Research on Nicholson's Blowflies Models with Delays and Patch Structure

Ahmadjan Muhammadhaji, and Jinting Zhao

Abstract—**Nicholson's blowflies model stands as a classic mathematical framework for characterizing population dynamics involving nonlinear growth and time delays, celebrated for its ability to capture the periodic fluctuations observed in insect populations. In this study, the traditional Nicholson's blowflies model is extended by integrating both time delays and patch structure, with the goal of investigating how inter-patch migration between patches and the combined effects of continuous and discrete delays collectively influence population stability and dynamics. By leveraging advanced inequality techniques, the Lyapunov method, and the comparison principle, novel criteria are derived for the periodicity, extinction, permanence, and global attractiveness of these models. Furthermore, two illustrative examples are presented to validate the effectiveness of the main results reported in this work.**

Index Terms—**Nicholson's blowflies model; extinction; time delay; permanence; global attractiveness**

I. Introduction

POPULATION dynamics, a core topic in ecology, focuses on understanding the mechanisms driving fluctuations in species abundance over time and space. Nicholson's blowflies experiment [1] was a landmark study, demonstrating that insect populations can exhibit stable periodic oscillations due to nonlinear feedbacks between reproduction and mortality. To characterize fluctuations in the reproductive population of Australian sheep blowflies and align with the experimental data reported in [2], Gurney and colleagues initially proposed Nicholson's blowflies model in 1980:

$$\dot{x}(t) = -dx(t) + \beta x(t-\tau)e^{-\alpha x(t-\tau)}. \tag{1}$$

The corresponding mathematical model, known as Nicholson's blowflies model, where time delays account for the lag between reproduction and maturation. Since its inception, this model has garnered extensive attention from researchers seeking to investigate its dynamic behavior. Over the past decades, extensions of Nicholson's model have been widely studied to incorporate realistic ecological factors [3-8]. Time delays, such as those in development, gestation, or resource regeneration, are critical in shaping population dynamics, as they can induce instability and periodic behavior [9]. Meanwhile, spatial structure (e.g., patchiness) is ubiquitous in nature, as populations often inhabit discrete patches connected by migration. Migration between patches can promote gene flow, reduce local extinction risks, and alter population stability [10]. On the other hand, in ecological systems, resource constraints and environmental fluctuations often drive species in multiple patches to interact and migrate, leading to inter-patch competition or cooperation and subsequent population migration [11-13]. However, the combined effects of delays and patch structure on Nicholson's blowflies dynamics remain underexplored, particularly regarding how migration modulates delay-induced instabilities.

As noted in [14], recent studies have revealed that patch structure substantially influences Nicholson's blowflies model. Patch structure accounts for heterogeneous environments arising from multiple factors, where resource heterogeneity among patches governs the spatial distribution of blowflies through both inter-patch migration and local population growth. Additionally, time delays often reflect the maturity of biological species and can induce fundamental shifts in a systems dynamic behavior. Consequently, the interplay between patch structure and time delays enables more precise investigations into Nicholson blowflies models. Several researchers have explored Nicholson blowflies models incorporating delays and patch structures (e.g., [15-21]). For example, Teresa Faria [15] studied the n-dimensional Nicholson's blowflies model with patch structure and multiple discrete delays:

$$\begin{aligned}\dot{x}_i(t) = &-d_i x_i(t) + \sum_{k=1}^{m} \beta_{ik} x_i(t-\tau_{ik}) e^{-x_i(t-\tau_{ik})} \\ &+ \sum_{j=1}^{n} a_{ij} x_j(t),\ i = 1, 2, \cdots, n,\end{aligned} \tag{2}$$

where $a_{11} = 0, a_{22} = 0, \cdots, a_{nn} = 0$. The authors investigated the global asymptotic properties of solutions and established the uniform boundedness of solutions. Additionally, under specific conditions, they proved the existence and global attractivity of the positive equilibrium. In light of (2), Bingwen Liu [16] investigated an n-dimensional Nicholson blowflies model incorporating patch structure and variable delays:

$$\begin{aligned}\dot{x}_i(t) = &\sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{m} \beta_{ij} x_i(t-\tau_{ij}(t)) e^{-x_i(t-\tau_{ij}(t))} \\ &-dx_i(t), i = 1, 2, \cdots, n.\end{aligned} \tag{3}$$

By constructing a Lyapunov functional, a novel sufficient condition is derived to ensure the global stability of the positive equilibrium solution of (3). Building upon systems (2) and (3), the authors in [17] investigated the following non-autonomous delay Nicholson-type system with patch

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Ahmadjan Muhammadhaji is an associate professor of College of Mathematics and System Sciences, Xinjiang University, Urumqi 830017, P. R. China (Corresponding author to provide e-mail: ahmatjanam@aliyun.com).

