A Comprehensive Analysis of Clique Vertex Neighborhood Numbers

Swarna J B, K Arathi Bhat and Smitha Ganesh Bhat*

***Abstract*—The open neighborhood $N(w)$ of a vertex $w \in V$ consists of all vertices adjacent to w in an undirected graph. The closed neighborhood $N[w]$, includes w and all vertices reachable from it. A complete maximal subgraph of G is a clique. A clique $k \in K(G)$ cv-covers a vertex v if $v \in \langle N[k] \rangle$, where $\langle N[k] \rangle$ is the subgraph induced by the closed neighborhood of k. A set $S \subseteq K(G)$ is a cv-neighborhood set if every vertex v is cv-covered by some $k \in S$, that is, $G = \bigcup_{k \in K(G)} \langle N[k] \rangle$. The minimum cardinality of such a set is the *clique vertex neighborhood number* $n_{cv}(G)$. In this paper, we establish bounds for n_{cv}, characterize graphs attaining these bounds, and compute n_{cv} for various graph products.**

***Index Terms*—Neighborhood number, clique coverings, corona product, Cartesian product, join graph, clique vertex neighborhood number.**

I. Introduction

To establish a clear foundation, we introduce the basic graph-theoretic terminology used throughout this work, following standard references such as [1] and [18]. We consider a graph G to be a connected, finite, and simple. A clique is a maximal complete subgraph. The vertex clique covering number $\theta_0(G)$ is the minimum number of cliques needed to cover all vertices [1], while the edge clique covering number $\theta_1(G)$ covers all edges [5]. The vertex covering number $\alpha_0(G)$ is the size of the smallest vertex set covering all edges, while the independence number $\beta_0(G)$ is the size of the largest independent set. A dominating set is a vertex set such that every vertex in G is either in the set or adjacent to it; the domination number $\gamma(G)$ is the size of a smallest dominating set. Minimum number of vertices that cover all the cliques of G is called clique transversal number, $\tau_c = \tau_c(G)$.

The study of domination concepts in graphs has been extensive; see, for example, [6], [17], [10]. Teffany et al.[8] investigated clique-dominating sets in various graph operations, including the join, corona, composition, and Cartesian product, and determined the corresponding clique domination numbers for the resulting graphs. Margaret et al. [7] provided forbidden subgraph characterizations for graphs possessing a dominating clique or a connected dominating set of size three. The problem of dominating cliques in interval graphs has been explored by Sudhakaraiah et al. [16], motivated by the relevance of interval graphs in applications such as scheduling and genetics; their work identifies dominating cliques within specific classes of interval graphs. Mohanaselvi et al. [14] determined the exact clique neighborhood domination numbers for several graph families, including complete graphs, complete bipartite graphs, star graphs, wheel graphs, fan graphs, banana trees, book graphs, n-barbell graphs, and friendship graphs. In addition, Edward et al. [12] demonstrated that for any $k, n \in \mathbb{Z}^+$ with $n \leq 4$ and $1 \leq k \leq n$, there exists a connected graph G with $|V(G)| = n$ and clique secure domination number $\gamma_{cls}(G) = k$. They further showed that for any $k, n, m \in \mathbb{Z}^+$ satisfying $1 \leq k \leq m$, there exists a connected graph G with $|V(G)| = n$, $\gamma_{cls}(G) = m$, and clique domination number $\gamma_{cl}(G) = k$, along with a characterization of the clique-secure dominating set resulting from the join of two graphs.

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Swarna J B is a Research Scholar in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India-576104 (email: swarnajb366@gmail.com).

K Arathi Bhat is an Associate Professor in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India-576104 (e-mail: arathi.bhat@manipal.edu).

Smitha Ganesh Bhat is an Associate Professor in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India-576104 (corresponding author to provide phone: 9844061970, e-mail: smitha.holla@manipal.edu).

II. Clique Vertex Neighborhood Numbers

Another notable graph invariant is the neighborhood covering (n-covering) number, introduced by Sampathkumar and Neeralagi [15] and further explored in [2], [4], [9], [11], [13]. A vertex v, n-covers an edge e, if $e \in \langle N[v] \rangle$, and a set $S \subseteq V$ is an n-covering set if it n-covers all edges of G. The lower and upper n-covering numbers, $n_0(G)$ and $N_0(G)$, are the minimum and maximum sizes of minimal n-coverings, respectively.

Motivated by the duality between the independence number $\beta_0(G)$ and the vertex covering number $\alpha_0(G)$, Surekha et al. [3] introduced the complementary notion of n-independent sets: a set $D \subseteq V$ is n-independent if every edge in $\langle D \rangle$ is n-covered by a vertex in $V \setminus D$. The corresponding parameters $\alpha_N(G)$ and $\alpha_n(G)$ denote the maximum and minimum orders of maximal n-independent sets, respectively.

Motivated by these definitions, we introduced a novel graph invariant called the clique vertex neighborhood number, which combines the ideas of neighborhood numbers and clique covering. This new invariant investigates the interaction between vertex neighborhoods and cliques, offering a fresh perspective on how these parameters function within different graph families. In essence, this paper highlights the concepts of the clique vertex neighborhood number and examines their connections with other established graph invariants.

