Implementing Benders Decomposition Method on Multi-objective Integer Adjustable Robust Counterpart Optimization Model with Polyhedral Uncertainty Set

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Abstract—**This paper investigates the use of the Benders Decomposition Method to solve a class of uncertain multi-objective optimization problems involving linear integer variables. The approach begins by modeling the problem as a multi-objective integer programming formulation. The Lexicographic Method is employed to prioritize and sequentially address the objectives according to a predefined hierarchy. This lexicographic formulation is then extended to an uncertain setting by incorporating polyhedral uncertainty sets, transforming the problem into an Adjustable Robust Optimization (ARO) framework. Given that the resulting robust formulation can be interpreted as a two-stage mixed-integer linear program (MILP), the Adjustable Robust Counterpart (ARC) is derived. This ARC problem remains a MILP, which is then efficiently solved using Benders Decomposition. The paper contributes both a methodological integration of lexicographic and robust optimization with decomposition techniques and a theoretical analysis ensuring the global optimality of the solution via convexity arguments. A mini-review is also presented to highlight the novelty of the approach.**

Index Terms—**Robust Optimization, Adjustable Robust Counterpart, Multi-objective, Lexicographic Method, Mixed Integer Linear Programming, Benders Decomposition Method**

I. Introduction

OPTIMIZATION models in real life often experience problems with data that cannot be known precisely [1]. This kind of data is termed data that contains uncertainty. This uncertainty can be caused by data measurement errors, such as measuring an object's dimensions and temperature, data estimation errors, or rounding numbers [2]. This becomes a drawback when using data in optimization problems. The methodology for dealing with the issue of data uncertainty in optimization is Robust Counterpart (RC), proposed by [3]. Referring to [3], Robust Optimization is a method for finding solutions to robust optimization problems against uncertain data in parameters where the uncertainty is in an uncertainty set. A recent survey on uncertain optimization can be seen in [4], which discusses surveying the current state of uncertain optimization models and methodologies.

Robust optimization can be categorized into two types, namely single-stage and multi-stage models. In single-stage Robust Optimization, all decision variables with here-and-now decisions are considered to be resolved immediately. In contrast, in a multi-stage Robust Optimization with wait-and-see decisions, the decision variables in the second stage are adjusted to the realization of the previous parameter uncertainty. This multi-stage Robust Optimization approach was first introduced in [5] by considering two variables. The first set must be determined before resolving the uncertainty, and the other set can be calculated after the uncertainty is resolved. This multi-stage Robust Optimization is the Adjustable Robust Counterpart (ARC) or two-stage RC methodology.

Multi-objective optimization problems can be solved using the Lexicographic Method (LM). Referring to [6], the lexicographic method is a method that sorts objective functions based on their interests or priorities determined by the researcher. The optimum solution x is obtained by minimizing the objective function, starting from the most important and continuing according to the number and level of importance. One of the implementations of the LM is presented in [7], which discusses how a hybrid metaheuristic algorithm is used for the bi-objective school bus routing problem. See also in [8], the application of LM on a robust optimization model using ellipsoidal and polyhedral uncertainty sets for the spatial land-use allocation problem.

Furthermore, three uncertainty parameters can be used in single-stage or multi-stage Robust Optimization, i.e., the uncertainty parameters in the Box, Ellipsoidal, and Polyhedral Uncertainty Sets. This study assumes that the uncertainty parameter is in the Polyhedral Uncertainty Set. The indeterminate data points mapped into the Polyhedral Uncertainty Set will produce a convex hull that guarantees a feasible solution. In addition, the Polyhedral Uncertainty Set assumes the best uncertainty set among the other two sets because it does not include additional data outside the set and does not discard the original data used.

Manuscript received December 7, 2024; revised July 22, 2025. This research is supported by Universitas Padjadjaran Research Grant within the Online References Research Grant or *Riset Data Pustaka dan Daring (RDPD)* under contract number 2217/UN6.3.1/PT.00/2024.

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A multi-objective integer ARC optimization model with a Polyhedral Uncertainty Set can also be obtained. The existence of the ARC methodology can overcome the problem of uncertainty in the multi-objective integer optimization model. The ARC method was chosen because this study worked on determining the integer decision variables in the first stage and determining other continuous decision variables in the second stage. Several studies have been conducted related to this topic with various problems. Research written by [9] discusses the comprehensive Mixed Integer Linear Programming (MILP) model for distributing energy reserves using the ARC methodology. In Chaerani et al. [10], ARC is used to determine the adjustable robust maximum flow problem with a parametric ellipsoidal and polyhedral uncertainty set.

Furthermore, the problem of multi-stage robust optimization (ARC) can be approached using various methods. Among the ways often used are Column-and-constraint Generating algorithms, as in the study of [11] on distribution networks. Research in [12] used the Benders Decomposition Method to determine facilities' construction. There is also a study written by [13] using the Cutting Plane Method to solve the Unit Commitment (UC) problem, [14] using the Branch and Bound Method to develop integer optimization problems by partitioning the set of uncertainty. A recent result in [15] presents a study on the Benders decomposition approach on an adjustable robust counterpart optimization model for multi-objective supply chain problems in sugar distribution.

Based on various methods that can solve the ARC optimization problem, a general multi-objective integer ARC optimization model is formulated using a polyhedral uncertainty set and the Benders Decomposition Method approach. Because the general model is obtained, the model can be applied according to the desired topic at the end of the study.

As in [16], [17] and [18], the Benders Decomposition Method is the basis of a mathematical model that is required to partition or divide the problem into linear or continuous parts that are easy to solve and nonlinear or integer parts that are difficult to solve. Then the Benders Decomposition algorithm is applied, which has the central concept of partitioning the variables into two sets x and y, solving the problem on the complex variable Y. This is an optimization method to solve problems with feasible subproblems, see [16] and [17]. For a robust mixed integer problem, see [18].

