Almost Periodic Dynamics of a Delayed Hematopoiesis Model with Feedback Control

Keke Min, Lian Duan, and Ping Chu

***Abstract*—The article introduces a delayed hematopoiesis model incorporating feedback control. By utilizing the differential inequality technique from functional analysis, several sufficient conditions for model permanence are demonstrated. Furthermore, by employing the Banach fixed-point theorem, the existence of the unique almost periodic solutions of the model were established. Additionally, by constructing appropriate Lyapunov functions and applying analysis techniques, sufficient criteria for the global asymptotic stability of the considered the almost periodic model have been presented. Finally, the feasibility of the consequence was confirmed through analog simulation.**

***Index Terms*—Hematopoiesis model, feedback control, permanence, almost periodic solutions, global asymptotic stability.**

I. Introduction

To characterize red blood cells dynamics, Mackey and Glass proposed a nonlinear delayed differential equation in 1977 [1]:

$$N_1'(t) = -\alpha N_1(t) + \frac{\beta}{1 + N_1^n(t-d)}, \quad \alpha, \beta, d, n > 0, \quad (1)$$

where n is a normal number, $N_1(t)$ represents the amount of mature cells present in the bloodstream, α means the death rate of circulating cells, and d stands for the maturation delay from the production of immature cells in the bone marrow to their entry into the circulatory system; the term $\frac{\beta}{1+N_1^n(t-d)}$ characterizes the influx of cells from the stem cell compartment into the circulation, which depends on the cell count $N_1(t-d)$ at the time $t-d$. Subsequently, the rich dynamics of model (1) and its modified versions have garnered extensive scholarly interest, as evidenced by numerous studies (refer to [2]–[6] and the associated literature).

In real-world ecosystems, unforeseen external forces frequently affect ecosystems, such as survival rates, causing these parameters to change. Ecology places a greater emphasis on examining the capacity of ecosystems to endure ongoing, unforeseen disruptions within a constrained temporal context. In control theory, this type of disturbance is defined as control variable. In 1993, the Logistic model was introduced to incorporate feedback control variables by Gopalsamy and Weng. The asymptotic behavior of the solutions of the Logistic model with feedback control was discussed in [7]. In [8], Fan, Yu and Wang extended the half-cycle concept to the time scale, established differential inequalities on the time scale, and studied the applications in the feedback control systems. Besides, a discrete N-species cooperative system with time delay and feedback control was proposed by Chen in [9], sufficient conditions for the system's permanence were obtained by applying the comparison theorem of the differential equation. In addition, references [10]–[19] and further research on feedback control systems in the cited literature are also consulted.

On the other hand, due to the multiple impacts of environmental factors in real life environment, environmental changes such as weather, reproduction, food supply, resource availability and other seasonal factors play an important role in ecosystem dynamics [20], [21]. Moreover, because ecosystem selectivity varies between fluctuating and stable environments. In particular, the periodically fluctuating environment significantly influences the dynamics of the model [22]–[25]. Therefore, incorporating the periodicity of biological parameters into the population model is both rational and significant. Additionally, investigating the stability of the hematopoiesis model incorporating feedback control and time-varying delay holds equal theoretical and practical significance . The system's permanence is critical for predicting long-term population dynamics, numerous authors in [8], [9], [26]–[28] have studied the persistence of biological systems. Hematopoiesis model is a kind of differential dynamic model whose coefficients depend on the state. The model often has discontinuous phenomenon, so it is easy to produce complex nonlinear behavior, and often has a certain uncertainty of state switching. Therefore, investigating the stability of the hematopoiesis model with feedback control and time-varying delay holds equal theoretical and practical significance [29]–[31]. However, to the authors' knowledge, very little work has been done to date on the persistence and global asymptotic stability of the almost periodic hematopoiesis model with feedback control and time-varying delays.

Motivated by the aforementioned discussions, in this paper, we consider the following delayed hematopoiesis model with feedback control:

$$\begin{cases} N_1'(t) = -\alpha(t)N_1(t) + \beta(t)\dfrac{1}{1+N_1^n(t-d(t))} \\ \qquad\qquad - c(t)N_1(t)N_2(t-\eta(t)), \\ N_2'(t) = -\lambda(t)N_2(t) + b(t)N_1(t-\delta(t)), \end{cases} \quad (2)$$

where $N_2(t)$ means an indirect control variable at time t, $c(t)$

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quantifies its inhibitory effect on the circulating mature cell density $N_1(t)$, the parameter $\lambda(t)$ measures the inhibition rate of $N_2(t)$ at time t, whereas $b(t)$ determines its controllability, the system incorporates two distinct delays: $\eta(t)$ for feedback regulation and $\delta(t)$ for maturation processes. Respectively, all other time-dependent parameters maintain the same biological interpretations as specified in (1). Our investigation focuses on analyzing the dynamic properties of model (2), specifically addressing three fundamental properties: (1) system permanence, (2) existence of almost periodic solutions, and (3) global asymptotic stability.

This paper is structured as follows: Section 2 outlines essential preliminaries. Section 3 elaborates on the primary findings along with their proofs. A numerical example is provided in Section 4 to verify the theoretical results.

II. Preliminaries

In this section, we deliver the key notations, definitions, and lemmas that will be needed for carrying out the subsequent analysis.