Jinting Zhao is a postgraduate student of College of Mathematics and System Sciences, Xinjiang University, Urumqi 830017, P. R. China (e-mail: 842472754@qq.com).

structure:

$$\begin{aligned}\dot{z}_1(t) &= -a_1(t)z_1(t) + c_1(t)z_1(t-\gamma)e^{-z_1(t-\gamma)} \\ &\quad + b_1(t)z_2(t), \\ \dot{z}_2(t) &= -a_2(t)z_2(t) + c_2(t)z_2(t-\gamma)e^{-z_2(t-\gamma)} \\ &\quad + b_2(t)z_1(t).\end{aligned} \tag{4}$$

By applying inequality techniques, the comparison principle, and Lyapunov functional construction, sufficient conditions are derived for the boundedness, persistence, extinction, existence, and global attractivity of positive periodic solutions. Furthermore, at the conclusion of this paper, the authors pose an intriguing question: investigate the dynamic properties of the following non-autonomous Nicholson blowflies model:

$$\begin{aligned}\dot{z}_a(t) &= -q_a(t)z_a(t) + \sum_{b=1, b\neq a}^{n} p_{ab}(t)z_b(t) \\ &\quad + k_a(t)z_a(t-r_a(t))e^{-z_a(t-r_a(t))},\end{aligned} \tag{5}$$

where $a = 1, 2, \cdots, n$. However, when modeling periodic population dynamics, the coefficients and time delays in the model are typically time-periodic, and few studies have investigated the scenario where coefficients are time-dependent functions in Nicholson blowflies models with patch structure. Building on the previous works and analyses, this paper aims to investigate the dynamic properties of the following Nicholson blowflies model with time-varying coefficients and continuous delays:

$$\begin{aligned}\dot{z}_a(t) &= -q_a(t)z_a(t) + \sum_{b=1, b\neq a}^{n} p_{ab}(t)z_b(t) \\ &\quad + \sum_{b=1}^{n} k_{ab}(t)z_a(t-r_b(t))e^{-z_a(t-r_b(t))},\end{aligned} \tag{6}$$

and the following Nicholson's blowflies model with time-varying coefficients and discrete delays

$$\begin{aligned}\dot{z}_a(t) &= -q_a(t)z_a(t) + \sum_{b=1, b\neq a}^{n} p_{ab}(t)z_b(t) \\ &\quad + \sum_{b=1}^{n} k_{ab}(t)z_a(t-r_b)e^{-z_a(t-r_b)},\end{aligned} \tag{7}$$

where $a = 1, 2, \cdots, n$. The aim of this study is to derive several conditions on the extinction, permanence, periodic solutions, and global attractivity for models (6) and (7). This study addresses this research gap by developing a Nicholson's blowflies model that integrates delays and patch structure. Specifically, we consider a system consisting of n patches, where the population dynamics in each patch are governed by nonlinear growth processes with an associated time delay, and migration between patches occurs at constant rates. Our objectives are threefold: to formally establish the model framework; to derive rigorous conditions characterizing the periodicity, extinction, permanence, and global attractiveness of the system; and to validate the theoretical findings through systematic numerical simulations.

In systems (6) and (7), we have that $z_a(t)(a = 1, 2, \cdots, n)$ denotes the density of the species in patch a at time t; $q_a(t)(a = 1, 2, \cdots, n)$ is the per capita daily adult death rate in patch a at time t; $p_{ab}(t)(a, b = 1, 2, \cdots, n, a \neq b)$ is the migration coefficient from patch b to patch a at time t, and the natural growth in each patch is of Nicholson-type; $k_{ab}(t)(a, b = 1, 2, \cdots, n)$ is the maximum per capita daily egg production at time t; $r_b(b = 1, 2, \cdots, n)$ and $r_b(t)(b = 1, 2, \cdots, n)$ denote time delay.

The basic assumptions for system (6) and system (7) are given by

(H_1) $q_a(t) > 0, r_b(t) > 0, p_{ab}(t) > 0 (a \neq b), k_{ab}(t) > 0 (a, b = 1, 2, \cdots, n)$ are all continuously positive T-periodic functions on $[0, \infty)$.

(H_2) $r_b > 0 (b = 1, 2, \cdots, n)$ are constants. $q_a(t) > 0, p_{ab}(t) > 0 (a \neq b), k_{ab}(t) > 0 (a, b = 1, 2, \cdots, n)$ are all continuously positive T-periodic functions on $[0, \infty)$.

The initial conditions for system (6) and system (7) take the form

$$\begin{cases}(z_1(t), z_2(t), \cdots, z_n(t)) = (\varsigma_1(t), \varsigma_2(t), \cdots, \varsigma_n(t)) \in C_+, \\ \varsigma_1(0), \varsigma_2(0), \cdots, \varsigma_n(0) > 0,\end{cases} \tag{8}$$

where

$$C_+ = C([-r, 0], R^n_{+0}) \text{ and } r = \max_{1\leq b\leq n} \sup_{t\in[0,\infty)} r_b(t).$$

Define

$$F^+ = \sup_{t\in[0,\infty)} \{f(t)\} \text{ and } F^- = \inf_{t\in[0,\infty)} \{f(t)\},$$

where $F(t) \in [0, +\infty)$ is a continuous and bounded function.

II. Positivity, permanence, extinction and periodic solution

Theorem 1. All solutions of system (6) with the initial condition (8) are positive for all $t \geq 0$.