Previous studies that examined the multi-objective integer RC optimization model using the Benders Decomposition Method approach were more than ARC. This is because ARC is a methodology to handle two-stage Robust Optimization that still needs to be developed [19]. Previous studies that have focused on developing mathematical methods regarding the RC multi-objective integer optimization model using the Benders Decomposition Method approach include (1) Research in [20] on a large-scale MIP optimization model with an objective function and a convex constraint function, (2) Research in [21] regarding the MILP optimization model for distribution problems with the first stage using the Cutting Plane Method, and (3) Research in [22] regarding the MIP optimization model containing inequalities, local branching, In-out Variant Method, and scenario-based aggregated cuts. As [23] is included in research references that focus on developing mathematical methods regarding the multi-objective integer ARC optimization model using the Benders Decomposition Method, this article solves the linear Robust Optimization problem with mixed integer decision variables as non-adjustable variables that are solved in the first stage and continuous resource decision variables as adjustable variables which are solved in the second stage. Furthermore, the article focuses on the uncertainty on the right-hand side. Then, it provides a generalization of the development of mathematical methods if there is uncertainty on the left side. The technique used in [23] is a generating constraint algorithm. The literature review and its novelty will be discussed later. In this research, the development of mathematical methods are presented. The analytical studies based on the convexity of the general model is also obtained. The general model obtained contains integer and continuous variables, so it has two stages of completion assisted by the ARC method to handle its uncertainty. Therefore, the Benders Decomposition Method approach can be used.

II. Materials and Methods

This section presents an overview of the materials and methods employed in the study. The foundational model used is a deterministic Mixed Integer Linear Programming (MILP) formulation introduced by [23], which serves as the initial input to our framework. The model involves two types of decision variables: integer variables determined in the first stage (here-and-now decisions), and continuous variables determined in the second stage (wait-and-see decisions). This two-stage structure supports the development of a more advanced multi-objective Adjustable Robust Counterpart (ARC) model, which incorporates polyhedral uncertainty sets to handle parameter ambiguity.

To address the computational challenges posed by the presence of integer variables, the Benders Decomposition Method is applied (see [16]). This approach decomposes the problem into a master problem and a subproblem, effectively managing complexity by isolating the integer and continuous components. As a result, a general ARC formulation for multi-objective mixed-integer optimization under polyhedral uncertainty is obtained.

Additionally, the mathematical properties of the resulting model are analyzed by examining its convexity. This analysis, following the principles outlined in [24], ensures that the locally optimal solutions identified are also globally optimal, thereby enhancing the robustness and reliability of the solution methodology.

A. Mixed-integer linear problem

Research in [23] describes an integer optimization model that is a single objective function that is formulated as follows:

$$\begin{aligned} \min_{x,y}\ & \alpha^T x + \beta^T y, \\ \text{s.t.: } & Ax + By \geq d, \\ & Cx \geq b, \\ & x, y \geq 0, \end{aligned} \tag{1}$$

with $A \in \mathbb{Q}^{T\times q}$, $B \in \mathbb{Q}^{T\times q}$, $C \in \mathbb{Q}^{l\times p}$, $d \in \mathbb{Q}^{l}$, $\alpha_+^p, \beta \in \mathbb{Q}_+^q$, and $\mathbb{Q}$ is the set of rational numbers. Optimization problems whose optimization is viewed from more than one point of view require the multi-objective characteristic of the model

B. Lexicographic method for multi-objective mixed-integer linear problem

This section discusses the formulation of the multi-objective integer optimization model, which refers to [23] with minor changes. An additional formulation of the integer optimization model (1) is needed to support the multi-objective. Referring to [6], the priority scale on the minimization objective function is denoted by additional indexes i and j, with j being $(i-1)$. The optimal solution is obtained in the calculation process by minimizing the objective function from the most important and then continuing according to the number and level of importance [6]. In other words, this calculation process is called the Lexicographic Method. The reformulation of the integer optimization model in (1) with the addition of multi-objective properties with i objective functions is as follows:

$$\begin{aligned} \min_{x,y}\ & (f_i(x,y) = \alpha_i^T x + \beta_i^T y), \forall i = 1,2,...,m, \\ \text{s.t.: } & Ax + By \geq d, \\ & Cx \geq b, \\ & f_i(x,y) = f_j^*, \forall j = 1,2,...,(m-1), \\ & x, y \geq 0,\ x \in \mathbb{R}^p, y \in \mathbb{N}^q, \end{aligned} \tag{2}$$

with $A \in \mathbb{Q}^{T\times p}$, $B \in \mathbb{Q}^{T\times q}$, $C \in \mathbb{Q}^{l\times p}$, $d \in \mathbb{Q}^T$, $b \in \mathbb{Q}^l$, $\alpha \in \mathbb{Q}_+^p$, $\beta \in \mathbb{Q}_+^q$, and $\mathbb{Q}$ is the set of rational numbers.

C. Benders Decomposition Method

Refers to J.F Benders in 1962 [16] and Bischop in 2006 [17], the partitioning procedures for solving mixed-integer variable programming problems is known as the Benders Decomposition Method. The method is an optimization method for solving problems with feasible subproblems. Consider the formulation of the minimization problem called the initial problem $P(x,y)$ as follows.

$$\begin{aligned} \min\ & c^T x + f(y), \\ \text{s.t } & Ax + F(y) = b, \\ & x \geq 0, y \in Y, \end{aligned} \tag{3}$$

where $A \in \mathbb{R}^{m\times n}$, $x, c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $y \in Y \subset \mathbb{R}^p$, in case $f(y)$ and $F(y)$ could be nonlinear and Y could be discrete or continuous range. Partition the variable into two sets, i.e., x and y is the main concept in the Benders Decomposition Algorithm. First, then solve the problem on the difficult variable Y.

It is mentioned in [17], the step-by-step Benders Decomposition Method is proceeded as follows.

1) First, formulate the fixed value of $y \in Y$ becomes a Linear Programming problem in terms of x. Let $P(x|y)$ be the notation of a feasible subproblem. Assume that $P(x|y)$ has a finite optimal solution $x\ \forall y \in Y$. Thus equation (3) can be reformulated into a constraint equivalent to $P_1(x,y)$:

$$\min_y \left\{ f(y) + \min_x \{c^T x \ : \ Ax = b - F(y), x \geq 0\} \right\}, \tag{4}$$

where

$$\min_x \{c^T x \ : \ Ax = b - F(y), x \geq 0\}, \tag{5}$$

is an inner optimization problem, and it is assumed to have an optimal solution x for every $y \in Y$.