Definition 2.1: Let $K(G)$ denote the set of all cliques in a graph G. A clique $k \in K(G)$ is said to cv-covers a vertex v if $v \in \langle N[k] \rangle$, where $\langle N[k] \rangle$ denotes the subgraph induced by the $N[k]$. A set of cliques $S \subseteq K(G)$ is called a *cv-neighborhood set*, if every vertices v in G is cv-covered by at least one clique in S, that is, $G = \bigcup_{k \in K(G)} \langle N[k] \rangle$. The minimum cardinality of a cv-neighborhood set is called a *clique vertex neighborhood number*, $n_{cv} = n_{cv}(G)$.

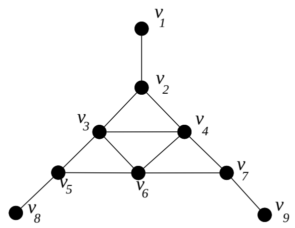

Fig. 1.

Example 2.1: For the graph G of Fig. 1, the cliques formed by vertices $\{v_2, v_3, v_4\}$, $\{v_5, v_8\}$ and $\{v_7, v_9\}$ form a n_{cv} set. Therefore $n_{cv}(G) = 3$.

Theorem 2.2:

- For any cycle, $n_{cv}(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil, & \text{if } n \geq 5 \\ \left\lfloor \frac{n}{3} \right\rfloor, & \text{if } n = 4 \end{cases}$
- For any path, $n_{cv}(P_n) = \left\lceil \frac{n-1}{3} \right\rceil$, $n \geq 2$.
- For any star, $n_{cv}(S_n) = 1$.
- For any wheel, $n_{cv}(W_n) = 1$.
- For any complete bipartite graph, $n_{cv}(K_{m,\ n}) = m$, $(m \leq n)$.

Theorem 2.3: Let G be a connected graph of order n. If $G \cong K_n$, then $n_{cv}(G) = 1$.

Proof: If $G \cong K_n$, then G itself is a clique. Let $k = V(G)$. The closed neighborhood $N[k]$ is also $V(G)$, and the induced subgraph $\langle N[k] \rangle = G$. Thus, the single clique k, cv-covers all the vertices, so $n_{cv}(G) = 1$. ∎

Corollary 2.3.1: For a connected graph G, $n_{cv}(G) = 1$ for the following graph types:

- Star Graph (S_n): For $n \geq 3$, $n_{cv}(S_n) = 1$.
  Proof: Let c be the central vertex of S_n. Consider any edge involving c, say $k_0 = \{c, v_i\}$. This is a clique. The closed neighborhood of the central vertex $N[c]$ includes c and all its neighbors, which are all vertices of S_n. Since k_0 contains c, $N[k_0] = N[c] \cup N[v_i]$. As $N[c]$ already covers all vertices, $N[k_0]$ also covers all vertices. Thus, $\langle N[k_0] \rangle = S_n$, and $n_{cv}(S_n) = 1$.
- Wheel Graph (W_n): For $n \geq 4$, $n_{cv}(W_n) = 1$.
  Proof: Let c be the central vertex of W_n. W_n consists of a central vertex connected to all vertices of a cycle C_{n-1}. Consider any triangle involving c and two adjacent cycle vertices, say $k_0 = \{c, v_i, v_{i+1}\}$. This is a clique. The closed neighborhood of the central vertex $N[c]$ includes c and all vertices on the cycle, i.e., all vertices of W_n. Since k_0 contains c, $N[k_0] = N[c] \cup N[v_i] \cup N[v_{i+1}]$. As $N[c]$ covers all vertices, $N[k_0]$ also covers all vertices. Thus, $\langle N[k_0] \rangle = W_n$, and $n_{cv}(W_n) = 1$.
- Fan Graph (F_n): For $n \geq 2$, $n_{cv}(F_n) = 1$.
  Proof: Let c be the central vertex of F_n. F_n consists of a central vertex connected to all vertices of a path P_n. Consider any triangle involving c and two adjacent path vertices, say $k_0 = \{c, v_i, v_{i+1}\}$. This is a clique. The closed neighborhood of the central vertex $N[c]$ includes c and all vertices on the path, i.e., all vertices of F_n. Since k_0 contains c, $N[k_0] = N[c] \cup N[v_i] \cup N[v_{i+1}]$. As $N[c]$ covers all vertices, $N[k_0]$ also covers all vertices. Thus, $\langle N[k_0] \rangle = F_n$, and $n_{cv}(F_n) = 1$.
- Friendship Graph (F_m): For $m \geq 1$, $n_{cv}(F_m) = 1$.
  Proof: Let c be the common central vertex of the m triangles in F_m. Consider any one of these triangles, say $k_0 = \{c, u, v\}$. This is a clique. The closed neighborhood of the central vertex $N[c]$ includes c and all other vertices in the graph (which are the $2m$ outer vertices of the triangles). Since k_0 contains c, $N[k_0] = N[c] \cup N[u] \cup N[v]$. As $N[c]$ already covers all vertices, $N[k_0]$ also covers all vertices. Thus, $\langle N[k_0] \rangle = F_m$, and $n_{cv}(F_m) = 1$.

Theorem 2.4: If G_1 and G_2 are isomorphic, then $n_{cv}(G_1) = n_{cv}(G_2)$.