For a bounded continuous function k defined on the real numbers $\mathbb{R}$, we denote its infimum and supremum as

$$k^- = \inf_{t\in\mathbb{R}} k(t), \quad k^+ = \sup_{t\in\mathbb{R}} k(t).$$

Assume that

$$\min\{\alpha^-, \beta^-, d^-, c^-, \eta^-, \lambda^-, b^-, \delta^-\} > 0,$$
$$w = \max\{d^+, \eta^+, \delta^+\} > 0.$$

Let $\mathbb{R}(\mathbb{R}_+)$ represent the set of all real numbers (resp. nonnegative real numbers). We work with the Banach space $C = C([-w, 0], \mathbb{R})$ of continuous functions supplimented with the supremum norm $\|\cdot\|$, and set $C_+ = C([-w, 0], \mathbb{R}_+)$. For any continuous functions $N_1(t)$ and $N_2(t)$ defined on the interval $[t_0 - w, \sigma)$, where $t_0, \sigma \in \mathbb{R}$, we define

$$N_t = (N_t^1, N_t^2) \in C \times C,$$
$$N_t^1(h) = N_1(t+h),$$
$$N_t^2(h) = N_2(t+h) \quad \forall h \in [-w, 0].$$

At the same time, let $N_t(t_0, \chi)(N(t; t_0, \chi))$ denote an admissible solution that satisfies the admissible Cauchy problem (2), with $[t_0, \eta(\chi))$ representing the maximal right-interval of existence for $N_t(t_0, \chi)$ starting at t_0. Furthermore, from a biological perspective, only positive-valued solutions of the model (2) are biologically significant, we impose the following initial conditions:

$$N_{t_0} = \chi, \quad \chi = (\chi_1, \chi_2) \in C_+ \times C_+,$$
$$\chi_1(0) > 0, \quad \chi_2(0) > 0. \tag{3}$$

Definition 1. *Model (2) is termed permanent if there exist positive constants a_i and A_i such that*

$$a_i \le \liminf_{t\to+\infty} N_i(t) \le \limsup_{t\to+\infty} N_i(t) \le A_i, \quad \text{for } i = 1, 2.$$

Lemma 1. *(see [32]) If $a^* > 0$, $q > 0$ and $\frac{dN}{dt} \ge q - a^*N$. Then, for $t \ge t'$ with $N(t') > 0$, it holds that*

$$\liminf_{t\to+\infty} N(t) \ge \frac{q}{a^*}.$$

Similarly, assume that $a^* > 0$, $q > 0$ and $\frac{dN}{dt} \le q - a^*N$, then for $t \ge t'$ with $N(t') > 0$, we have

$$\limsup_{t\to+\infty} N(t) \le \frac{q}{a^*}.$$

Definition 2 (see [33]). *Suppose that the function $\phi(s) : \mathbb{R} \to \mathbb{R}$ is continuous for $s \in \mathbb{R}$. The function $\phi(s)$ is termed almost periodic on $\mathbb{R}$ if, for every $\varepsilon > 0$, the set $P(\phi, \varepsilon) = \{\xi : |\phi(s+\xi) - \phi(s)| < \varepsilon, \text{for all } s \in \mathbb{R}\}$ is relatively dense, this means that for any $\varepsilon > 0$, there exists a positive real number $m = m(\varepsilon) > 0$, such that in any interval of length $m(\varepsilon)$, there is a number $\xi = \xi(\varepsilon)$ in this interval satisfying $|\phi(s+\xi) - \phi(s)| < \varepsilon$, for all $s \in \mathbb{R}$.*

Definition 3 (see [33]). *Suppose that $l \in \mathbb{R}^n$ and $A(t)$ is an $n \times n$ continuous matrix defined on $\mathbb{R}$. Considering the linear system*

$$l'(t) = A(t)l(t), \tag{4}$$

this systemis possess an exponential dichotomy on $\mathbb{R}$ if there exist positive constants i, j, a projection matrix P, and the fundamental solution matrix $L(t)$ of (3) satisfying

$$\left\|L(t)PL^{-1}(\mu)\right\| \le ie^{-j(t-\mu)}, \quad \text{for } t \ge \mu,$$
$$\left\|L(t)(I-P)L^{-1}(\mu)\right\| \le ie^{-j(\mu-t)}, \quad \text{for } t \le \mu,$$

where I denotes the identity matrix.

Lemma 2 (see [33]). *Assume the linear system (3) possesses an exponential dichotomy. Then for any almost periodic forcing term $k(t)$, the nonhomogeneous system*

$$l'(t) = A(t)l(t) + k(t)$$

possesses a uniquely almost periodic solution $l(t)$, which can be expressed as

$$l(t) = \int_{-\infty}^{t} L(t)PL^{-1}(\mu)g(\mu)\,d\mu - \int_{t}^{+\infty} L(t)(I-P)L^{-1}(\mu)g(\mu)\,d\mu,$$

where $L(t)$ is the fundamental solution matrix and P is the dichotomy projection.

Lemma 3 (see [33]). *Suppose that $\pi_i(t)$ is an almost periodic function on $\mathbb{R}$ and satisfies*

$$A[\pi_i] = \lim_{K\to+\infty} \frac{1}{K}\int_t^{t+K} \pi_i(\mu)d\mu > 0 \quad i = 1, 2, \ldots, n.$$

Then , the diagonal linear system

$$l'(t) = \text{diag}(-\pi_1(t), -\pi_2(t), \ldots, -\pi_n(t))l(t)$$

has an exponential dichotomy on $\mathbb{R}$.