2) Second, find a dual problem formulation for the inner optimization problem. Equation (3) can be rewritten as:

$$\min_y \left\{ f(y) + \max_u \left\{ (b - F(y))^T u \ : \ A^T y \leq c \right\} \right\}. \tag{6}$$

The constraint function in the inner optimization problem is independent of the y variable. As it is assumed that (5) has an optimal solution x for every $y \in Y$, the optimal solution for the inner optimization problem is finite. This optimal solution is the extreme point $u \in \mathcal{U}$. Thus, Equation (6) can be rewritten as follows:

$$\min_y \left\{ f(y) + \max_u (b - F(y))^T u \right\}. \tag{7}$$

3) Third, the Full Master Problem is determined by rewriting (7) as a simple minimization problem as follows.

$$\begin{aligned} \min\ & f(y) + m \ \text{ s.t } \quad (b - F(y))^T u \leq m, u \in U, \\ & y \in Y. \end{aligned} \tag{8}$$

From equation (8), the Relaxed Master Problem $M(y,m)$ is obtained as follows.

$$\begin{aligned} \min\ & f(y) + m \\ \text{s.t } & (b - F(y))^T u \leq m, u \in B \\ & y \in Y, \end{aligned} \tag{9}$$

where B is an empty set and m is initialized as 0. Benders' subproblem that solves an extreme point u with a fixed value $y \in Y$ so that it can be considered a maximization problem

$$S(u|y) = \max \left\{ (b - F(y))^T u \ : \ A^T y \leq c \right\}, \tag{10}$$

where $u \in R$, $S(u|y)$ has a finite optimal solution. The subproblem $S(u|y)$ is solved to get the value of u, i.e., given the value determined from solving the master problem $M(x,y)$. This means that a simple test was done to determine whether a constraint that includes u should be added to the master problem. If so, the next problem is solving the master problem to generate a new value of y. Then, input the new value of y to the subproblem, which is solved again. This iteration continues until the optimization is reached.

D. Robust Optimization

Referring to Ben-Tal *et al.* in [3], RO is a method to solve Optimization problems with data uncertainty and is only known in a set of uncertainties. The general form of the problem of indefinite linear optimization can be formulated as in equation (11) follows:

$$\begin{aligned} \min\ & c^T x, \\ \text{s.t } & Ax \leq b, \\ & (c, A, b) \in \mathcal{U}, \end{aligned} \tag{11}$$

where $c \in R^n$, $A \in R^{mxn}$, $b \in R^n$, the three decision variables are indefinite coefficients. $\mathcal{U}$ is a notation of the set of uncertainties.

There are three basic assumptions in RO, namely all decision variables state decisions "here and now", decision makers are fully responsible for the consequences of decisions made, if and only if the actual data has been determined in the set of uncertainties $\mathcal{U}$, and constraints on programming problems linear with uncertainty is "hard". In addition, referring to Gorissen *et al.* in [25], in dealing with Linear RO, three things are also assumed. First, the objective function is certainly valuable. Suppose there are uncertainties in the objective function. In that case, the problem can be formulated by replacing the objective function with a single-variable function, such that uncertainty arises in the constraint function. Second, the right vertex vector b is of a certain value. If b is uncertain, an extra variable x_{n+1} can be introduced. Third, robustness against $\mathcal{U}$ can be formulated as a constraint-wise problem, and the set of uncertainties $\mathcal{U}$ is a closed and convex set.

Assuming that $c \in R^n$ and $b \in R^m$ are of certain value, the Robust Counterpart (RC) formulation of equation (11) is equivalent to equation (12) below.

$$\begin{aligned} \min\ & c^T x, \\ \text{s.t}\ \ & a^T(\zeta)x \leq b, \\ & x \geq 0, \forall\ \zeta \in \mathcal{Z}. \end{aligned} \tag{12}$$

Note the uncertain constraint in equation (12) and define the uncertain parameter $a(\zeta) = \bar{a} + P\zeta$ where $\bar{a} \in R^n$ is a nominal value vector and $P \in R^{nxL}$ is a confounding matrix. The set $\mathcal{U}$ is defined as in equation (13).

$$\mathcal{U} = \{a | a = \bar{a} + P\zeta, \quad \zeta \in \mathcal{Z}\}, \tag{13}$$

where $\mathcal{Z} \subset R^L$ is an uncertain set of primitive factors, so equation (14) is obtained.

$$(\bar{a} + P\zeta)^T x \leq b, \quad \forall\ \zeta \in \mathcal{Z}. \tag{14}$$

The optimal solution from Robust Counterpart is called optimal robust. Furthermore, the following Theorem 1 as stated in [3], applies to reformulate the set of uncertainty $\mathcal{U}$ into a computationally tractable problem.

Theorem 1: Assume the set of uncertainty $\mathcal{U}$ is an affine image of the limited set $\mathcal{Z} = \{\zeta\} \subset R^n$, and $\mathcal{U}$ is:

1) The system of linear inequality constraints

$$P\zeta \leq p. \tag{15}$$

2) The system of conic quadratic inequality

$$\|P_i\zeta - p_i\|_2 \leq p_i^T\zeta - r_i, i = 1, ..., M. \tag{16}$$

3) Systems of linear matrix inequality

$$p_0 + \sum_{i=1}^{dim\zeta} \zeta_i P_i \geq 0. \tag{17}$$

In cases (2) and (3) it is also assumed that the system of the constraints defining $\mathcal{U}$ is strictly feasible. Then, the Robust Counterpart of equation (11) is equivalent to:

1) Linear Programming (LP) problems in the case (1),
2) Conic Quadratic Programming (CQP) problems in cases (2),
3) Semidefinite Programming (SDP) problems in cases (3).

As stated in Gorissen *et al.* in [25], the computational tractability of robust counterpart for different sets of uncertainties can be seen in Table I.

TABLE I: Tractable formulations for constraints with sets of uncertainties

Uncertainty Set	$\mathcal{Z}$	Robust Counterpart	Tractability
Box	$\|\zeta\|_\infty \leq 1$	$a^T x + \|P^T x\|_1 \leq b$	LP
Ellipsoidal	$\|\zeta\|_2 \leq 1$	$a^T x + \|P^T x\|_2 \leq b$	CQP
Polyhedral	$D\zeta + q \geq 0$	$\begin{cases} a^T x + q^T w \leq b \\ D^T w = -P^T x \\ w \geq 0 \end{cases}$	LP

E. Adjustable Robust Counterpart Optimization

Referring to Bental *et al.* in [3] and [5] also in Yanikouglu *et al.* [19], in multistage optimization, the basic paradigm of RO, namely the "here and now" decision, can be relaxed. Some decision variables can be adjusted later according to decision rules, which are a function of (some or all parts of) uncertain data. Adjustable Robust Counterpart (ARC) is given as in equation (18).

$$\begin{aligned} \min_{x,y(.)}\ & c^T x, \\ \text{s.t.}\ \ & A(\zeta)x + By(\zeta) \leq b, \\ & \forall \zeta \in \mathcal{Z}. \end{aligned} \tag{18}$$

where $x \in R^n$ is the first stage decision "here and now" made before $\zeta \in R^L$ is realized, $y \in R^k$ denotes a "wait and see" decision and $B \in R^{mxk}$ which shows a certain matrix coefficient.