Proof: Assume $G_1 \cong G_2$, with an isomorphism $\phi : V(G_1) \to V(G_2)$.
Let S_1 be a minimum cv-neighborhood set for G_1, so $|S_1| = n_{cv}(G_1)$ and $G_1 = \bigcup_{k_1 \in S_1} \langle N_{G_1}[k_1] \rangle$. Consider $S_2 = \{\phi(k_1) \mid k_1 \in S_1\}$, where $\phi(k_1)$ are the corresponding cliques in G_2. For any $w \in V(G_2)$, there exists $v = \phi^{-1}(w) \in V(G_1)$. Since S_1 is a cv-neighborhood set, $v \in \langle N_{G_1}[k_1] \rangle_{G_1}$ for some $k_1 \in S_1$. Due to the isomorphism, $\phi(v) = w \in \langle \phi(N_{G_1}[k_1]) \rangle_{G_2} = \langle N_{G_2}[\phi(k_1)] \rangle_{G_2}$. Thus, S_2 is a cv-neighborhood set for G_2 with $|S_2| = |S_1| = n_{cv}(G_1)$, implying $n_{cv}(G_2) \leq n_{cv}(G_1)$. By symmetry, if S_2' is a minimum cv-neighborhood set for G_2, then $S_1' = \{\phi^{-1}(k_2) \mid k_2 \in S_2'\}$ is a cv-neighborhood set for G_1 with $|S_1'| = |S_2'| = n_{cv}(G_2)$, implying $n_{cv}(G_1) \leq n_{cv}(G_2)$. Therefore, $n_{cv}(G_1) = n_{cv}(G_2)$. ∎

Theorem 2.5: Let G be a graph of order n. If $G_1, G_2, ..., G_k$ are the components of G, then $n_{cv}(G) = \sum_{i=1}^{k} n_{cv}(G_i)$.

Proof: Let G have components $G_1, G_2, ..., G_k$. For each component G_i, let S_i be a minimum cv-neighborhood set with $|S_i| = n_{cv}(G_i)$. Consider $S = \bigcup_{i=1}^{k} S_i$. Since components are disconnected, a clique in S_i is also a clique in G, and its closed neighborhood and induced subgraph are the same in G as in G_i. Thus, $\bigcup_{k \in S} \langle N_G[k] \rangle_G = \bigcup_{i=1}^{k} \left(\bigcup_{k \in S_i} \langle N_{G_i}[k] \rangle_{G_i} \right) = \bigcup_{i=1}^{k} G_i = G$.
So, S is a cv-neighborhood set for G, and $n_{cv}(G) \leq |S| = \sum_{i=1}^{k} n_{cv}(G_i)$. Let S be a minimum cv-neighborhood set for G, $|S| = n_{cv}(G)$. If a clique $k \in S$, cv-covers a vertex in G_i, then k must be entirely within G_i. Let $S_i = \{k \in S \mid k \subseteq V(G_i)\}$. S_i must cv-cover all vertices of G_i, so $|S_i| \geq n_{cv}(G_i)$. Since the $S_i' s$ partition S, $n_{cv}(G) = |S| = \sum_{i=1}^{k} |S_i| \geq \sum_{i=1}^{k} n_{cv}(G_i)$. Therefore, $n_{cv}(G) = \sum_{i=1}^{k} n_{cv}(G_i)$. ∎

Theorem 2.6: For a graph G and a proper subgraph G_1 of G such that $n_{cv}(G_1) \leq n_{cv}(G)$.

Proof: Consider a graph G and a proper induced subgraph G_1. Any set of cliques in G that cv-covers $V(G)$ also provides a means to cv-cover the subset $V(G_1)$ using the same cliques and their neighborhoods within G_1. Thus, the minimum number of cliques needed for G_1 cannot exceed that for G, so $n_{cv}(G_1) \leq n_{cv}(G)$. ∎

Theorem 2.7: There exist a graph G and a proper subgraph G_1 of G such that $n_{cv}(G_1) = n_{cv}(G)$.

Proof: Let G be a graph and G_1 be a proper subgraph of G. Here K be a clique which cv-covers all the vertices $V(G)$ also provides a means to cv-cover the subset $V(G_1)$. We have to consider the clique in such a way that, clique must contain a vertex where that particular vertex must be adjacent to all other vertices in graph G. Thus, the above equality $n_{cv}(G_1) = n_{cv}(G)$ holds. ∎

Note:- For star graph, wheel graph, fan graph, K_3 and friendship graph the above theorem holds.

Theorem 2.8: There exist a graph G and a proper subgraph G_1 of G such that $n_{cv}(G_1) \geq n_{cv}(G)$.

Proof: Consider a graph G where a few cliques efficiently cv-cover all vertices due to high connectivity. By removing edges to form a spanning subgraph G_1, these cliques significantly reduced, required more cliques for cv-coverage. Thus, $n_{cv}(G_1) \geq n_{cv}(G)$. ∎

Theorem 2.9: For any graph G, $n_{cv}(G) \leq \tau_c(G)$.

Proof: Let T be a vertex transversal of the maximal cliques of G with $|T| = \tau_c(G)$. For each $t \in T$, choose a maximal clique K_t that contains t. Put $S = \{K_t : t \in T\}$. We show that S is a cv-neighborhood set.

Take any vertex $u \in V(G)$. Let M be a maximal clique that contains u. Since T meets every maximal clique, there exists $t \in T \cap M$. Because $t, u \in M$ and M is a clique, u is adjacent to t, so $u \in N[t]$. As $t \in K_t$, we have $N[t] \subseteq N[K_t]$; hence $u \in N[K_t]$. Therefore u lies in the induced subgraph $\langle N[K_t] \rangle$.