Lemma 4. *The solution $N_t(t_0, \chi) \in C_+$ for all $t \in [t_0, \eta(\chi))$, the set of $\{N_t(t_0, \chi) : t \in [t_0, \eta(\chi))\}$ is bounded, and $\eta(\chi) = +\infty$. Moreover, $N_i(t; t_0, \chi) > 0$ for all $t \ge t_0$, $i = 1, 2$.*

Proof: Since $\chi \in C_+$, according to Theorem 5.2.1 in [35], it follows that $N_t(t_0, \chi) \in C_+$ for all $t \in [t_0, \eta(\chi))$. Set $N(t) = (N_1(t), N_2(t)) = N(t; t_0, \chi)$. By integrating the second equation of (2) from t_0 to t, one has

$$N_2(t) = e^{-\int_{t_0}^t \lambda(u)\,du} N_2(t_0) + e^{-\int_{t_0}^t \lambda(u)\,du} \int_{t_0}^t e^{\int_{t_0}^s \lambda(v)\,dv} b(s)N_1(s - \delta(s))\,ds,$$

for all $t \in [t_0, \eta(\chi))$. Since $N_2(t_0) = \chi_2(0) > 0$, it shows that $N_2(t) > 0$ for all $t \in [t_0, \eta(\chi))$. Again, according to (2) and the fact that $\sup_{N_1 \geq 0} \frac{1}{1+N_1^n(t-d(t))} = 1$,we can deduce

$$\begin{aligned} N_1'(t) =& -(\alpha(t) + c(t)N_2(t-\eta(t)))N_1(t) \\ &+ \beta(t)\frac{1}{1+N_1^n(t-d(t))} \\ \leq& -\alpha(t)N_1(t) + \beta(t)\frac{1}{1+N_1^n(t-d(t))} \\ \leq& -\alpha(t)N_1(t) + \beta(t) \\ \leq& -\alpha^- N_1(t) + \beta^+. \end{aligned}$$

Integrating both sides of the first equation of (2) over the interval $[t_0, t]$ yields the following formula

$$\begin{aligned} N_1(t) =& e^{-\int_{t_0}^t o(u)\,du} N_1(t_0) \\ &+ e^{-\int_{t_0}^t o(u)\,du} \int_{t_0}^t e^{\int_{t_0}^s o(v)\,dv} \beta(s) \\ &\times \frac{1}{1+N_1^n(s-d(s))}\,ds, \end{aligned}$$

in which

$$o(u) = \alpha(u) + c(u)N_2(u-\eta(u)).$$

Consequently, since $N_1(t_0) = \chi_1(0) > 0$, it follows that $N_1(t) > 0$. Furthermore, it can be easily derived that

$$\begin{cases} N_1(t) \leq N_1(t_0)e^{-\alpha^-(t-t_0)} + \dfrac{\beta^+}{\alpha^-}(1-e^{-\alpha^-(t-t_0)}), \\ N_2(t) \leq N_2(t_0)e^{-\lambda^+(t-t_0)} + \dfrac{b^+\beta^+}{\lambda^+\alpha^-}(1-e^{-\lambda^+(t-t_0)}). \end{cases}$$

Thus, $N_1(t)$ and $N_2(t)$ are bounded on $[t_0, \eta(\chi))$. According to Theorem 2.3.1 in [35], we can conclude that $\eta(\chi) = +\infty$. The proof is complete. ■

(H1) $\alpha(t), \beta(t), d(t), c(t), \eta(t), \lambda(t), b(t), \delta(t) : \mathbb{R} \rightarrow (0, +\infty)$ are almost periodic functions.

III. Main Results

This section develops the principal findings along with their mathematical proofs. We begin with the following symbols, which streamline subsequent analysis.

Denote

$$A_1 = \frac{\beta^+}{\alpha^-}, \qquad A_2 = \frac{b^+ A_1}{\lambda^-},$$

$$a_1 = \frac{\beta^-}{(\alpha^+ + c^+ A_2)(1+A_1^n)}, \qquad a_2 = \frac{b^- a_1}{\lambda^+},$$

$$g(n) = \begin{cases} \frac{n a_1^{n-1}}{(1+a_1^n)^2}, & \text{if } 0 < n \leq 1, \\ n, & \text{if } n > 1 \end{cases}$$

Theorem 1. *Assume (H1) holds, the model* (2) *is persistent.*

Proof: Consider an arbitrary positive solution $(N_1(t), N_2(t))$ for the dynamical system defined by equations (2) and (3). Given the solution's boundedness and positivity in model (2), it is easy to establish that $\sup_{N_1 \geq 0} \frac{1}{1+N_1^n(t-d(t))} = 1$. By examining the first dynamical equation of model (2), it immediately obtain the inequality

$$\begin{aligned} N_1'(t) &\leq -\alpha(t)N_1(t) + \beta(t)\frac{1}{1+N_1^n(t-d(t))} \\ &\leq -\alpha^- N_1(t) + \beta^+. \end{aligned}$$

According to Lemma 1, we can get

$$\limsup_{t\to+\infty} N_1(t) \leq \frac{\beta^+}{\alpha^-} := A_1. \qquad (5)$$