Proof. Let $(z_1(t), z_2(t), \cdots, z_n(t))$ be a any solution of system (6) with (8). By system (6), when $t \in [0, r], a = 1, 2, \cdots, n$, we obtain

$$\begin{aligned}\dot{z}_a(t) &= -q_a(t)z_a(t) + \sum_{b=1, b\neq a}^{n} p_{ab}(t)z_b(t) \\ &\quad + \sum_{b=1}^{n} k_{ab}(t)\varsigma_a(t-r_b(t))e^{-\varsigma_a(t-r_b(t))} \\ &\geq -q_a(t)z_a(t).\end{aligned}$$

Since $\varsigma_a(t)$ nonnegative on $t \in [-r, 0]$ and by comparison argument, we further have

$$z_a(t) \geq z_a(0)e^{-\int_0^t q_a(s)ds}.$$

Thus, $z_a(t) > 0$ for $t \in [0, r]$. Next, treat intervals $[r, 2r], \cdots, [nr, (n+1)r]$ in the same way, we have $z_a(t) > 0 (a = 1, 2, \cdots, n)$ for $t > 0$.

Corollary 1. All solutions of system (7) with the initial condition (8) are positive for all $t \geq 0$.

Theorem 2. If (H_1) and $B_a > 0 (a = 1, 2, \ldots, n)$ hold, then system (6) is ultimately bounded, where $B_a = q_a^- - \sum_{b=1, b\neq a}^{n} p_{ba}^+$.

Proof. Let $(z_1(t), z_2(t), \cdots, z_n(t))$ is any positive solution of system (6) with (8). Define a function $u(t) = \sum_{a=1}^n z_a(t)$.

Then from the derivative of $u(t)$ and by $\max_{\theta\geq 0}\theta e^{-\theta}\leq \frac{1}{e}$ for $t>r$, we obtain

$$\begin{aligned}\dot{u}(t) &\leq -(q_1^- - \sum_{b=2}^{n} p_{b1}^+)z_1(t) - (q_2^- - \sum_{b=1,b\neq 2}^{n} p_{b2}^+) \\ &\quad \times z_2(t) - \cdots - (q_n^- - \sum_{b=1,b\neq n}^{n} p_{bn}^+)z_n(t) \\ &\quad +(\sum_{b=1}^{n} k_{1b}^+ + \sum_{b=1}^{n} k_{2b}^+ + \cdots + \sum_{b=1}^{n} k_{nb}^+)\frac{1}{e} \\ &= -B_1z_1(t) - B_2z_2(t) - \cdots - B_nz_n(t) \\ &\quad + \sum_{a=1}^{n}\sum_{b=1}^{n} k_{ab}^+\frac{1}{e} \\ &\leq G_1 - G_2u(t),\end{aligned}$$

where

$$G_1 = \sum_{a=1}^{n}\sum_{b=1}^{n} k_{ab}^+\frac{1}{e}, G_2 = \min\{B_1, B_2, \cdots B_n\}.$$

By Lemma 2.1 in [22], we derive

$$u(t) \leq \frac{G_1}{G_2} + \big(u(0) - \frac{G_1}{G_2}\big)e^{-G_2t}.$$

Then

$$\lim_{t\to+\infty} u(t) = \lim_{t\to+\infty}(\sum_{a=1}^{n} z_a(t)) = \frac{G_1}{G_2}.$$

Hence, there exists a real number $T_0 > r$ such that

$$z_1(t) < \frac{G_1}{G_2}, z_2(t) < \frac{G_1}{G_2}, \cdots, z_n(t) < \frac{G_1}{G_2},$$

as $t > T_0$.

Corollary 2. If (H_1) holds and $B_a > 0$ for $a = 1, 2, \ldots, n$, then system (6) is ultimately bounded, where $B_a = q_a^- - \sum_{b=1,b\neq a}^{n} p_{ba}^+$.

Theorem 3. If for $a = 1, 2, \cdots, n$, there are three constants: $m \in (0, 1)$, $l \in (1, +\infty)$, M, such that

$$me^{-m} = le^{-l}, m < M \leq l, \tag{9}$$

and

$$-q_a^- M + \sum_{b=1,b\neq a}^{n} p_{ab}^+ M + \sum_{b=1}^{n} k_{ab}^+\frac{1}{e} < 0, \tag{10}$$

$$-q_a^+ + \sum_{b=1,b\neq a}^{n} p_{ab}^- + \sum_{b=1}^{n} k_{ab}^- e^{-m} > 0, \tag{11}$$

then system (6) is permanent.

Proof. Suppose that $(z_1(t), z_2(t), \cdots, z_n(t))$ is any positive solution of system (6) with (8). From Theorem 1, we have

$$z_a(t) > 0,\ \forall t > 0, a = 1, 2, \cdots, n. \tag{12}$$

We first prove

$$z_a(t) < M,\ \forall t > 0, a = 1, 2, \cdots, n. \tag{13}$$

If not, there exist $\xi' \in (0, +\infty)$ and $a \in \{1, 2, \cdots, n\}$, such that

$$z_a(\xi') = M, z_b(t) < M, \forall t \in [-r, \xi'), b = 1, 2, \cdots, n. \tag{14}$$

From (10),(14) and $\sup_{\theta\geq 0}\theta e^{-\theta} = \frac{1}{e}$, we obtain

$$\begin{aligned}0 &\leq \dot{z}_a(\xi') \\ &= -q_a(\xi')z_a(\xi') + \sum_{b=1,b\neq a}^{n} p_{ab}(\xi')z_b(\xi') \\ &\quad + \sum_{b=1}^{n} k_{ab}(\xi')z_a(\xi' - r_b(\xi'))e^{-z_a(\xi' - r_b(\xi'))} \\ &\leq -q_a(\xi')M + \sum_{b=1,b\neq a}^{n} p_{ab}(\xi')M + \sum_{b=1}^{n} k_{ab}(\xi')\frac{1}{e} \\ &\leq -q_a^- M + \sum_{b=1,b\neq a}^{n} p_{ab}^+ M + \sum_{b=1}^{n} k_{ab}^+\frac{1}{e} < 0.\end{aligned}$$