Since this holds for every $u \in V(G)$, the union $\bigcup_{k \in S} \langle N[k] \rangle$ equals G, so S is a cv-neighborhood set. Thus $|S| \leq |T| = \tau_c(G)$, and by the minimality of $n_{cv}(G)$ we obtain

$$n_{cv}(G) \leq \tau_c(G).$$

∎

Theorem 2.10: For any graph G, $n_{cv}(G) \leq \theta_0(G)$.

Proof: Let $C = (c_1, c_2, ..., c_{\theta_0})$ be a minimum vertex clique cover of G. By the definition, every vertex $v \in V(G)$ belongs to at least one clique $c_i \in C$. Consider the closed neighborhood of each clique c_i, denoted by $N[c_i]$. Since $c_i \subseteq N[c_i]$, it follows that every vertex $v \in c_i$ is also in $N[c_i]$, and therefore, v is in $\langle N[c_i] \rangle$. Since every vertex $v \in V(G)$ is in at least one clique c_i of the vertex clique cover C. Therefore, the set of cliques $C = (c_1, c_2, ..., c_{\theta_0})$ forms a cv-neighborhood set for G. Thus, $n_{cv}(G) \leq |C| = \theta_0(G)$. Therefore, $n_{cv}(G) \leq \theta_0(G)$. ∎

Corollary 2.10.1: For any graph G, $n_{cv}(G) \leq \theta_0(G) \leq \theta_1(G)$.

Corollary 2.10.2: For a complete graph G, $n_{cv}(G) = \theta_0(G) = \theta_1(G)$.

Theorem 2.11: For any graph G, $n_{cv}(G) \leq n_0(G)$.

Proof: Let G be a graph with given $n_{cv}(G)$ and $n_0(G)$. Suppose in one of the clique which is required for n_{cv}, if none of the vertices of that clique are contained in n_0 then we show that we are getting n_0 with less number of vertices, which is a contradiction. Let us construct a graph with n_o set without containing any vertices of the cliques of n_{cv}. Let v be any vertex. In order to cover the edges of the complete graph K_r, v can be maximum adjacent to $(r-2)$ vertices. We get complete graph K_{r-1} . Then to cover the remaining edges, that is, $(2r-3)$ edges of the complete graph k_r, we need another vertex say w. Again this w can be adjacent to maximum $(r-2)$ vertices. Then the graph remaining is C_4. To cover the C_4 one more vertex x needed. With this we are getting no vertices from the clique K_r. From the original set, we remove $[v, w, x]$ and add the other 2 vertices which are adjacent to v, w form a n_0 set with lesser vertices, hence a contradiction. ∎

Theorem 2.12: For any infinite class of graph G, $n_{cv}(G) < n_0(G)$.

Proof: We construct an infinite class of triangle free graphs $G(k, n)$, $n \geq 4$,, with the following steps:
Consider 2 copies of a cycle C_n for $n \geq 5$, join its i^{th} vertex of first copy of C_n (inner cycle) with the i^{th} vertex of second copy of C_n (outer cycle). We denote graph as $H = C_n \oplus C_n$. Now obtain k-copies of a graph H as shown in Fig. 2. Here the joined edge nothing but a clique. Here that cliques are considered in n_{cv}-set. We are going to name the joined clique as c_{ij}, $1 \leq i \leq k$, $j = 1, 2, ..., n$ respectively. Let u_{ij} and v_{ij} for $1 \leq i \leq k$, $j = 1, 2, ..., n$ respectively be the labels of the vertices of inner and outer cycles C_n in the i^{th} copy of H_i of H.

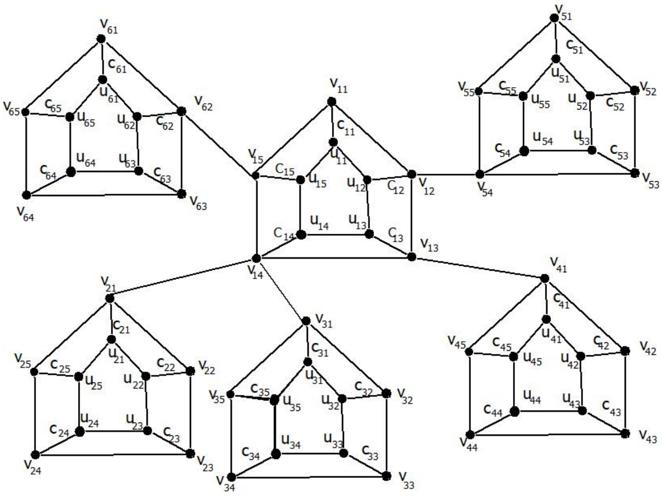

Fig. 2. Infinite Class of graph G.