Subsequently, for any constant $\epsilon > 0$ that is sufficiently small, there exists $T_1 > t_0$ such that

$$N_1(t) < A_1 + \varepsilon, \qquad t \in (T_1, +\infty).$$

Consequently, combining the above inequality with the second equation of model (2), we derive

$$N_2'(t) < -\lambda^- N_2(t) + b^+(A_1 + \varepsilon), \quad t \in (T_1 + w, +\infty).$$

Applying Lemma 1 again, it can be deduced that

$$\limsup_{t\to+\infty} N_2(t) \leq \frac{b^+(A_1+\varepsilon)}{\lambda^-}.$$

The above inequality suggests that when $\varepsilon \to 0$, the result inferred from the inequality implies

$$\limsup_{t\to+\infty} N_2(t) \leq \frac{b^+ A_1}{\lambda^-} := A_2. \qquad (6)$$

For any sufficiently small positive constants ε, from equation (6), there exists a specific $T_2 > T_1 + w$ such that

$$N_2(t) \leq A_2 + \varepsilon, \quad t \in [T_2, +\infty). \qquad (7)$$

Subsequently, we proceed to prove

$$\liminf_{t\to+\infty} N_1(t) > 0. \qquad (8)$$

By contradiction, suppose that $\liminf_{t\to+\infty} N_1(t) = 0$. For every $t \geq t_0$, define

$$A(t) = \max\left\{\iota : \iota \leq t, N_1(\iota) = \min_{t_0 \leq s \leq t} N_1(s)\right\}.$$

Note that $A(t) \to +\infty$, as $t \to +\infty$ and

$$\lim_{t\to+\infty} N_1(A(t)) = 0. \qquad (9)$$

By the concept of $A(t)$, one gets $N_1(A(t)) = \min_{t_0 \leq s \leq t} N_1(s)$ and $N_1'(A(t)) \leq 0$. Subsequently, By analyzing the first governing equation in model (2) and (7), we derive the following key dynamical relationship:

$$\begin{aligned} 0 \geq& N_1'(A(t)) \\ =& -\alpha(A(t))N_1(A(t)) \\ &+ \beta(A(t))\frac{1}{1+N_1^n(A(t)-\tau(A(t)))} \\ &- c(A(t))N_1(A(t))N_2(A(t)-\eta(A(t))) \\ \geq& -\alpha^+ N_1(A(t)) + \beta^- \\ &- c^+(A_2+\varepsilon)N_1(A(t)), \quad t \geq \max\{A(t), T_2 + w\}. \end{aligned}$$

A direct calculation yields

$$(\alpha^+ + c^+(A_2+\varepsilon))N_1(A(t)) \geq \beta^-,$$

Considering the positivity and boundedness of the solution, combined with (9), we deduce

$$\beta^- \leq 0.$$

Which contradicts with $\beta^- > 0$. Therefore, (8) holds.

Given $A_1 = \limsup_{t\to+\infty} N_1(t)$, we have $A_1^n \geq N_1^n(t-d(t))$, and it easily obtains that $\frac{1}{1+N_1^n(t-d(t))} \geq \frac{1}{1+A_1^n}$. From the first equation of (2) and (7), one leads to

$$\begin{aligned} N_1'(t) \geq &- \alpha^+ N_1(t) + \beta^- \frac{1}{1+A_1^n} - c^+ N_1(t)(A_2+\varepsilon) \\ =&- \left(\alpha^+ + c^+(A_2+\varepsilon)\right) N_1(t) \\ &+ \beta^- \frac{1}{1+A_1^n}, \quad \text{for } t \in [T_2+w, +\infty). \end{aligned}$$

By Lemma 1, and setting $\varepsilon \to 0$, this inequality implies

$$\begin{aligned} \liminf_{t\to+\infty} N_1(t) \geq &\frac{\beta^- \frac{1}{1+A_1^n}}{\alpha^+ + c^+ A_2} = \frac{\beta^-}{(\alpha^+ + c^+A_2)(1+A_1^n)} \\ :=& a_1. \end{aligned} \tag{10}$$

For a sufficiently small positive number ε, (10) implies that there exists $T_3 > T_2 + w$ such that

$$N_1(t) > a_1 - \varepsilon, \quad t \in [T_3, +\infty).$$

Combining this with the second equation of (2), one follows

$$N_2'(t) > -\lambda^+ N_2(t) + b^-(a_1 - \varepsilon), \quad t \in [T_3 + w, +\infty).$$

By Lemma 1 again, one obtains

$$\liminf_{t\to+\infty} N_2(t) \geq \frac{b^-(a_1-\varepsilon)}{\lambda^+}.$$

Set $\varepsilon \to 0$, this inequality entails

$$\liminf_{t\to+\infty} N_2(t) \geq \frac{b^- a_1}{\lambda^+} := a_2. \tag{11}$$

Finally, (5), (6), (10), and (11) imply that the system (2) is persistent. With these arguments, we have successfully proven the result. ■

Theorem 2. *Under hypothesis* $(H1)$ *and the additional assumption that*

(H2) $0 < a_1 \leq \frac{1}{\alpha^+}\left(\beta^- \frac{1}{1+A_1^n} - c^+ A_1 A_2\right)$;