This leads to a contradiction; therefore, relation (13) holds. Next, we proceed to prove that

$$z_a(t) > m, \forall t > 0, a = 1, 2, \cdots, n. \tag{15}$$

If not, we can find $\xi'' \in (0, +\infty)$ and $a \in \{1, 2, \cdots, n\}$, which satisfy the following formula

$$z_a(\xi'') = m, z_b(t) > m, \forall t \in [-r, \xi''), b = 1, 2, \cdots, n. \tag{16}$$

From (9),(13) and (16), we obtain

$$m \leq z_a(\xi'' - r_b(\xi'')) \leq M \leq l.$$

So

$$z_a(\xi'' - r_b(\xi''))e^{-z_a(\xi'' - r_b(\xi''))} \geq \min\left\{\frac{m}{e^m}, \frac{l}{e^l}\right\} = \frac{m}{e^m},$$

where $a, b = 1, 2, \cdots, n$. From (11) and (16), we have

$$\begin{aligned}0 &\geq \dot{z}_a(\xi'') \\ &= -q_a(\xi'')z_a(\xi'') + \sum_{b=1,b\neq a}^{n} p_{ab}(\xi'')z_b(\xi'') \\ &\quad + \sum_{b=1}^{n} k_{ab}(\xi'')z_a(\xi'' - r_b(\xi''))e^{-z_a(\xi'' - r_b(\xi''))} \\ &\geq -q_a(\xi'')m + \sum_{b=1,b\neq a}^{n} p_{ab}(\xi'')m + \sum_{b=1}^{n} k_{ab}(\xi'')\frac{m}{e^m} \\ &\geq m(-q_a^+ + \sum_{b=1,b\neq a}^{n} p_{ab}^- + \sum_{b=1}^{n} k_{ab}^- e^{-m}) > 0.\end{aligned}$$

This leads to a contradiction; therefore, equation (15) holds. Consequently, every positive solution $(z_1(t), z_2(t), \cdots, z_n(t))$ of system (6) with (8) satisfies

$$m \leq z_1(t) \leq M, m \leq z_2(t) \leq M, \cdots, m \leq z_n(t) \leq M,$$

for $t > 0$.

Corollary 3. If for $a = 1, 2, \cdots, n$, there are three constants: $m_0 \in (0, 1)$, $l_0 \in (1, +\infty)$ and M_0, such that

$$m_0e^{-m_0} = l_0e^{-l_0}, m_0 < M_0 \leq l_0, \tag{17}$$

and

$$-q_a^- M_0 + \textstyle\sum_{b=1,b\neq a}^{n} p_{ab}^+ M_0 + \sum_{b=1}^{n} k_{ab}^+\frac{1}{e} < 0, \tag{18}$$

$$-q_a^+ + \textstyle\sum_{b=1,b\neq a}^{n} p_{ab}^- + \sum_{b=1}^{n} k_{ab}^- e^{-m_0} > 0, \tag{19}$$

then system (7) is permanent.

Theorem 4. Assume that (H$_1$) holds and $\sum_{b=1,b\neq a}^{n} p_{ab}^{+} + \sum_{b=1}^{n} k_{ab}^{+} < q_a^{-} (a = 1, 2, \cdots, n)$, then system (6) is extinct.
Proof. Let $\chi_a(x)$ denote the continuous,where

$$\chi_a(x) = x - q_a^{-} + \sum_{b=1,b\neq a}^{n} p_{ab}^{+} + \sum_{b=1}^{n} k_{ab}^{+} e^{xr}, a = 1, 2, \cdots, n.$$

Based on the conditions of Theorem 4, we obtain

$$\chi_a(0) = -q_a^{-} + \sum_{b=1,b\neq a}^{n} p_{ab}^{+} + \sum_{b=1}^{n} k_{ab}^{+} < 0, a = 1, 2, \cdots, n.$$

From the continuity of $\chi_a(x)$, we can find $\lambda > 0$ such that for each $a = 1, 2, \cdots, n$, we have

$$\chi_a(\lambda) = \lambda - q_a^{-} + \sum_{b=1,b\neq a}^{n} p_{ab}^{+} + \sum_{b=1}^{n} k_{ab}^{+} e^{\lambda r} < 0.$$

Let

$$h_a(t) = z_a(t) e^{\lambda t}, a = 1, 2, \cdots, n.$$

By differentiating $h_a(t)$, we obtain

$$\begin{aligned} \dot{h}_a(t) &= \dot{z}_a(t) e^{\lambda t} + z_a(t) \lambda e^{\lambda t} \\ &= \lambda h_a(t) + e^{\lambda t}[-q_a(t) z_a(t) + \sum_{b=1,b\neq a}^{n} p_{ab}(t) z_b(t) \\ &\quad + \sum_{b=1}^{n} k_{ab}(t) z_a(t - r_b(t)) e^{-z_a(t-r_b(t))}] \\ &= \lambda h_a(t) - q_a(t) h_a(t) + \sum_{b=1,b\neq a}^{n} p_{ab}(t) h_b(t) \\ &\quad + \sum_{b=1}^{n} k_{ab}(t) e^{\lambda r_b(t)} h_a(t - r_b(t)) e^{-z_a(t-r_b(t))}. \end{aligned} \tag{20}$$