- When n is an even cycle. In this graph the alternate c_{ij} cliques are enough for the cv-neighborhood set. Here, $\{c_{ij} | 1 \leq i \leq k, j = 2, 4, 6, ..., n\}$ is a n_{cv}-set.
Here, $\{u_{ij} | 1 \leq i \leq k, j = 1, 3, 5, ..., n+1\} \cup \{v_{ij} | 1 \leq i \leq k, j = 2, 4, 6..., n\}$ is a n_0-set of $G(k, n)$.

$$n_{cv}(C_{2m} \oplus C_{2m}) = mk, \quad m \geq 3. \tag{II.1}$$

where k = number of copies.

$$n_0(C_{2m} \oplus C_{2m}) = 2n_{cv}(C_{2m} \oplus C_{2m}) = 2mk. \tag{II.2}$$

- When n is an odd cycle. In this graph the alternate c_{ij} cliques and additional one clique is enough for the cv-neighborhood set. Here, $\{c_{ij} | 1 \leq i \leq k, j = 1, 3, 5, ..., n, n-1\}$ is a n_{cv}-set.
Here, $\{u_{ij} | 1 \leq i \leq k, j = 1, 3, 5, ..., n\} \cup \{v_{ij} | 1 \leq i \leq k, j = 2, 4, 6..., n, n-1\}$ is a n_0-set of $G(k, n)$.

$$n_{cv}(C_{2m+1} \oplus C_{2m+1}) = k \left\lceil \frac{2m+1}{2} \right\rceil, \quad m \geq 2. \tag{II.3}$$

Therefore,

$$\begin{aligned} n_0(C_{2m+1} \oplus C_{2m+1}) &= 2n_{cv}(C_{2m+1} \oplus C_{2m+1}) \\ &= 2k \left\lceil \frac{2m+1}{2} \right\rceil. \end{aligned} \tag{II.4}$$

∎

Corollary 2.12.1: For C_4 graph, where k = number of copies.

$$n_{cv}(C_4 \oplus C_4) = k \left\lceil \frac{4}{3} \right\rceil \quad \text{(II.5)}$$

$$n_0(C_4 \oplus C_4) = 2kn_{cv}(C_4 \oplus C_4). \quad \text{(II.6)}$$

Remark:- Let N be an even positive number, where $N \geq 10$. We can determine the number of cliques in an n_{cv}-set by applying the above theorem

Proof: As N is an even integer and $N \geq 10$, we can write $N = 2n$, where $n = N/2$. Since $N \geq 10$, it follows that $n \geq 5$. We will analyze the number of cliques in an n_{cv}-set based on the properties of n.

Case 1: n is a prime number.

In this case, since $n \geq 5$ and n is prime, n must be an odd number (e.g., 5, 7, 11, etc., as 2 is the only even prime, and $n \neq 2$). We calculate the number of cliques in the n_{cv}-set by constructing a graph isomorphic to C_n as specified by the above theorem. This construction involves only one copy of H. The number of cliques can then be directly computed using the odd cycle formula.

Case 2: n is a composite number.

If n is composite, it can be factored as $n = rs$ for integers $r, s \geq 2$. Consequently, $N = 2rs$. We subdivide this case further based on the values of r and s.

Subcase 2.1: One factor is small ($r \leq 4$) and the other is large ($s \geq 5$).: Without loss of generality, let $r \leq 4$ and $s \geq 5$. We interpret r as the number of k-copies of some graph component. The factor s represents the number of vertices in the graph structure we are constructing, following the principles of the above theorem. The relevant formula is then applied accordingly to determine the number of cliques in the n_{cv}-set.

Subcase 2.2: Both factors are small ($r \leq 4$ and $s \leq 4$).: In this situation, $n = rs$ will be a relatively small composite number (e.g., if $r = 2, s = 3$, then $n = 6$; if $r = 3, s = 3$, then $n = 9$). We treat $m = rs$ as a single effective size for the graph construction. The calculation proceeds similarly to Case 1, likely by constructing a C_m or similar small graph as per the above theorem and applying the relevant formula.

Subcase 2.3: Both factors are large ($r \geq 5$ and $s \geq 5$).: Here, r can be considered as the number of k-copies of a base structure, and s as the number of vertices in the main graph construction, or vice-versa. The formulas derived from the above theorem are applied.

Specific Example within Subcase 2.3:: If $r \geq 5$ and $s \geq 5$, and r is even while s is odd (or vice versa):

$$n_{cv}(C_{rs} \oplus C_{rs}) = n_{cv}(C_r \oplus C_r) < n_{cv}(C_s \oplus C_s).$$

■

The maximum strength of a clique (Δ_s) is the number of vertices in $\langle N[k] \rangle$. Now we obtain a bound for $n_{cv}(G)$ in terms of maximum strength Δ_s.

Theorem 2.13: For any graph, $\frac{n}{\Delta_s} \leq n_{cv}(G) \leq n - \Delta_s + 1$.

Proof: Let k be the clique of maximum strength Δ_s. Then k, n_{cv}-cover all the vertices in $\langle N[k] \rangle$. Let S be the set of cliques that are needed to cv-covers all the vertices. Then to cover all the vertices Δ_s cliques are needed. Then remaining that many vertices from n results in a graph $(V(G) - \langle N[k] \rangle)$. Thus $|V(G) - \langle N[k] \rangle| = n - \Delta_s$. Thus S, n_{cv} covers all the vertices in $(V(G) - \langle N[k] \rangle)$ of every clique formed by choosing one end. Hence $(S \cup k)$ is a n_{cv} -covering of graph G.
Therefore, $n_{cv} \leq |S \cup k| \leq n - \Delta_s + 1$. So to cover all the vertices outside the graph of $\langle N[k] \rangle$. We need maximum $n - \Delta_s + 1$ provided every vertex is of degree 1. Since a clique can n_{cv}-cover at most $\Delta_s(G)$ vertices to n_{cv}-cover all the vertices, we needed at least $\frac{n}{\Delta_s(G)}$. Therefore, $\frac{n}{\Delta_s} \leq n_{cv}(G)$. ■

