, pp. 335–351, 2013.

[2] J. DeMaio, and J. Jacobson, "Fibonacci number of the tadpole graph," *Electronic Journal of Graph Theory and Applications*, vol. 2, pp. 129-138, 2014.

[3] S. D'Souza, H. J. Gowtham, P. G Bhat, "Energy of Generalized Complements of a Graph," *Engineering Letters*, vol. 28, no. 1, pp. 131-136, 2020.

[4] F. Harary, "Graph Theory," *CRC Press*, 2018.

[5] S. Nayak, S. D'Souza, P. G. Bhat, "Color Laplacian Energy of generalised complements of a Graph," *Engineering Letters*, vol. 29, no. 4, pp. 1502–1510, 2021.

[6] A. Pai, H. A. Rao, S. D'Souza, P. G. Bhat and S. Upadhyay, "δ-complement of a graph," *Mathematics*, vol. 10, no. 8, pp. 1203, 2022.

[7] D. B. West, "Introduction to Graph Theory, 2nd Edition," *Prentice Hall*, 2001.