Let $M^* > 0$ and $h_a(t) < M^*, \forall t \in [-r, 0], a = 1, 2, \cdots, n$, then we have

$$h_a(t) < M^*, \forall t > 0, a = 1, 2, \cdots, n. \tag{21}$$

If not, there exist $t^* \in (0, +\infty)$ and $a \in \{1, 2, \cdots, n\}$, such that

$$h_a(t^*) = M^*, h_b(t) < M^*, \forall t < t^*, b = 1, 2, \cdots, n.$$

Then, from (20), we have

$$\begin{aligned} 0 &\leq \dot{h}_a(t^*) \\ &= \lambda h_a(t^*) - q_a(t^*) h_a(t^*) + \sum_{b=1,b\neq a}^{n} p_{ab}(t^*) h_b(t^*) \\ &\quad + \sum_{b=1}^{n} k_{ab}(t^*) e^{\lambda r_b(t^*)} h_a(t^* - r_b(t^*)) e^{-z_a(t^*-r_b(t^*))} \\ &\leq \lambda M^* - q_a(t^*) M^* + \sum_{b=1,b\neq a}^{n} p_{ab}(t^*) M^* \\ &\quad + \sum_{b=1}^{n} k_{ab}(t^*) e^{\lambda r} M^* \\ &\leq (\lambda - q_a^{-} + \sum_{b=1,b\neq a}^{n} p_{ab}^{+} + \sum_{b=1}^{n} k_{ab}^{+} e^{\lambda r}) M^* < 0. \end{aligned}$$

This leads to a contradiction; consequently, relation (21) is established, and

$$z_a(t) = h_a(t) e^{-\lambda t} \leq M^* e^{-\lambda t}, a = 1, 2, \cdots, n.$$

Hence

$$\lim_{t\to+\infty} z_a(t) = 0, a = 1, 2, \cdots, n.$$

Corollary 4. Assume that (H$_2$) holds and $\sum_{b=1,b\neq a}^{n} p_{ab}^{+} + \sum_{b=1}^{n} k_{ab}^{+} < q_a^{-}, (a = 1, 2, \cdots, n)$, then system (7) is extinct.

By applying the result of Lemma 4 in [23], together with Theorem 5 and Corollary 6, we can derive the following corollaries.

Corollary 5. If the conditions of Theorem 3 are satisfied and the coefficients of system (6) are T-periodic functions on the interval $[0, T]$, then system (6) is permanent and admits at least one positive T-periodic solution.

Corollary 6. If the conditions of Corollary 3 are satisfied and the coefficients of system (7) are T-periodic functions on the interval $[0, T]$, then system (7) is permanent and admits at least one positive T-periodic solution.

III. Global Attractivity

Theorem 5. If the conditions of Theorem 5 hold and

$$\sum_{b=1,b\neq a}^{n} p_{ab}^{+} + \sum_{b=1}^{n} k_{ab}^{+} \frac{1}{e^2} < q_a^{-} (a = 1, 2, \cdots, n), \tag{22}$$

then system (6) is globally attractive.
Proof. Suppose that $(z_1(t), z_2(t), \cdots, z_n(t))$ and $(z_1^*(t), z_2^*(t), \cdots, z_n^*(t))$ are any two positive solutions of system (6) with (8). Define the continuous functions $\chi_a(x)$ by setting

$$\chi_a(x) = x - q_a^{-} + \sum_{b=1,b\neq a}^{n} p_{ab}^{+} + \sum_{b=1}^{n} k_{ab}^{+} \frac{1}{e^2} e^{xr},$$

where $x \in [0, 1], a = 1, 2, \cdots, n$. According to the conditions of Theorem 11, we obtain

$$\chi_a(0) = -q_a^{-} + \sum_{b=1,b\neq a}^{n} p_{ab}^{+} + \sum_{b=1}^{n} k_{ab}^{+} \frac{1}{e^2} < 0, a = 1, 2, \cdots, n.$$

From the continuity of $\chi_a(x)$, we can find $\mu > 0$ and $\lambda \in (0, 1]$, such that

$$\begin{aligned} \chi_a(\lambda) &= \lambda - q_a^{-} + \sum_{b=1,b\neq a}^{n} p_{ab}^{+} + \sum_{b=1}^{n} k_{ab}^{+} \frac{1}{e^2} e^{\lambda r} \\ &< -\mu < 0, a = 1, 2, \cdots, n, t > 0. \end{aligned} \tag{23}$$

Let $h_a(t) = z_a(t) - z_a^*(t), t \in [-r, +\infty), a = 1, 2, \cdots, n$, then

$$\begin{aligned} \dot{h}_a(t) = \dot{z}_a(t) - \dot{z}_a^*(t) &= -q_a(t)(z_a(t) - z_a^*(t)) \\ &+ \sum_{b=1,b\neq a}^{n} p_{ab}(t)(z_b(t) - z_b^*(t)) \\ &+ \sum_{b=1}^{n} k_{ab}(t)(z_a(t - r_b(t)) e^{-z_a(t-r_b(t))} \\ &- z_a^*(t - r_b(t)) e^{-z_a^*(t-r_b(t))}). \end{aligned}$$