The clique vertex neighborhood number n_{cv} of a complete graph K_n is 1, and same holds for the line graph of a complete graph, that is, $n_{cv}(L(K_n)) = 1$. For any graph G, $n_{cv}(L(G)) = n_{cv}(G)$, except for path. In a cycle graph, the number of vertices is equal to the number of cliques formed by its edges. The clique vertex neighborhood number n_{cv} remains unchanged in the total graph transformation, where $n_{cv}(G) = n_{cv}(T(G))$, except for C_4. Here, total graph $T(G)$ of a graph G is the graph obtained by combining G and its line graph $L(G)$. Its vertex set is $V(G) \cup E(G)$, and two vertices in $T(G)$ are adjacent if they are adjacent in G, adjacent in $L(G)$, or if a vertex and an edge are incident in G.

Theorem 2.14: For any graph G, $n_{cv}(T(G)) \leq n_{cv}(G) + n_{cv}(L(G))$.

Proof: Suppose a clique k is said to be n_{cv} cover a vertex v if $v \subseteq \langle N(k) \rangle$. Here, $T(G)$ contains both G and $L(G)$ as subgraphs. Let S_G be a cv-neighborhood set of G, $|S_G| = n_{cv}(G)$ and S_L be a cv-neighborhood set of $L(G)$, $|S_L| = n_{cv}(L(G))$. $T(G)$ consist of both the vertices of G and the edges of G, we can combine the clique cover of G and $L(G)$ to form a valid covering for $T(G)$. The union $S = S_G \cup S_L$ form a valid cv-neighborhood set for $T(G)$.

$$|S| \leq |S_G| + |S_L|.$$

which gives:

$$n_{cv}(T(G)) \leq n_{cv}(G) + n_{cv}(L(G)).$$

■

Corollary 2.14.1: The bound is sharp for C_4 graph.

The corona product of two graphs G and H, denoted by $G \circ H$, is formed by taking one copy of G and $|V(G)|$ copies of H. For each vertex u in G, every vertex in the i-th copy of H is connected by an edge to u. The total number of vertices in $(G \circ H)$ is:

$$|V(G \circ H)| = |V(G)| + |V(G)| \cdot |V(H)|. \quad \text{(II.7)}$$

Theorem 2.15: If G and H be any graph, n is the number of vertices in G and l is the number of vertices in the cliques forming a minimum cv-neighborhood set in G, then $n_{cv}(G \circ H) \leq n_{cv}(G) + (n - l)$.

Proof: Let S be a minimum cv-neighborhood set of G, with $|S| = n_{cv}(G)$. For each $s \in S$, let C_s be a maximal clique in G containing s, and let $L = \bigcup_{s \in S} V(C_s)$ with $|L| = l$. Consider $S' = S \cup (V(G) \setminus L)$ in $G \circ H$, with $|S'| = n_{cv}(G) + (n - l)$. We show S' is a cv-neighborhood set for $G \circ H$. Let $v \in V(G \circ H)$.

Case 1: If $v \in V(G)$: Since S cv-covers G, v is in $n_{cv}(s)$ for some $s \in S \subseteq S'$, both in G and in the G part of $G \circ H$.

Case 2: If $v \in V(H_i)$ (attached to $g_i \in V(G)$):

- If $g_i \in L$, then g_i is near a maximal clique related to some $s \in S$, so v is covered by $n_{cv}(s)$ in $G \circ H$ due to the adjacency between g_i and H_i.
- If $g_i \notin L$, then $g_i \in S'$. The cv-neighborhood of g_i includes all of H_i, so v is covered.

Thus, S' is a cv-neighborhood set for $G \circ H$, and $n_{cv}(G \circ H) \leq |S'| = n_{cv}(G) + (n - l)$.

■

Theorem 2.16: If G is a tree on n vertices, H be any graph and l is the number of vertices involved in the cliques forming a minimum cv-neighborhood set of G, then $n_{cv}(G \circ H) = n - l$.

Proof: Since G is a tree, its only maximal clique are edges (K_2). Suppose we select t cliques in G, since each clique cover exactly 2 vertices, the total number of vertices covered is:

$$l = 2t.$$

The number of remaining uncovered vertices in G is

$$n - l = n - 2t. \tag{II.8}$$

Thus the remaining vertices will determine $n_{cv}(G \circ H)$. Since the added copies of H are only connected to their respective vertex in G, we must include cliques covering these original vertices to extend coverage to all new vertices in H.
From general inequality,

$$n_{cv}(G \circ H) \leq n_{cv}(G) + (n - l).$$

Since,

$$n_{cv}(G) = t, \quad l = 2t$$
$$n_{cv}(G \circ H) \leq n - t$$
$$n_{cv}(G \circ H) \leq n - 2t = n - l$$
$$n_{cv}(G \circ H) \leq n - l. \tag{II.9}$$

We have to show that, $n_{cv}(G \circ H) \geq n - l$.
Here the number of additional clique neighborhoods required by the number of uncovered vertices in G after selecting the minimal clique set. Since the uncovered vertices required clique neighborhoods, we must add at least one clique per uncovered vertex. This gives us an lower bound for $n - k$ additional clique.
Thus, we conclude,

$$n_{cv}(G \circ H) \geq n - k. \tag{II.10}$$

Therefore, $n_{cv}(G \circ H) = n - k$.