(H3) $\max\left\{\frac{1}{\alpha^-}\left(\beta^+ g(n) + c^+(A_1+A_2)\right), \frac{b^+}{\lambda^-}\right\} < 1$

hold, the system (2) *admits a unique positive almost periodic solution within the region*

$$\begin{aligned} H^* = \Big\{ N \,|N \in AP(\mathbb{R};\mathbb{R}) \times AP(\mathbb{R};\mathbb{R}), \\ a_i \leq N_i(t) \leq A_i, t \in \mathbb{R}, i = 1,2 \Big\}. \end{aligned}$$

Proof: For arbitrary initial conditions $\chi_i \in AP(\mathbb{R};\mathbb{R})$, we investigate the almost periodic dynamical system described by:

$$\begin{cases} N_1'(t) = - \alpha(t)N_1(t) + \beta(t)\dfrac{1}{1+\chi_1^n(t-d(t))} \\ \qquad\qquad - c(t)\chi_1(t)\chi_2(t-\eta(t)), \\ N_2'(t) = - \lambda(t)N_2(t) + b(t)\chi_1(t-\delta(t)). \end{cases} \tag{12}$$

Given that $A[\alpha] > 0$ and $A[\lambda] > 0$, by applying Lemma 3, the linear system

$$\begin{cases} N_1'(t) = -\alpha(t)N_1(t), \\ N_2'(t) = -\lambda(t)N_2(t), \end{cases}$$

exhibits an exponential dichotomy on $\mathbb{R}$. Consequently, by Lemma 2, the system (12) possesses a unique almost periodic solution $N^\chi(t) = (N^{\chi_1}(t), N^{\chi_2}(t))$ expressed by

$$\begin{cases} N^{\chi_1}(t) = \displaystyle\int_{-\infty}^t e^{-\int_s^t \alpha(u)\,du}\Bigg(\beta(s)\frac{1}{1+\chi_1^n(s-d(s))} \\ \qquad\qquad -c(s)\chi_1(s)\chi_2(s-\eta(s))\Bigg)\,ds, \\ N^{\chi_2}(t) = \displaystyle\int_{-\infty}^t e^{-\int_s^t \lambda(u)\,du}\Big(b(s)\chi_1(s-\delta(s))\Big)\,ds. \end{cases} \tag{13}$$

Now, define a mapping $\Gamma : H^* \to H^*$

$$\Gamma(\chi(t)) = N^\chi(t), \quad \forall \phi \in H^*.$$

The set H^* is clearly closed in $AP(\mathbb{R};\mathbb{R}) \times AP(\mathbb{R};\mathbb{R})$. Regarding any $\chi \in H^*$, since $m_1 \leq \chi_1(t) \leq M_1$, it follows that $\sup\limits_{\chi_1 \geq 0} \frac{1}{1+\chi_1^n(t-\tau(t))} = 1$. Combining this with (13), we get

$$\begin{cases} N^{\chi_1}(t) \leq \displaystyle\int_{-\infty}^t e^{-\int_s^t \alpha(u)\,du}\beta(s)\frac{1}{1+\chi_1^n(t-d(t))}\,ds \\ \qquad\quad \leq \displaystyle\int_{-\infty}^t e^{-\int_s^t \alpha(u)\,du}\beta^+\,ds \\ \qquad\quad \leq \dfrac{\beta^+}{\alpha^-} = A_1, \\ N^{\chi_2}(t) \leq \displaystyle\int_{-\infty}^t e^{-\int_s^t \lambda(u)\,du} b^+ A_1\,ds \\ \qquad\quad \leq \dfrac{b^+A_1}{\lambda^-} = A_2. \end{cases} \tag{14}$$

Since $\chi_1 \in H^*$ and $\chi_1 \leq A_1$, we have $\frac{1}{1+\chi_1^n(s-d(s))} \geq \frac{1}{1+A_1^n}$. According to $(H2)$ and combining with (12), we get

$$\begin{cases} N^{\chi_1}(t) \geq \displaystyle\int_{-\infty}^t e^{-\int_s^t \alpha(u)\,du}\Bigg(\beta(s)\frac{1}{1+A_1^n} \\ \qquad\qquad -c(s)\chi_1(s)\chi_2(s-\eta(s))\Bigg)\,ds \\ \qquad\quad \geq \displaystyle\int_{-\infty}^t e^{-\int_s^t \alpha(u)\,du}\left(\frac{\beta^-}{1+A_1^n} - c^+A_1A_2\right)ds \\ \qquad\quad \geq \dfrac{1}{\alpha^+}\left(\dfrac{\beta^-}{1+A_1^n} - c^+A_1A_2\right) \geq a_1, \\ N^{\chi_2}(t) \geq \displaystyle\int_{-\infty}^t e^{-\int_s^t \lambda(u)\,du} b^- a_1\,ds \\ \qquad\quad \geq \dfrac{b^-a_1}{\lambda^+} = a_2. \end{cases} \tag{15}$$

Equations (14) and (15) imply that Γ maps H^* into H^*.