In practice, $y(\zeta)$ is often through an approach with affine or linear decision rules $y(\zeta) = y^0 + Q\zeta$ with $y^0 \in R^k$ and $Q \in R^{kxL}$ is the coefficient in the decision rule, which is to be optimized. Thus, the reformulation of equation (18) is equation (19).

$$\begin{aligned} \min_{x,y^0,Q}\ & c^T x, \\ \text{s.t.}\ \ & A(\zeta) + By^0 + BQ\zeta \leq b, \\ & \forall \zeta \in \mathcal{Z}. \end{aligned} \tag{19}$$

III. Results

This section discusses some of the results of the mini-review for topics Benders Decomposition and Multi-objective Integer Adjustable Robust Counterpart (ARC) Optimization Model, the formulation of the initial multi-objective integer optimization model, the multi-objective Adjustable Robust Counterpart (ARC) optimization model with Polyhedral Uncertainty Sets and their convexity analysis of the application of the Benders Decomposition Method to the model.

A. Mini-review for topics Benders Decomposition and Multi-objective Integer Adjustable Robust Counterpart (ARC) Optimization Model

In this paper, the Benders Decomposition and Multi-objective Integer Adjustable Robust Counterpart

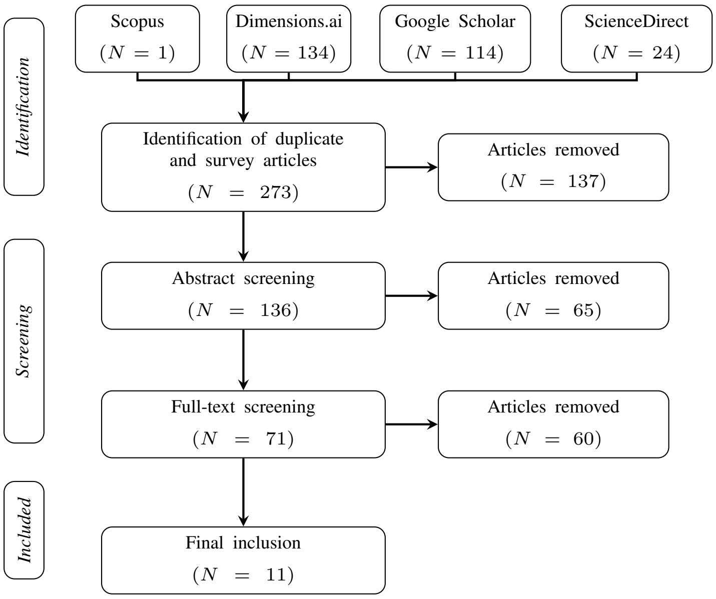

Fig. 1: Selection process based on the PRISMA framework.

(ARC) Optimization Model have been reviewed for the period 2014 to 2025. The literature search was performed using the combined keywords Benders Decomposition, Adjustable Robust Optimization, and Multi-objective Optimization. The number N of results varied across databases: Scopus ($N = 1$), ScienceDirect ($N = 24$), dimensions.ai ($N = 134$), and Google Scholar ($N = 114$). All retrieved records were then subjected to the PRISMA screening process to ensure that only articles meeting our inclusion criteria were carried forward. The complete flow of this selection process is illustrated in Figure 1.

This study employed a systematic twophase data analysis approach. The initial bibliometric analysis was conducted on Dataset 1, which comprised 136 articles remaining after duplicates and survey articles were removed. Dataset 1 served as the basis for mapping overall research trends in Benders Decomposition and Adjustable Robust Optimization.

For the indepth systematic literature review, a rigorous PRISMA-guided screening of Dataset 1 records produced eleven articles, referred to as Dataset 2. These selected articles were analyzed for their specific contributions to Adjustable Robust Optimization, Benders Decomposition, Multiobjective Optimization, and the types of uncertainty sets employed. A thematic map of Dataset 1 was generated using the Bibliometrix package in R Software, plotting each theme by development (density) and relevance (centrality).

Table II presents this strategic diagram, where the motor themes mean that the result is in high density and high centrality. The niche themes mean high density and low centrality, the basic themes mean low density and high centrality, and the emerging/declining themes refer to low density and low centrality. The interpretation of the thematic map reveals in the following indications.

1) *Benders Decomposition* appears as a *niche theme*, indicating methodological maturity but specialized application areas, notably renewable energy planning.
2) *Multiobjective Optimization* is positioned among the *motor themes*, reflecting both high maturity and centrality, and driving developments in microgrid management and demandside optimization.
3) *Robust Optimization*, including the Adjustable Robust variant, falls into the *basic themes* quadrant, signifying fundamental relevance for handling uncertainty, with innovation progressing at a steadier pace.

The initial findings and a detailed analysis is conducted from the eleven selected articles in Dataset 2. As can be seen in Table III summarizes their methodological characteristics, with particular focus on the types of uncertainty sets employed and the implementation frameworks. A notable pattern emerges: polyhedral uncertainty sets dominate the literature, appearing in seven out of the eleven studies reviewed [10], [15], [26], [27], [28], [29], [30], [31]. The remaining works employ alternative constructs such as budget-based [32], interval [33], norm-based data-driven [34], and machine learning-derived uncertainty sets [35]. The popularity of polyhedral sets is largely attributable to their convex structure and representation via linear constraints. These properties allow efficient reformulation techniquessuch as dualization and Benders decompositionand often lead to globally optimal solutions [15], [28].

For instance, ARC-ISOP and sugar-distribution models [10], [15] employ polyhedral bounds to model uncertainty in delivery times and price fluctuations, ensuring robustness without excessive conservatism. In hydro-thermal-wind unit commitment models [29], polyhedral sets capture spatial-temporal inflow uncertainty, enabling scalable resolution using Column-and-Constraint Generation (C&CG) methods while preserving system interdependencies. Similarly, in hydrogen infrastructure planning [30], representative polyhedral days facilitate adjustable risk-conservatism tradeoffs.

In contrast, [35] proposes a machine learning-based uncertainty set using Support Vector Clustering (SVC). Their study benchmarks the performance of SVC-based sets against traditional polyhedral constructs in hub-routing problems. Results indicate that SVC-based models yield more robust and less conservative outcomes while effectively

TABLE II: Thematic Map Based on Density and Centrality

		Centrality	
		Low	High
Density	High	**Niche Themes** −Distributed generation, planning, renewable −Benders decomposition, demand-side management, improved non-dominated sorting genetic algorithm	**Motor Themes** −Demand response, microgrid, multi-objective optimization −Renewable energy, electric vehicles, energy management
	Low	**Emerging or Declining Themes** −Optimization, stochastic programming, project management −Surgery scheduling	**Basic Themes** −Robust optimization, uncertainty, facility location −Distributionally robust optimization, energy hub, integrated energy system

TABLE III: Robust Optimization Methods in Dataset 2.