■

Theorem 2.17: For any graph G, $n_{cv}(G) \leq n_{cv}(G \circ G)$.

Proof: Let G be any graph, the total number of vertices in $G \circ G$ is $n + n^2$.

Let S be a clique vertex neighborhood cover for G, $|S| = n_{cv}(G)$. This cover partially cover $G \circ G$, but additional clique is needed to cover a vertex. Since each of the additional $|V(G)|$ copies of G must be cv-covered, at least some additional cliques must be added to S. Therefore the clique vertex neighborhood number cannot decrease when constructing $G \circ G$. Thus, we conclude:

$$n_{cv}(G) \leq n_{cv}(G \circ G). \tag{II.11}$$

■

Theorem 2.18: For a cycle and a complete graph with n vertices, $n_{cv}(C_n \circ K_n) = n_{cv}(C_n) + n_{cv}(K_n)$.

Proof: The total number of vertices in $C_n \circ K_n$ is $n + n^2$. In C_n, number of cliques is equal to number of vertices, $k = n$.

The minimum number of cliques needed to cover all the vertices in a cycle C_n is

$$n_{cv}(C_n) = \left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{k}{3} \right\rceil, \quad n \geq 5.$$

Since K_n is a complete graph, a single clique covers all its vertices so

$$n_{cv}(K_n) = 1.$$

Thus to cover $C_n \circ K_n$

$$n_{cv}(C_n) + n_{cv}(K_n) = \left\lceil \frac{k}{3} \right\rceil + 1.$$

Hence the result follows. ■

Theorem 2.19: For a cycle, $n_{cv}(C_m) \leq n_{cv}(C_m \circ C_n)$.

Proof: The total number of vertices in $C_m \circ C_n$ is $m + m \cdot n$.

The clique vertex neighborhood number for a cycle on m vertices is $n_{cv}(C_m) = \lceil \frac{m}{3} \rceil$ for $m \geq 5$, and $n_{cv}(C_4) = \lfloor \frac{m}{3} \rfloor = 1$. Since $C_m \circ C_n$ contains C_m as a subgraph, at least all the cliques required for C_m must also be present in $C_m \circ C_n$. In $n_{cv}(C_m \circ C_n)$ the total clique count cannot be smaller than that of $n_{cv}(C_m)$. Therefore

$$n_{cv}(C_m) \leq n_{cv}(C_m \circ C_n). \tag{II.12}$$

■

The Cartesian product $G \square H$ combines two graphs G and H. Its vertices are ordered pairs (u, v) where u is a vertex in G and v is a vertices in H. Two vertices (u, v) and (u', v') are adjacent if either $u = u'$ and v is adjacent to v' in H, or $v = v'$ and u is adjacent to u' in G.

Theorem 2.20: For a path graph, $n_{cv}(P_m \square P_n) \geq n_{cv}(P_m) + n_{cv}(P_n)$, except for $m = 2$ and $n = 2, 3, 5$.

Proof: Let

$$n_{cv}(P_n) = \left\lceil \frac{n-1}{3} \right\rceil. \tag{II.13}$$

The $P_m \square P_n$ forms a grid graph where each vertex (u, v) connects to vertices along its row (u', v) and column (u, v'). To cv-cover all vertices, cliques must be selected to cover both rows and columns independently.
For a path graph P_k, the cv-neighborhood number $n_{cv}(P_k)$ equals $\lceil \frac{k-1}{3} \rceil$, the minimum cliques needed to cv-cover all vertices. In $P_m \square P_n$, every vertex in a row of P_m and every vertex in a column of P_n must be cv-covered independently. Therefore the total cv-neighborhood set size cannot be smaller than the sizes of P_m and P_n. Hence, $n_{cv}(P_m \square P_n) \geq n_{cv}(P_m) + n_{cv}(P_n)$. ■
The graph join operation combines two or more graphs into a single graph. It involves taking the original vertices and edges, plus adding new edges to connect every vertex in the first graph to every vertex in the second graph.

Theorem 2.21: For any two non-empty graphs, G_1 and G_2, the clique vertex neighborhood number of their join, denoted as $G_1 \vee G_2$, is 1. That is, $n_{cv}(G_1 \vee G_2) = 1$.

Proof: Let $G = G_1 \vee G_2$. By definition, the vertex set of G is $V(G) = V_1 \cup V_2$, and its edge set includes all edges connecting every vertex in V_1 to every vertex in V_2.

To find $n_{cv}(G)$, we need to find the minimum number of cliques whose closed neighborhoods cover all vertices in G.

Consider any single vertex v from the graph G. A single vertex is a clique of size 1. Let's examine the closed neighborhood of this vertex.

- If $v \in V_1$, its neighbors in G are all of its neighbors in G_1 plus all vertices in V_2. Therefore, the closed neighborhood of v in G is $N_G[v] = V_1 \cup V_2 = V(G)$.
- If $v \in V_2$, a similar argument shows that the closed neighborhood of v in G is also $N_G[v] = V_1 \cup V_2 = V(G)$.