Now, we verify the contraction condition of the Banach fixed-point theorem. It is easy to calculate that $\frac{n\theta^{n-1}}{(1+\theta^n)^2}$ is decreasing on $(0, +\infty)$ for $0 < n \leq 1$. Therefore, for any $a_1 < x < y < A_1$, employing the mean value theorem, we can observe that

$$\begin{aligned} \left|\frac{1}{1+x^n} - \frac{1}{1+y^n}\right| &= \left|\frac{-n\theta^{n-1}}{(1+\theta^n)^2}\right||x-y| \\ &\leq g(n)|x-y|, \end{aligned}$$

where $\theta \in (x, y)$. For any $\xi, \zeta \in H^*$, we can infer

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} |\Gamma(\xi_1(t)) - \Gamma(\zeta_1(t))| \\
=&\sup_{t\in\mathbb{R}} \Bigg| \int_{-\infty}^{t} e^{-\int_s^t \alpha(u)\,du} \Bigg\{ \beta(s) \Bigg(\frac{1}{1+\xi_1^n(s-d(s))} \\
&- \frac{1}{1+\zeta_1^n(s-d(s))} \Bigg) - c(s) \Bigg(\xi_1(s)\xi_2(s-\eta(s)) \\
&- \zeta_1(s)\zeta_2(s-\eta(s)) \Bigg) \Bigg\} ds \Bigg| \\
\le& \frac{\beta^+ g(n)}{\alpha^-} \|\xi - \zeta\| + \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\int_s^t \alpha(u)\,du} \Bigg\{ \\
&c(s) \Bigg[|\xi_1(s)| \Big| \xi_2(s-\eta(s)) - \zeta_2(s-\eta(s)) \Big| \\
&+ |\zeta_2(s-\eta(s))| \Big| \xi_1(s) - \zeta_1(s) \Big| \Bigg] \Bigg\} ds \\
\le& \frac{\beta^+ g(n)}{\alpha^-} \|\xi - \zeta\| + \frac{c^+}{\alpha^-} (A_1 + A_2) \|\xi - \zeta\| \\
=& \frac{1}{\alpha^-} \left(\beta^+ g(n) + c^+ (A_1 + A_2) \right) \|\xi - \zeta\|,
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} |\Gamma(\xi_2(t)) - \Gamma(\zeta_2(t))| \\
=&\sup_{t\in\mathbb{R}} \Bigg| \int_{-\infty}^{t} e^{-\int_s^t \lambda(u)\,du} b(s) \Bigg(\xi_1(s-\delta(s)) \\
&-\zeta_1(s-\delta(s)) \Bigg) ds \Bigg| \\
\le& \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\int_s^t \lambda(u)\,du} b^+ \Big| \xi_1(s-\delta(s)) - \zeta_1(s-\delta(s)) \Big| ds \\
\le& \frac{b^+}{\lambda^-} \|\xi - \zeta\|.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\Gamma\xi - \Gamma\zeta\| \le \max \Bigg\{ & \frac{1}{\alpha^-} \Big(\beta^+ g(n) + c^+(A_1 + A_2) \Big), \\
& \frac{b^+}{\lambda^-} \Bigg\} \|\xi - \zeta\|.
\end{aligned}$$

This, together with (H3), demonstrates that the operator Γ is a contraction mapping on the space H^*. According to the Banach fixed-point theorem, there exists a unique fixed point $\phi^* \in H^*$ satisfying $\Gamma\chi^* = \chi^*$. Consequently, χ^* represents the unique almost periodic solution of (2) and (3) within the function space H^*. Thus, this completes the proof of the theorem. ■

We now analyze the asymptotic stability of the almost periodic solution $z^*(t) = (N_1^*(t), N_2^*(t))$ for the dynamical system described by equations (2).

Theorem 3. *Assuming the conditions of Theorem 2 are fulfilled and further suppose*

(H4) *The delay functions $d(t), \delta(t), \eta(t)$ are continuously differentiable, and there exist non-negative constants d^*, δ^*, η^* such that*

$$0 \le d'(t) \le \tau^* < 1, \quad 0 \le \delta'(t) \le \delta^* < 1,$$
$$0 \le \eta'(t) \le \eta^* < 1.$$

(H5)

$$\begin{aligned}
K_1 \triangleq& 2\alpha^- - \beta^+ g(n) - c^+ A_1 - 2c^+ A_2 \\
&- \beta^+ \frac{1}{1-d^*} g(n) - \frac{b^+}{1-\delta^*} > 0, \\
K_2 \triangleq& 2\lambda^- - b^+ - \frac{c^+ A_1}{1-\eta^*} > 0.
\end{aligned}$$

Then, the almost periodic solution of model (2) *is global asymptotic stability.*

Proof: Let $z^*(t) = (N_1^*(t), N_2^*(t))$ be the positive almost periodic solution of model (2), and $z(t) = (N_1(t), N_2(t))$ be any other solution to model (2). Define

$$p_1(t) = N_1(t) - N_1^*(t), \quad p_2(t) = N_2(t) - N_2^*(t).$$

For convenience, denote

$$\begin{aligned}
r(t) =& \beta(t) \left(\frac{1}{1+N_1^n(t-d(t))} - \frac{1}{1+(N_1^*)^n(t-d(t))} \right) \\
&- c(t) \left(N_1(t) N_2(t-\eta(t)) - N_1^*(t) N_2^*(t-\eta(t)) \right).
\end{aligned}$$

Then, model (2) simplifies to

$$\begin{cases} p_1'(t) = -\alpha(t) p_1(t) + r(t), \\ p_2'(t) = -\lambda(t) p_2(t) + b(t) p_1(t-\delta(t)). \end{cases} \tag{16}$$