Article	Topic	Adjustable Robust Optimization	Benders Decomposition	Multi Objective Optimization	Uncertainty Set
[15]	Sugar Distribution Supply Chain	✓	✓	✓	Polyhedral
[26]	ARC Model with Polyhedral Uncertainty in ISOP	✓	-	-	Polyhedral
[32]	Dynamic Programming for Natural Gas Networks	-	✓	✓	Budget-based
[27]	Benders Decomposition for ARC in Internet Shopping Problem	✓	✓	-	Polyhedral
[28]	Analysis Multi-objective Linear Robust Optimization	-	-	✓	Polyhedral
[29]	C&CG algorithm for two-stage robust UC	-	✓	✓	Polyhedral
[34]	Deep Peak Shaving and Renewable Energy	-	✓	-	Data-driven
[30]	Data-driven robust model for hydrogen infrastructure	✓	-	✓	Polyhedral
[35]	Hub Location-Routing Problem	-	-	-	Machine learning-based (SVC) compared with polyhedral and Box
[33]	Time-Cost-Environment Project Scheduling	-	✓	✓	Interval
[31]	Integrated energy system with multi-scale hydrogen energy management.	✓	✓	-	Polyhedral
	This research	✓	✓	✓	Polyhedral

capturing multimodal and nonlinear travel-time distributions. These findings collectively highlight the strength of polyhedral sets in ensuring tractability and global optimality. However, they also underscore the growing potential of machine learning to learn richer, data-driven uncertainty representations, particularly within multi-objective adjustable robust optimization frameworks [15].

B. Multi-objective Integer Adjustable Robust Counterpart (ARC) Optimization Model with Polyhedral Uncertainty Set

This section discusses the formulation of a multi-objective integer Adjustable Robust Counterpart (ARC) optimization model with a Polyhedral Uncertainty Set. In the previous section, the ARC methodological approach with Polyhedral Uncertainty Sets is carried out on the multi-objective integer optimization model. The ARC methodology can be applied to multi-stage optimization problems, so the first thing to do is formulate the model and determine the uncertainty parameters and two-stage decision variables. In the preparation of the model, the uncertain parameters (containing the assumption of uncertainty) are matrix A, while the other parameters d are certain. Furthermore, the here-and-now decision variable (a non-adjustable variable determined in the first stage) is a continuous value variable x.

In contrast, the wait-and-see decision variable (an adjustable variable specified in the second stage) is discrete. Therefore, it can be assumed that $A \in \mathcal{U}_1$ and $y \in \mathcal{U}_2$, with $\mathcal{U}_1,\ \mathcal{U}_2 \in \mathcal{U}$ and $\mathcal{U}$ is a primitive uncertainty set. The formulation (2), if adjusted into the general form of the ARC methodology, is as can be seen in formulation (20).

$$\begin{aligned}
\min_{x,y} \quad & (f_i(x,y) = \alpha_i^T x + \beta_i^T y), \forall i = 1,2,...,m, \\
\text{s.t.:} \quad & -Ax - By \leq -d, \\
& -Cx \leq -b, \\
& f_j(x,y) = f_j^*, \forall j = 1,2,...,(m-1), \\
& x, y \geq 0, \\
& A \in \mathcal{U}_1, y \in \mathcal{U}_2, \\
& x \in \mathbb{R}^p, y \in \mathbb{N}^q.
\end{aligned} \tag{20}$$

Based on the first basic assumption related to the general model of the Robust Optimization problem, if uncertainty arises in the objective function, it is necessary to define a single variable function in the form of $t_i \in \mathbb{R}$ with $t \geq f_i(x,y)$ and $f_i\,(x,y) = \alpha_i^T x + \beta_i^T y$, $\forall i = 1,2,\ldots,m$, so that the reformulation form is obtained as follows (vector y and constraint matrix A expressed in a primitive uncertainty parameter $\zeta_i, \zeta_2 \in \mathcal{Z}$, with $\zeta_1 \in \mathbb{R}^w$ and $\zeta_2 \in \mathbb{R}^z$:

$$\begin{aligned}
\min_{t_1} \quad & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} t_i \\ x \\ y \end{bmatrix}, \\
\text{s.t.:} \quad & \begin{bmatrix} -A(\zeta) & -B \\ -C & 0 \\ \alpha_i & \beta_i \end{bmatrix} \begin{bmatrix} x \\ y(\zeta) \end{bmatrix} \leq \begin{bmatrix} -d \\ -b \\ t_i \end{bmatrix}, \\
& \begin{bmatrix} x \\ y \end{bmatrix} \geq 0, \\
& t_i \text{ unrestricted.}
\end{aligned} \tag{21}$$

Given the constraint-wise assumption of uncertainty in Robust Optimization, two constraints containing an uncertainty vector (first and third constraints) in the formulation (21) can be focused on a single constraint as follows:

$$\begin{aligned} -\left(\bar{a}+P\zeta_2\right)^T x-b^T\left(\bar{y}+Q\zeta_1\right) &\leq -d, \\ \alpha^T x+\beta^T\left(\bar{y}+Q\zeta_1\right) &\leq t, \end{aligned} \tag{22}$$

with $(\bar{a}+P\zeta_2)$ and $(\bar{y}+Q\zeta_1)$ are an affine function over primitive uncertain parameters $\zeta_i, \zeta_2 \in \mathcal{Z}, \bar{a} \in \mathbb{Q}^p$, and $\bar{y} \in \mathbb{N}^q$.

Next is determining ζ, which maximizes the two constraints in (22) and satisfies these constraints. The formulation of this optimization model assumes that the uncertain parameter A and the adjustable variable y are in the Polyhedral Uncertainty Set, which is defined as follows:

$$\begin{aligned} \mathcal{Z}_1 &= \left\{\zeta_1 : r-R\zeta_1 \geq 0\right\}, \\ \mathcal{Z}_2 &= \left\{\zeta_2 : h-H\zeta_2 \geq 0\right\}, \end{aligned} \tag{23}$$

with $r \in \mathbb{R}^m$, $R \in \mathbb{R}^{(w\times m)}$, $\zeta_1 \in \mathbb{R}^w$, $h \in \mathbb{R}^n$, $H \in \mathbb{R}^{z\times n}$, and $\zeta_2 \in \mathbb{R}^z$.