We consider the Lyapunov function

$$V_a(t) = |h_a(t)|e^{\lambda t}, \lambda > 0.$$

Then we get

$$\begin{aligned} D^- V_a(t) \le& \lambda |h_a(t)|e^{\lambda t} - q_a(t)\text{sign}(z_a(t) - z_a^*(t)) \\ &\times (z_a(t) - z_a^*(t))e^{\lambda t} + \sum_{b=1,b\neq a}^{n} p_{ab}(t) \\ &\times \text{sign}(z_b(t) - z_b^*(t))(z_b(t) - z_b^*(t))e^{\lambda t} \\ &+ \sum_{b=1}^{n} k_{ab}(t)|z_a(t - r_b(t))e^{-z_a(t-r_b(t))} \\ &- z_a^*(t - r_b(t))e^{-z_a^*(t-r_b(t))}|e^{\lambda t}. \end{aligned} \tag{24}$$

From Theorem 3, let

$$\psi \in C^0 = \{\psi | \psi \in C, \psi(\theta) \in (m, M), \theta \in [-r, 0]\},$$

then for $\forall t > t_{\psi,z^*}, a = 1, 2, \cdots, n$, we have

$$\begin{aligned} &V_a(t) = |h_a(t)|e^{\lambda t} < e^{\lambda t_{\psi,z^*}} (\max_{a=1,2,\cdots,n} \\ &\max_{t\in[-r,t_{\psi,z^*}]} |z_a(t) - z_a^*(t)| + 1) := K_{\psi,z^*}. \end{aligned} \tag{25}$$

If the aforementioned formula fails to hold, we can identify $t_* > t_{\psi,z^*}$ and $a \in \{1, 2, \cdots, n\}$, such that for each $b = 1, 2, \cdots, n$ the conditions

$$V_a(t_*) = K_{\psi,z^*}, V_b(t) < K_{\psi,z^*}, \forall t \in [-r, t_*). \tag{26}$$

are satisfied. From (23),(24),(26), and

$$|se^{-s} - te^{-t}| \le \frac{1}{e^2}|s - t|, \quad s, t \in (0, +\infty],$$

we have

$$\begin{aligned} 0 \le& D^- V_a(t_*) \le \lambda|h_a(t_*)|e^{\lambda t_*} - q_a(t_*)\text{sign}(z_a(t_*) \\ &- z_a^*(t_*))(z_a(t_*) - z_a^*(t_*))e^{\lambda t_*} + \sum_{b=1,b\neq a}^{n} p_{ab}(t_*) \\ &\times \text{sign}(z_b(t_*) - z_b^*(t_*))(z_b(t_*) - z_b^*(t_*))e^{\lambda t_*} \\ &+ \sum_{b=1}^{n} k_{ab}(t_*)|z_a(t_* - r_b(t_*))e^{-z_a(t_*-r_b(t_*))} \\ &- z_a^*(t_* - r_b(t_*))e^{-z_a^*(t_*-r_b(t_*))}|e^{\lambda t_*} \\ \le& \lambda V_a(t_*) - q_a(t_*)V_a(t_*) + \sum_{b=1,b\neq a}^{n} p_{ab}(t_*)V_b(t_*) \\ &+ \sum_{b=1}^{n} k_{ab}(t_*)\frac{1}{e^2}V_a(t_* - r_b(t_*))e^{\lambda r_b(t_*)} \\ \le& [-(q_a^- - \lambda) + \sum_{b=1,b\neq a}^{n} p_{ab}^+ + \sum_{b=1}^{n} k_{ab}^+ \frac{1}{e^2}e^{\lambda r}]K_{\psi,z^*} < 0. \end{aligned}$$

This leads to a contradiction; therefore, equation (25) holds. It follows that $|h_a(t)| < K_{\psi,z^*}e^{-\lambda t}$ for all $t > t_{\psi,z^*}$ and $a = 1, 2, \cdots, n$. Hence

$$\lim_{t\to+\infty} |h_a(t)| = \lim_{t\to+\infty} |z_a(t) - z_a^*(t)| = 0, a = 1, 2, \cdots, n.$$

Corollary 7. If the conditions of Corollary 5 are satisfied, and for each $a = 1, 2, \cdots, n$ the inequality $\sum_{b=1,b\neq a}^{n} p_{ab}^+ + \sum_{b=1}^{n} k_{ab}^+ \frac{1}{e^2} < q_a^-$ holds, then system (6) admits a positive T-periodic solution that is globally attractive.

Corollary 8. If the conditions of Corollary 6 are satisfied, and for each $a = 1, 2, \cdots, n$ the inequality $\sum_{b=1,b\neq a}^{n} p_{ab}^+ + \sum_{b=1}^{n} k_{ab}^+ \frac{1}{e^2} < q_a^-$ holds, then system (7) admits a positive T-periodic solution that is globally attractive.

IV. Examples

In this section, we undertake numerical simulations to substantiate the results from the aforementioned theoretical analysis.