Employing the mean value theorem, we can infer

$$\begin{aligned}
\left| \frac{1}{1+x^n} - \frac{1}{1+y^n} \right| =& \left| \frac{-n\theta^{n-1}}{(1+\theta^n)^2} \right| |x-y| \\
\le& g(n)|x-y|,
\end{aligned}$$

where $\theta \in (x, y)$, thus,

$$\begin{aligned}
|r(t)| \le& \beta(t) \left| \frac{1}{1+N_1^n(t-d(t))} - \frac{1}{1+(N_1^*)^n(t-\tau(t))} \right| \\
&+ c(t) \Big(|N_1(t)| \, |N_2(t-\eta(t)) - N_2^*(t-d(t))| \\
&+ |N_2^*(t-\eta(t))| \, |N_1(t) - N_1^*(t)| \Big) \\
\le& \beta^+ g(n) \, |p_1(t-d(t))| + c^+ \big(A_1 \, |p_2(t-\eta(t))| \\
&+ A_2 \, |p_1(t)| \big).
\end{aligned} \tag{17}$$

Consider the following Lyapunov function:

$$V(t) = V_1(t) + V_2(t),$$

where

$$V_1(t) = p_1^2(t) + p_2^2(t).$$

and

$$\begin{aligned}
V_2(t) =& \beta^+ g(n) \frac{1}{1-d^*} \int_{t-d(t)}^{t} p_1^2(s)\,ds \\
&+ \frac{b^+}{1-\delta^*} \int_{t-\delta(t)}^{t} p_1^2(s)\,ds \\
&+ \frac{c^+ A_1}{1-\eta^*} \int_{t-\eta(t)}^{t} p_2^2(s)\,ds.
\end{aligned}$$

According to (16) and (H5), compute the derivatives of $V_1(t)$ and $V_2(t)$ respectively,

$$\begin{aligned}\frac{\mathrm{d}V_1(t)}{\mathrm{d}t} =&2p_1(t)p_1'(t)+2p_2(t)p_2'(t)\\ =&2p_1(t)\Big[-\alpha(t)p_1(t)+y(t)\Big]\\ &+2p_2(t)\Big[-\lambda(t)p_2(t)+b(t)p_1(t-\delta(t))\Big]\\ \le&-2\alpha^-p_1^2(t)+2|p_1(t)||y(t)|\\ &-2\lambda^-p_2^2(t)+2b^+|p_2(t)||p_1(t-\delta(t))|\\ \le&-2\alpha^-p_1^2(t)+\beta^+g(n)\Big(p_1^2(t)+p_1^2(t-\tau(t))\big)\\ &+c^+A_1\big(p_1^2(t)+p_2^2(t-\eta(t))\Big)\\ &+2c^+A_2p_1^2(t)-2\lambda^-p_2^2(t)\\ &+b^+\big(p_2^2(t)+p_1^2(t-\delta(t))\big)\\ =&\left(-2\alpha^-+\beta^+g(n)+c^+A_1+2c^+A_2\right)p_1^2(t)\\ &+\beta^+g(n)p_1^2(t-\tau(t))+b^+p_1^2(t-\delta(t))\\ &+\left(-2\lambda^-+b^+\right)p_2^2(t)+c^+A_1p_2^2(t-\eta(t)),\end{aligned} \tag{18}$$

and

$$\begin{aligned}\frac{\mathrm{d}V_2(t)}{\mathrm{d}t} =&\beta^+g(n)\frac{1}{1-d^*}\Big[p_1^2(t)-p_1^2(t-d(t))(1-d'(t))\Big]\\ &+\frac{b^+}{1-\delta^*}\Big[p_1^2(t)-p_1^2(t-\delta(t))(1-\delta'(t))\Big]\\ &+\frac{c^+M_1}{1-\eta^*}\Big[p_2^2(t)-p_2^2(t-\eta(t))(1-\eta'(t))\Big]\\ =&\beta^+g(n)\frac{1}{1-d^*}p_1^2(t)\\ &-\beta^+g(n)\frac{1-d'(t)}{1-\tau^*}p_1^2(t-d(t))\\ &+\frac{b^+}{1-\delta^*}p_1^2(t)-b^+\frac{1-\delta'(t)}{1-\delta^*}p_1^2(t-\delta(t))\\ &+\frac{c^+A_1}{1-\eta^*}p_2^2(t)-c^+A_1\frac{1-\eta'(t)}{1-\eta^*}p_2^2(t-\eta(t))\\ \le&\beta^+g(n)\frac{1}{1-d^*}p_1^2(t)-\beta^+g(n)p_1^2(t-d(t))\\ &+\frac{b^+}{1-\delta^*}p_1^2(t)-b^+p_1^2(t-\delta(t))\\ &+\frac{c^+A_1}{1-\eta^*}p_2^2(t)-c^+A_1p_2^2(t-\eta(t)).\end{aligned} \tag{19}$$

According to (18) and (19), we get

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t} =&\frac{\mathrm{d}V_1(t)}{\mathrm{d}t}+\frac{\mathrm{d}V_2(t)}{\mathrm{d}t}\\ \le&\Big(-2\alpha^-+\beta^+g(n)+c^+M_1+2c^+M_2\\ &+\beta^+g(n)\frac{1}{1-d^*}+\frac{b^+}{1-\delta^*}\Big)p_1^2(t)\\ &+\left(-2\lambda^-+b^++\frac{c^+A_1}{1-\eta^*}\right)p_2^2(t)\\ =&-K_1p_1^2(t)-K_2p_2^2(t)\\ <&0.\end{aligned}$$