The first step in applying the Polyhedral Uncertainty Set to the model is to reformulate the second left-hand side of the constraints in (22). The following are obtained.

$$\begin{aligned} -\bar{a}^T x \quad &- \max_{\zeta_2 : h-H\zeta_2 \geq 0} \left(P^T x\right)^T \zeta_2 - \\ &b^T\bar{y} - \max_{\zeta_1 : r-R\zeta_1 \geq 0} \left(b^T Q\right)^T \zeta_1 \leq d, \end{aligned} \tag{24}$$

$$\alpha^T x+\beta^T\bar{y} \max_{\zeta_1 : r-R\zeta_1 \geq 0} \left(b^T Q\right)^T \zeta_1 \leq t. \tag{25}$$

Next, the discussion is focused on changing the primal form to the dual form. The maximization problem in both constraints in (24) and (25) is a primal form of inequality $\leq$ and has unrestricted uncertain variables ζ_1 and ζ_2, so the dual formulation is as follows:

$$\begin{aligned} -\bar{a}^T x - \min_{\gamma} \left\{h^T\gamma : H^T\gamma = P^T x, \gamma \geq 0\right\} - b\bar{y} \\ - \min_{\lambda} \left\{r^T\lambda : R^T\lambda = b^T Q, \lambda \geq 0\right\} \leq -d, \\ \alpha^T x+\beta^T\bar{y}+\min_{\lambda} r^T\lambda : R^T\beta^T Q, \lambda \geq 0 \leq t. \end{aligned} \tag{26}$$

The primal-dual relationship used is strong duality. Therefore, the optimum value of the primal and dual problems in the (24) and (26) formulations is the same. In other words, the primal-dual relationship can be written as follows:

$$\max\{\left(P^T x\right)^T \zeta_2 \quad : \quad h-H\zeta_2 \geq 0\} \tag{27}$$

is equivalent with

$$\min\left\{h^T\gamma : H^T\gamma = P^T x, \gamma \geq 0\right\}. \tag{28}$$

As well as

$$\max\{\left(b^T Q\right)^T \zeta_1 : r-R\zeta_1 \geq 0\} \tag{29}$$

is equivalent with

$$\min\left\{r^T\lambda : R^T\lambda = b^T Q, \lambda \geq 0\right\}, \tag{30}$$

and

$$\max\{\left(\beta^T Q\right)^T \zeta_1\{r-R\zeta_1 \geq 0\} \tag{31}$$

is equivalent with

$$\min\left\{r^T\lambda : R^T\lambda = \beta^T Q, \lambda \geq 0\right\}, \tag{32}$$

which is satisfactory for a feasible solution γ and λ contained in the following feasible set:

$$\begin{aligned} G_1 &= \left\{\gamma \middle| H^T\gamma = P^T x, \gamma \geq 0\right\} \rightarrow \exists\gamma \geq 0 \ni H^T\gamma = P^T x, \\ G_2 &= \left\{\lambda \middle| R^T\lambda = b^T Q, \lambda \geq 0\right\} \rightarrow \exists\lambda \geq 0 \ni R^T\lambda = b^T Q, \\ G_3 &= \left\{\lambda \middle| R^T\lambda = \beta^T Q, \lambda \geq 0\right\} \rightarrow \exists\lambda \geq 0 \ni R^T\lambda = \beta^T Q. \end{aligned} \tag{33}$$

Thus, the formulation of the multi-objective integer ARC optimization model with the overall Polyhedral Uncertainty Set can be written as:

$$\begin{aligned} \min_{t_i} \; & f_i(t) = t_i, \forall 1, 2, ..., m, \\ \text{s.t.:} \; & -\bar{a}^T x - h^T\gamma - b\bar{y} - r^T\lambda \leq -d, \\ & H\gamma = Px, \\ & R\lambda = b^T Q, \\ & -Cx \leq -b, \\ & \alpha_i^T x + \beta_i^T\bar{y} + r^T\lambda \leq t_i, \\ & R^T\lambda = \beta^T Q, \\ & f_j(t) = f_j^*, j = 1, 2, ..., (m-1), \\ & x, \bar{y}, \gamma, \lambda, Q \geq 0, \\ & t_i \text{ unrestricted,} \end{aligned} \tag{34}$$

with $\alpha_i \in \mathbb{Q}_+^p$, $\bar{a} \in \mathbb{Q}^p$, $h \in \mathbb{R}^n$, $r \in \mathbb{R}^m, d \in \mathbb{Q}^T$, $H \in \mathbb{R}^{z\times n}$, $P \in \mathbb{R}^{z\times p}$, $R \in \mathbb{R}^{w\times m}$, $b \in \mathbb{Q}^q$, $C \in \mathbb{Q}^{l\times p}$, $b \in \mathbb{Q}^q$, $C \in \mathbb{Q}^{l\times p}$, $b \in \mathbb{Q}^l$, $\beta_i \in \mathbb{Q}_+^q$, $Q \in \mathbb{R}^{q\times w}$, $x \in \mathbb{R}^p$, $t_i \in \mathbb{R}$, $\gamma \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$, and $\bar{y} \in \mathbb{N}^q$. Furthermore, the multi-objective integer ARC optimization model with the whole Polyhedral Uncertainty Set (34) can be expressed in the form of sigma and index, i.e.

$$\min_{t_i} \quad f_i(t) = t_i, \forall 1, 2, ..., m, \tag{35}$$

$$\text{s.t.:} \quad \sum_{p\in P} \bar{a}_p \mathbf{x}_p + \sum_{n\in N} h_n\gamma_n + \sum_{q\in Q} b_q\bar{y}_q + \sum_{m\in M} \mathbf{r}_m\lambda_m \geq d_T, \quad \forall t \in T, \tag{36}$$

$$\sum_{n\in N} H_{zn}\gamma_n = \sum_{p\in P} P_{zp}x_p, \quad \forall z \in Z, \tag{37}$$

$$\sum_{m\in M} R_{wm}\lambda_m = \sum_{q\in Q} b_q Q_{qw}, \quad \forall w \in W, \tag{38}$$

$$\sum_{p\in P} C_{lp}x_p \geq b_l, \quad \forall l \in L, \tag{39}$$

$$\sum_{p\in P} \alpha_p^{(i)} x_p + \sum_{q\in Q} \beta_q^{(i)}\bar{y}_q + \sum_{m\in M} r_m\lambda_m \leq t_i, \tag{40}$$

$$\sum_{m\in M} R_{wm}\lambda_m = \sum_{q\in Q} \beta_q Q_{qw}, \quad \forall w \in W, \tag{41}$$

$$f_j(t) = f_j^*, \quad j = 1, 2, \ldots, (m-1), \tag{42}$$

$$x_p, \bar{y}_q, \gamma_n, \lambda_m, Q_{qw} \geq 0, \tag{43}$$

$$t_i \text{ unrestricted.} \tag{44}$$

IV. Discussion

A. Mathematical Characterization Based on Convexity in the Multi-objective Integer Adjustable Robust Counterpart (ARC) Optimization Model with Polyhedral Uncertainty