In both cases, the closed neighborhood of a single vertex covers the entire vertex set of the graph, $V(G)$. A single vertex is a clique, so the subgraph induced by the closed neighborhood of this clique covers all vertices. Since we only need one clique to form a cv-neighborhood set, the minimum cardinality of such a set is 1. Thus, $n_{cv}(G_1 \vee G_2) = 1$. ■

Corollary 2.21.1: For any finite collection of non-empty graphs, $G_1, G_2, ..., G_n$, $n_{cv}(G_1 \vee G_2 \vee ... \vee G_n) = 1$.

III. Clique independence number

A set of cliques are *clique independent* if no two cliques have a vertex in common. A set $L \subseteq K(G)$ is said to be a *clique independent set* if no two cliques in L have a vertex in common. The *clique independence number* $\beta_{cc} = \beta_{cc}(G)$ is the maximum number of cliques in a clique independent set of G. A *clique graph* $K_G(G)$ is a graph with vertex set $K(G)$ and any two vertices in $K_G(G)$ are adjacent if corresponding cliques in G have a vertex in common. A graph G is called a *clique path*, if $K_G(G)$ is a path. A graph G is called a *clique cycle*, if $K_G(G)$ is a cycle. A graph G is called a *clique complete*, if $K_G(G)$ is a complete. A graph G is called a *clique star*, if $K_G(G)$ is a clique complete graph.

Now we give clique independence number of some standard graphs.

Proposition 3.1:

1) If G is a clique path, $\beta_{cc}(G) = \left\lceil \frac{k}{2} \right\rceil$.
2) If G is a clique cycle, $\beta_{cc}(G) = \left\lceil \frac{k}{2} \right\rceil$.
3) If G is a clique complete graph, $\beta_{cc}(G) = 1$.
4) If G is a clique star with c cut-vertices, $\beta_{cc}(G) = c$.

Proposition 3.2: For any connected graph G,

$$\beta_{cc} \leq \left\lceil \frac{k}{2} \right\rceil.$$

Further, the bounds are sharp.

Proof: Let $\beta_{cc} = t$. If G has k number of cliques, out of which t cliques are independent and at least $t-1$ cliques are connecting k cliques. Therefore

$$k \geq t + (t-1)$$
$$k \geq 2t - 1$$
$$\text{Thus,} \quad t \leq \frac{k+1}{2}$$

The bound is sharp for any block graph. ■

Let ϑ be the minimum clique number, then we have the following proposition.

Proposition 3.3: For any graph G,

$$\beta_{cc} \leq \frac{p}{\vartheta}.$$

Further, the bound is sharp.

Proof: Let $\beta_{cc} = t$. Let $L = \{k_1, k_2, \ldots, k_t\} \subseteq K(G)$ be the β_{cc}-set of G. Let $|k_i|$ denote the number of vertices in the clique k_i. Then $|k_i| \geq \vartheta(G)$, $1 \leq i \leq t$. Since each k_i, $1 \leq i \leq t$ is an independent clique, any two cliques are mutually disjoint and $|k_1| \cup |k_2| \cup \cdots \cup |k_t| \subseteq V$. Therefore, $t\vartheta(G) \leq |k_1| \cup |k_2| \cup \cdots \cup |k_t| \leq |V| = p$ proving the desired inequality.

The bound is sharp for a complete graph and an even cycle C_{2n}. ■

Proposition 3.4: For any graph G,

$$\beta_{cc} \leq min\{\beta_0, \beta_1\}.$$

Proof: Let $\beta_{cc} = k$ and suppose $L = \{l_1, l_2, \ldots, l_k\}$ be the β_{cc}-set of G. Since all the cliques in L are mutually independent, we can choose an edge $x_i \in l_i$ for each $i = 1, 2, \ldots, k$. Then $L_1 = \{x_1, x_2, \ldots, x_k\}$ is an edge independent set of cardinality k. Then $k = \beta_{cc} = |L| = |L_1| \leq \beta_1$.

Let $L = \{l_1, l_2, \ldots, l_k\}$ be the β_{cc}-set of G. Clearly, we can select a vertex v_i from each clique l_i in L such that $\{v_1, v_2, \ldots, v_k\}$ is an independent set of vertices. Thus $\beta_{cc} \leq \beta_0$. ■

Let δ_{cc} be the minimum clique clique degree of a graph G.

Proposition 3.5: For any graph G,

$$\beta_{cc}(G) \leq k - \delta_{cc}(G).$$

Further, the bound is sharp.

Proof: Let L be the set of all cliques with cc-degree δ_{cc}. Then we can find a β_{cc} set S such that $L \cap S \neq \phi$. Hence there exists a $l \in L \cap S$ with cc-degree δ_{cc}. Then $S \subseteq K(G) - N_{cc}(l)$. Therefore $\beta_{cc}(G) \leq k - \delta_{cc}(G)$. The bound is attained for any clique complete graph. ■

IV. Application

Application of clique vertex neighborhood number (n_{cv}) to network analysis $n_{cv}(G)$, the minimum cliques needed to cv-cover all vertices, offers a novel lens for network analysis, particularly in assessing vulnerability and robustness.

A low n_{cv} suggests efficient coverage through local clique structures, potentially indicating better resilience to failures. Conversely, a high n_{cv} might imply fragmented coverage and increased susceptibility.

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