Example 1. In correspondence with system (6) and system (7), we first examine the subsequent system:

$$\begin{aligned} \dot{z}_1(t) =& -q_1(t)z_1(t) + p_{12}(t)z_2(t) + k_{11}(t)z_1(t - r_1(t)) \\ &\times e^{-z_1(t-r_1(t))} + k_{12}(t)z_1(t - r_2(t))e^{-z_1(t-r_2(t))}, \\ \dot{z}_2(t) =& -q_2(t)z_2(t) + p_{21}(t)z_1(t) + k_{21}(t)z_2(t - r_1(t)) \\ &\times e^{-z_2(t-r_1(t))} + k_{22}(t)z_2(t - r_2(t))e^{-z_2(t-r_2(t))}. \end{aligned} \tag{27}$$

where $n = 2$, $q_1(t) = 1.25 + 0.21\cos(t), p_{12}(t) = 0.052 + 0.03\cos(t), k_{11}(t) = 1.3 + 0.01\cos(t), k_{12}(t) = 1.35 + 0.03\cos(t), q_2(t) = 1.87 + 0.38\cos(t), p_{21}(t) = 0.045 + 0.002\cos(t)$, $k_{21}(t) = 1.32 + 0.01\cos(t), k_{22}(t) = 1.34 + 0.02\cos(t), r_1(t) = r_1 = 0.25, r_2(t) = r_2 = 0.25$. First, from the parameters of system (27), it can be observed that these parameters are consistent with assumptions (H_1) and (H_2). Next, by direct calculation of conditions (10) and (11) in Theorem 5, we can get

$$\begin{aligned} -1.04M + 0.082M + 2.69e^{-1} < 0, \\ -1.49M + 0.047M + 2.69e^{-1} < 0, \end{aligned}$$

and

$$\begin{aligned} -1.46 + 0.022 + 2.61e^{-m} > 0, \\ -2.25 + 0.043 + 2.63e^{-m} > 0, \end{aligned}$$

where $1.0329 < M < +\infty$, $0 < m < 0.1755$ and $T = 2\pi$. Consequently, the conditions of Theorem 5, Corollary 5, Corollary 7, and Corollary 8 are fulfilled. Invoking the conclusions of these results, we deduce that system (27) is permanent and there exists at least one 2π-periodic solution. Furthermore, through direct computation of condition (22) in Theorem 5, we derive that

$$\begin{aligned} -q_1^- + p_{12}^+ + k_{11}^+ \tfrac{1}{e^2} + k_{12}^+ \tfrac{1}{e^2} &= -1.04 + 0.082 + 2.69\frac{1}{e^2} \\ &< 0, \\ -q_2^- + p_{21}^+ + k_{21}^+ \tfrac{1}{e^2} + k_{22}^+ \tfrac{1}{e^2} &= -1.49 + 0.047 + 2.69\frac{1}{e^2} \\ &< 0. \end{aligned}$$

Consequently, the conditions of Theorem 5, Corollary 5, Corollary 7, and Corollary 8 are fulfilled. Invoking the conclusions of these results, we deduce that system (27) has a globally attractive 2π-periodic solution.

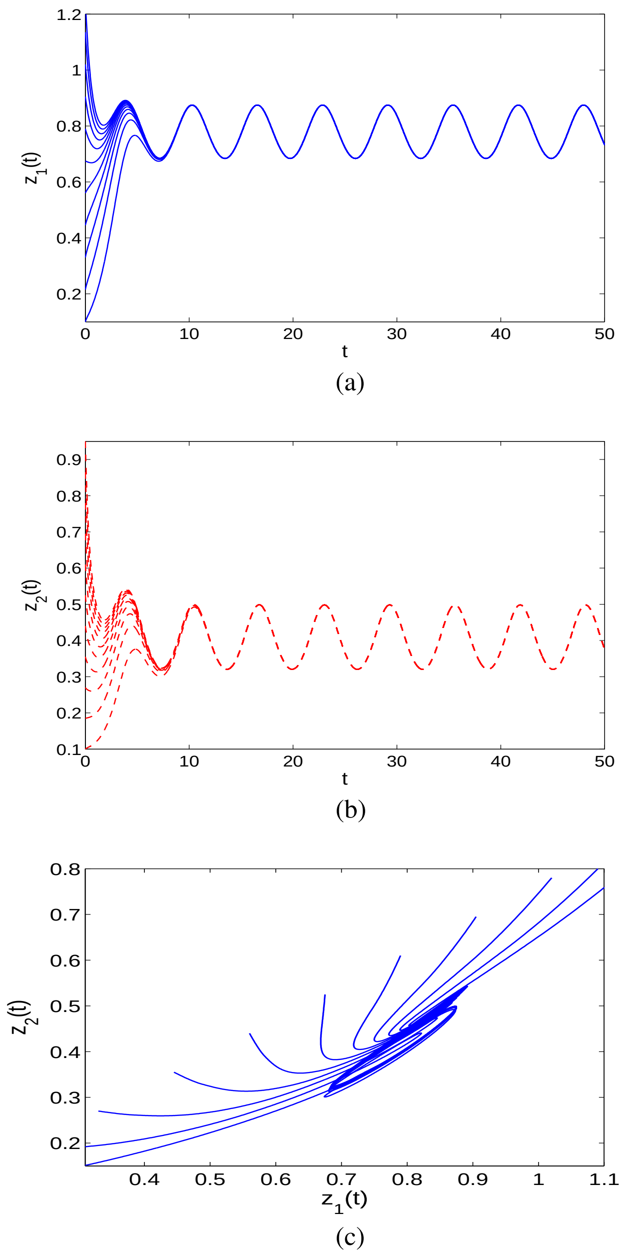

Fig 1: Dynamical behaviors of system (27)

As depicted in Fig. 1(a,b,c), numerical simulation results strongly indicate that system (27) has a globally attractive 2π-periodic solution. Here, we have selected multiple sets of different initial conditions.