This subsection refers to Theorem 1 by presenting the proof of the convex set on the entire model solution set (35)-(44). There are five types of solution sets for the non-negative decision variables x_p, $\bar{y}_q$, γ_n, λ_m, and Q_{qw} as follows:

$$\begin{aligned} x_p &= (x_1, x_2, \ldots, x_n) \geq 0, \\ \bar{y}_q &= (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_i) \geq 0, \\ \lambda_m &= (\lambda_1, \lambda_2, \ldots, \lambda_i) \geq 0, \\ \gamma_n &= (\gamma_1, \gamma_2, \ldots, \gamma_i) \geq 0, \\ Q_{qw} &= (Q_1, Q_2, \ldots, Q_n) \geq 0. \end{aligned} \tag{45}$$

The first set of solutions is S_1 which is defined as

$$S_1 = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | x_i \geq 0,\ i = 1, 2, \ldots, n\}. \tag{46}$$

It will be proved that S_1 is a convex set.

To this end, take any $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in S_1$ with

$$\mathbf{x}^{(1)} = (x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}) \tag{47}$$

and

$$\mathbf{x}^{(2)} = \left(x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}\right), \tag{48}$$

with $\lambda \in [0, 1]$, so that:

$$\mathbf{x}^{(2)} + (1 - \lambda)\, \mathbf{x}^{(1)} \geq \lambda \cdot 0 + (1 - \lambda) \cdot 0 = 0. \tag{49}$$

Therefore, it is clear that $\lambda \mathbf{x}^{(2)} + (1 - \lambda)\, \mathbf{x}^{(1)} \in S_1$, and S_1 proved to be a convex set. Similar proofs are carried out for the entire solution set, proving that the whole solution set is a convex set.

The convexity analysis of the objective function can be shown as follows. Referring to the model (35)-(44), the objective function in equation (35) is $f(t) = t$. Further analysis was carried out to check whether the objective function was convex. Note that $f(t) = t$. Take any $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$, so that:

$$f(\lambda x + (1 - \lambda)\, y) = \lambda x + (1 - \lambda) = \lambda f(x) + (1 - \lambda)\, f(y). \tag{50}$$

Thus, based on (50), it is clear that the objective function $f(t) = t$ is convex.

B. Multi-objective Integer Adjustable Robust Counterpart (ARC) Optimization Model with Polyhedral Uncertainty Set Using Benders Decomposition Method

This subsection discusses the application of the Benders Decomposition Method as a calculation process in solving numerical experiments on an integer multi-objective Adjustable Robust Counterpart (ARC) optimization model with a Polyhedral Uncertainty Set. Duality Theory is needed to apply the Benders Decomposition Method to decompose the obtained model into a Master Problem and Dual Subproblem.

Referring to the model (35)-(44), the first thing to do is to decompose the decision variables and the constraint function into two parts. The first part is the constraint function, which only contains the discrete decision variable $\bar{y} \in \mathbb{N}^q$ in (36) and (40). The second part is the constraint function which only includes the continuous decision variables $t_i \in \mathbb{R}$, $\gamma \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$, $Q \in \mathbb{R}^{q \times w}$, and $x \in \mathbb{R}^p$ in (37), (38), (39), (41), and (42), which represents the linear part that needs to be duplicated. Furthermore, σ, π, τ, μ, and θ are dual variables for the constraints (37), (38), (39), (41), and (42), respectively.

The steps to determine the Master Problem or $M(\bar{y}, m)$ as the result of the first decomposition and the Dual Subproblem or $S(\sigma, \pi, \tau, \mu, \pi | \bar{y})$ as the result of the second decomposition are to create a Relaxed Master Problem or $M(\bar{y}, m = 0)$. Formulation of $M(\bar{y}, m = 0)$ is a separate model that only contains the discrete decision variable $\bar{y} \in \mathbb{N}^q$. If, in the model, there are continuous decision variables simultaneously, then suppose the continuous decision variables become equal to one. The formulation is obtained as follows:

$$\begin{aligned} &\min 0, \\ &\text{s.t.: } \sum_{p \in P} \bar{a}_p x_p + \sum_{n \in N} h_n \gamma_n + \sum_{q \in Q} b_{qT} \bar{y}_q \\ &\qquad + \sum_{m \in M} r_m \lambda_m \geq d_T,\ \forall t \in T, \\ &\qquad \sum_{p \in P} \alpha_p^{(i)} x_p + \sum_{q \in Q} \beta_q^{(i)} \bar{y}_q + \sum_{m \in M} r_m \lambda_m \leq t_i, \\ &\qquad \bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_i) \geq 0, \\ &\qquad x_p = (x_1, x_2, \ldots, x_p) = 1, \\ &\qquad \gamma_n = (\gamma_1, \gamma_2, \ldots, \gamma_n) = 1, \\ &\qquad \lambda_m = (\lambda_1, \lambda_2, \ldots, \lambda_m) = 1, \\ &\qquad t_i = (t_1, t_2, \ldots, t_i) = 1. \end{aligned} \tag{51}$$