Correspondingly,

$$\begin{aligned}V(t)+K_1\int_0^t p_1^2(s)ds+K_2\int_0^t p_2^2(s)ds\\ \le V(0),\quad t\ge 0.\end{aligned} \tag{20}$$

According to Lemma 4, the solutions of model (2) remain bounded on the interval $[0,+\infty)$. Consequently, this implies that $\frac{\mathrm{d}p_1(t)}{\mathrm{d}t}$ and $\frac{\mathrm{d}p_2(t)}{\mathrm{d}t}$ are bounded on $[0,+\infty)$, which guarantees the uniform continuity of $p_1(t)$ and $p_2(t)$ on $[0,+\infty)$. Furthermore, equation (20) also implies that $p_i\in L^1[0,+\infty)$. Applying Barbalat's lemma [37], we therefore conclude that

$$\lim_{t\to+\infty}p_1(t)=0,\quad \lim_{t\to+\infty}p_2(t)=0.$$

This leads to the conclusion that the solution $z^*(t)$ is globally asymptotically stable. The demonstration is accomplished. ■

IV. A Numerical example

This section provides a numerical example to validate the validity of our theoretical results.

Example 1. We examine a delayed hematopoiesis model with feedback regulation, described by the following system:

$$\begin{cases}N_1'(t)=-\alpha(t)N_1(t)+\beta(t)\dfrac{1}{1+N_1^n(t-d(t))}\\ \qquad\qquad -c(t)N_1(t)N_2(t-\eta(t)),\\ N_2'(t)=-\lambda(t)N_2(t)+b(t)N_1(t-\delta(t)),\end{cases} \tag{21}$$

where

$$\begin{aligned}&d(t)=0.5,\quad \eta(t)=0.3,\quad \delta(t)=0.4,\quad n=2,\\ &\alpha(t)=1+0.5|\cos(\sqrt{2}t)|,\qquad \beta(t)=0.2+0.1|\sin(\sqrt{3}t)|,\\ &c(t)=0.01+0.005\sin(t),\qquad \lambda(t)=1+0.2\cos(2t),\\ &b(t)=0.5+0.1\sin(\sqrt{2}t).\end{aligned}$$

Clearly, we have $\alpha^-=1,\alpha^+=1.5,\beta^-=0.2,\beta^+=0.3,c^-=0.005,c^+=0.015,\lambda^-=0.8,\lambda^+=1.2,b^-=0.4,b^+=0.6$. By calculation, we obtain

$$\begin{aligned}&A_1=\frac{\beta^+}{\alpha^-}=0.3,\qquad A_2=\frac{b^+A_1}{\lambda^-}=0.225,\\ &a_1=\frac{\beta^-}{(\alpha^++c^+A_2)(1+A_1^n)}=0.122,\\ &a_2=\frac{b^-a_1}{\lambda^+}=0.041,\\ &0<a_1=\frac{1}{\alpha^+}\left(\beta^-\frac{1}{1+A_1^n}-c^+A_1A_2\right)=0.122,\\ &\frac{1}{\alpha^-}\left(\beta^+g(n)+c^+(A_1+A_2)\right)=0.61<1,\\ &\frac{b^+}{\lambda^-}=0.75<1,\\ &K_1=0.202>0,\qquad K_2=0.997>0.\end{aligned}$$

This demonstrates that the criteria stated in Theorem 2 and Theorem 3 hold. Consequently, it is concluded that model (21) maintains persistent and possesses a unique almost periodic solution that exhibits global asymptotic stability. Different initial values have been selected for numerical simulation (see Figure 1), which effectively substantiates the feasibility of the obtained theoretical results.

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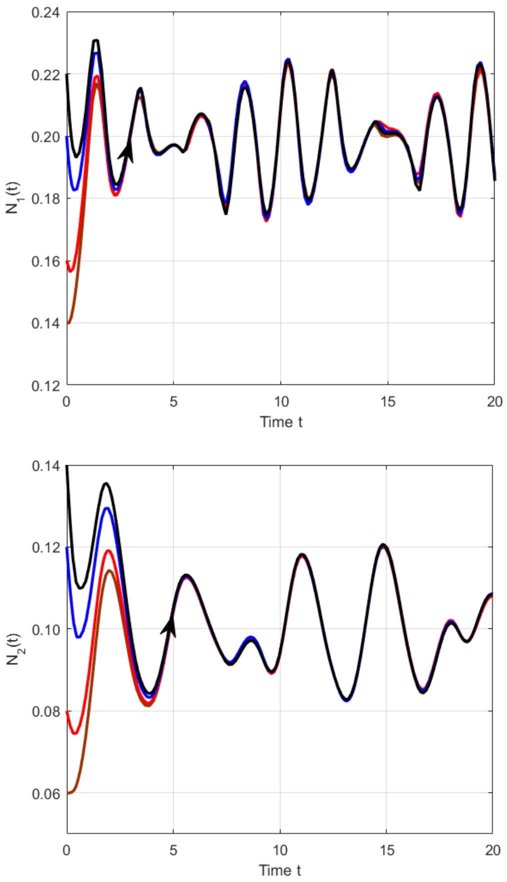

Fig. 1. Simulated trajectories of model (21) with different initial values.

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