Example 2. In correspondence with systems (6) and (7), we next examine the subsequent system:

$$\begin{aligned}\dot{z}_1(t) =& -q_1(t)z_1(t) + p_{12}(t)z_2(t) + k_{11}(t)z_1(t-r_1(t))\\ &\times e^{-z_1(t-r_1(t))} + k_{12}(t)z_1(t-r_2(t))e^{-z_1(t-r_2(t))},\\ \dot{z}_2(t) =& \ -q_2(t)z_2(t) + p_{21}(t)z_1(t) + k_{21}(t)z_2(t-r_1(t))\\ &\times e^{-z_2(t-r_1(t))} + k_{22}(t)z_2(t-r_2(t))e^{-z_2(t-r_2(t))}.\end{aligned} \tag{28}$$

where $n = 2$, $q_1(t) = 2.16 + 0.77\sin(t), p_{12}(t) = 0.36 + 0.08\sin(t), k_{11}(t) = 0.001 + 0.003\sin(t), k_{12}(t) = 0.003 + 0.005\sin(t), q_2(t) = 2.73 + 0.45\sin(t), p_{21}(t) = 0.54 + 0.03\sin(t)$, $k_{21}(t) = 0.004 + 0.002\sin(t)$, $k_{22}(t) = 0.007 + 0.004\sin(t), r_1(t) = r_1 = 0.5, r_2(t) = r_2 = 0.55$. First, from the parameters of system (28), it can be observed that these parameters are consistent with assumptions (H_1) and (H_2). Next, through direct computation of the conditions in Theorem 4 and Corollary 4, we derive that

$$\begin{aligned}-q_1^- + p_{12}^+ + k_{11}^+ + k_{12}^+ &= -1.39 + 0.44 + 0.004 + 0.008\\ &= -0.938 < 0,\\ -q_2^- + p_{21}^+ + k_{21}^+ + k_{22}^+ &= -2.28 + 0.57 + 0.006 + 0.011\\ &= -1.693 < 0.\end{aligned}$$

Clearly, the conditions of Theorem 4 and Corollary 4 hold. Consequently, invoking the conclusions of these results, we deduce that system (28) becomes extinct.

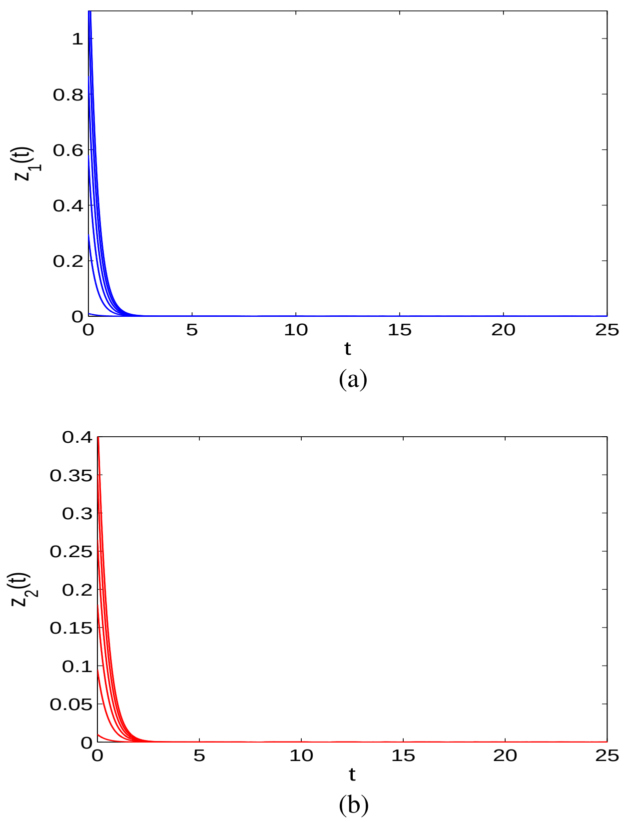

Fig 2: Dynamical behaviors of system (28)

As shown in Fig 2(a,b), numerical simulation suggests that system (28) is extinct.Here, we have selected multiple sets of different initial conditions.

V. Conclusions

In this study, we analyze a non-autonomous Nicholson blowflies model with time-varying coefficients and continuous delays. To the best of our knowledge, this is the first investigation into the dynamic properties of non-autonomous system (6). Our findings reveal that both time delays and patch migration profoundly impact Nicholson's blowflies dynamics. First, using differential inequality techniques and the comparison principle, we derive novel conditions ensuring the system's permanence, extinction, and existence of positive periodic solutions. Second, by constructing an appropriate Lyapunov functional and applying analytical inequality methods, we establish sufficient conditions for the system's global attractivity. Finally, we provide a numerical example to demonstrate the feasibility of our main results.

Additionally, we derive sufficient conditions for the permanence, extinction, existence of periodic solutions, and global attractivity of system (7). By extending systems (2)-(5) to system (6) and deriving conditions for the dynamic behaviors described above, this study generalizes and expands upon previous works [12-16] and other relevant studies.

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