Next, a formulation of the Inner Optimization Problem or $P(t_i, \gamma, \lambda, Q, x, \bar{y})$, where the formulation is a separate model containing only continuous decision variables t_i, γ_n, λ_m, Q_{qw}, and x_p. If the model includes discrete decision variables simultaneously, then suppose the discrete decision variables are equal to one so that the following formulation is obtained:

$$\begin{aligned} &\min_{t_i} f_i(t) = t_i, \forall i = 1, 2, ..., m, \\ &\text{s.t.: } \sum_{n \in N} H_{zn} \gamma_n = \sum_{p \in P} P_{zp} x_p,\ \forall z \in Z, \\ &\qquad \sum_{m \in M} R_{wm} \lambda_m = \sum_{q \in Q} b_{qT} Q_{qw}, \forall w \in W, t \in T, \\ &\qquad \sum_{p \in P} C_{lp} x_p \geq b_l,\ \forall l \in L, \\ &\qquad \sum_{m \in M} R_{wm} \lambda_m = \sum_{q \in Q} \beta_q Q_{qw}, \forall w \in W, \\ &\qquad f_j(t) = f_j^*,\ j = 1, 2, \ldots, (m - 1), \\ &\qquad x_p = (x_1, x_2, \ldots, x_p) \geq 0, \\ &\qquad \gamma_n = (\gamma_1, \gamma_2, \ldots, \gamma_n) \geq 0, \\ &\qquad \lambda_m = (\lambda_1, \lambda_2, \ldots, \lambda_m) \geq 0, \\ &\qquad Q_{qw} = (Q_1, Q_2, \ldots, Q_i) \geq 0, \\ &\qquad t_i \text{ unrestricted}, \end{aligned} \tag{52}$$

with $\alpha_i \in \mathbb{Q}_+^p$, $\bar{a} \in \mathbb{Q}^p$, $h \in \mathbb{R}^n$, $r \in \mathbb{R}^m$, $d \in \mathbb{Q}^V$, $H \in \mathbb{R}^{z \times n}$, $P \in \mathbb{R}^{z \times p}$, $R \in \mathbb{R}^{w \times m}$, $b \in \mathbb{Q}^{q \times V}$, $C \in \mathbb{Q}^{l \times p}$, $b \in \mathbb{Q}^l$, $\beta_i \in \mathbb{Q}_+^q$, $Q \in \mathbb{R}^{q \times w}$, $x \in \mathbb{R}^p$, $t_i \in \mathbb{R}$, $\gamma \in \mathbb{R}^n$,

$\lambda \in \mathbb{R}^m$, $\bar{y} \in \mathbb{N}^q$. See that $P(\bar{y}, t, \gamma, \lambda, x)$ model in (52) is a model to be duplicated with a dual variable σ, π, τ, μ, for each constraint. The dual result of the model is called Dual Subproblem or $S(\sigma, \pi, \tau, \mu, \theta|\bar{y})$ the result of the second decomposition, which is formulated as follows:

$$\begin{aligned}
\max_{\tau,\theta} \quad & \tau_l b_l + \theta_j f_j^*, \quad \forall l \in L,\ j \in J, \\
\text{s.t.:} \quad & -\sum_{p\in P} P_{zp}\tau_p + \sum_{p\in P} C_{lp}\tau_p \leq 0, \\
& \sum_{n\in N} H_{zn}\sigma_n \leq 0, \\
& \sum_{m\in M} R_{wm}\pi_m + \sum_{m\in M} R_{wm}\mu_m \leq 0, \\
& -\sum_{q\in Q} b_{qT}\pi_{qw} - \sum_{q\in Q} \beta_q\mu_{qw} \leq 0, \\
& (f_j(t))\,\theta_j = 1, \quad j = 1, 2, \ldots, (m-1), \\
& \tau_p = (\tau_1, \tau_2, \ldots, \tau_p) \geq 0, \\
& \tau_l = (\tau_1, \tau_2, \ldots, \tau_l) \geq 0, \\
& \sigma_n,\ \pi_{qw},\ \pi_m,\ \mu_{qw},\ \mu_m, \text{ and } \theta_j \text{ unrestricted.}
\end{aligned} \tag{53}$$

The next step is to determine the Benders Cut obtained directly from the formulation objective function $S(\sigma, \pi, \tau, \mu, \theta|\bar{y})$ in (53). Benders Cut is a new constraint added to the Relaxed Master Problem or $M\,(\bar{y}, m = 0)$, so as that produces a Full Master Problem or $M(\bar{y}, m)$ as the result of the first decomposition. Benders Cut is formulated as follows:

$$\tau_l b_l + \theta_j f_j^* \leq m,\ \forall l \in L, j \in J, \tag{54}$$

with b_l and f_j^* are parameters, τ_j a dual variable, and an unknown number. Furthermore, the formulation of the Full Master Problem or $M(\bar{y}, m)$ as the result of the first decomposition is as follows:

$$\begin{aligned}
& \min\ m, \\
& \text{s.t.:} \sum_{p\in P} \bar{a}_p x_p + \sum_{n\in N} h_n\gamma_n + \sum_{q\in Q} b_{qT}\bar{y}_q \\
& \quad + \sum_{m\in M} r_m\lambda_m \geq d_T, \\
& \quad \sum_{p\in P} \alpha_p^{(i)} x_p + \sum_{q\in Q} \beta_q^{(i)}\bar{y}_q + \sum_{m\in M} r_m\lambda_m \leq t_i, \\
& \quad \bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_i) \geq 0, \\
& \quad x_p = (x_1, x_2, \ldots, x_p) = 1, \\
& \quad \gamma_n = (\gamma_1, \gamma_2, \ldots, \gamma_n) = 1, \\
& \quad \lambda_m = (\lambda_1, \lambda_2, \ldots, \lambda_m) = 1, \\
& \quad t_1 = (t_1, t_2, \ldots, t_i) = 1.
\end{aligned} \tag{55}$$

Thus, the implementing Benders Decomposition Method is done. The Benders Decomposition method is said to be successful if it can be applied to the model so that it is partitioned into two parts: the Dual Subproblem Model (the result of the first decomposition) and the Full Master Problem Model (the result of the second decomposition). The two decomposition results must produce an optimal and computationally tractable solution using the ARC methodology.

V. CONCLUSIONS

The Adjustable Robust Counterpart (ARC) optimization model for multi-objective integer programming problems with a polyhedral uncertainty set can be effectively solved using the Benders Decomposition Method. This is justified by the inherent two-stage structure of the ARC frameworkcomprising the here-and-now (first-stage) and wait-and-see (second-stage) decisionsand the mixed nature of variables (integer and continuous), which aligns well with the decomposition principles of Benders methodology. The convexity analysis conducted on the feasible set, objective functions, and constraints confirms that the local optimal solution obtained from the ARC model is also globally optimal, ensuring solution reliability under uncertainty.

For future research, the integration of machine learning techniques presents a promising direction for enhancing the tractability of ARC models. As demonstrated by Lee *et al.* [36], machine learning can be employed to estimate uncertainty sets and accelerate the solution process for robust mixed-integer linear programs. Additionally, dynamic programming approachessuch as those discussed by Shapiro [37] may offer alternative strategies for solving ARC models, particularly in high-dimensional or stage-dependent problem structures.

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