The Effect of Gravity and Surface Tension on Free Surface Flows Over a Submerged Obstacles

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***Abstract*—Free-surface two-dimensional flows past successive triangular obstacles are considered. We suppose that the fluid is incompressible and non-viscous. The flow is assumed to be steady and irrotational. The gravity and the surface tension are included in the free surface condition. The problem is solved numerically by employing series-truncation method. The numerical solutions exist for various values of the Weber number and the Froude number. When the surface tension tends to zero, It is shown that there are solutions for which the flow is supercritical and sub-critical both upstream and downstream. The free surface profiles are plotted for different sizes of successive triangles.**

***Index Terms*—Free surface flow; potential flow; Weber number; surface tension; Froude number.**

I. Introduction

THe problem of free-surface flows over a submerged obstacle has been studied by many researchers. Various mathematical techniques have been employed to study free-surface flows over different kinds of obstacles situated at the bottom of a channel. For example, Forbes [8] considered flow over a semi-circular obstruction, Dias [6] studied the free surface flows over a triangular obstacle with gravity effect, Hanna [10] considered super-critical free-surface flow over a trapezoidal obstacle of a finite depth by using a series-truncation method. In the case of flows over two obstacles, several authors have solved this problem using the boundary integral equation method [4], [5]. In this work, we study the problem of flows over two triangular obstacles by applying a series-truncation technique. This method has been used extensively by many researchers, Birkhoff [3], Dias [6], Lee [11], Daboussy [7], Vanden-Broeck [16], [17] and others. The fluid is assumed to be inviscid, incompressible and the flow is irrotationnal. The flow is supposed to be uniform, with a constant velocity U and constant depth H, far upstream. Both of surface tension and gravity are considered. In this case, the problem is characterized by the Weber number α defined by

$$\alpha = \frac{\rho U^2 H}{T} \tag{1}$$

and the Froude number Fr defined by

$$Fr = \frac{U^2}{\sqrt{gH}} \tag{2}$$

Where T is the surface tension, g is the acceleration of gravity and ρ is the density of the fluid. The upstream flow is said to be supercritical when $(Fr > 1)$ and sub-critical when $(Fr < 1)$. The problem of free surface flow over an obstacle under the effect of gravity and surface tension has been treated by many authors, for example, Forbes [8], Lee [12] considered the bubbles rising in an inclined tube and others. It is shown that there are solutions for each value of the Weber number $\alpha \geq \alpha_0$ where α_0 is a critical value, and different values of the Froude number Fr. It is found that solutions exist for triangular obstacles of arbitrary size. When $T = 0$, we calculate these solutions for supercritical and sub-critical flow. In section 2, we formulate the problem. The numerical scheme is described in section 3. The results are presented and discussed in section 4.

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II. Mathematical Formulation

We consider steady two-dimensional flow over an isosceles triangle. The angle between the wall BC and the horizontal is γ, where $-\frac{\pi}{2} < \gamma < 0$. The fluid domain is bounded below by the rigid wall AG, the two triangles BCD and DEF, and above by the free surface $A'G'$ (see figure 1). We introduce Cartesian coordinates such that the $x-$axis is parallel to the wall AG and the $y-$axis is vertically. Far upstream, $x \to \infty$, the flow is uniform with a constant velocity U and a constant depth H. Gravity is acting vertically downwards, and the surface tension is considered. We choose H as the unit length and U is the unit velocity. The fluid is assumed to be inviscid and incompressible, and the flow is irrotational. We introduce the complex potential $f = \varphi + i\psi$, which is a function of the complex variable $z = x + iy$. The function ψ is the stream function and ϕ is the potential function. Without loss of generality, we choose $\varphi = 0$ at point D on streamline $\psi = 0$. The free surface $A'G'$ defines a streamline on $\psi = 1$. The upper half of the flow region in the $z-$plane will be mapped via the potential function f onto the infinite strip $0 < \psi < 1$ (see figure 2). The problem is formulated in terms of the velocity potential ϕ. This function satisfies Laplace's equation

$$\Delta\phi = 0 \qquad \text{in the fluid domain} \tag{3}$$

We denote by $u - iv$ the complex velocity. The function $u - iv$ is an analytic function of z everywhere in the fluid domain. We define the function

$$\xi = u - iv = \frac{df}{dz} \tag{4}$$

On the free streamline (free surface) $A'G'$, the Bernoulli equation yields

$$\frac{1}{2}q^2 + \frac{p}{\rho} + gy = \frac{1}{2}U^2 + \frac{p_0}{\rho} + gH = const \quad on \quad A'G' \tag{5}$$

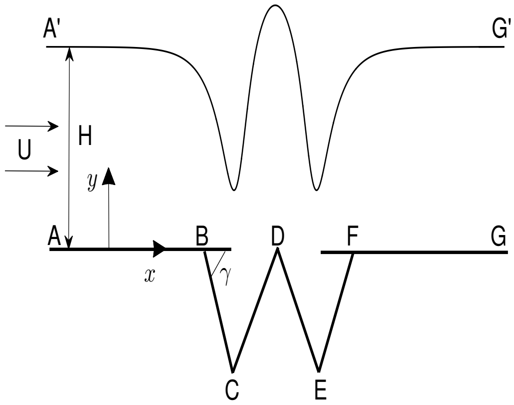

Fig. 1. Sketch of the flow domain in the physical $z-$plane $z = x + iy$.

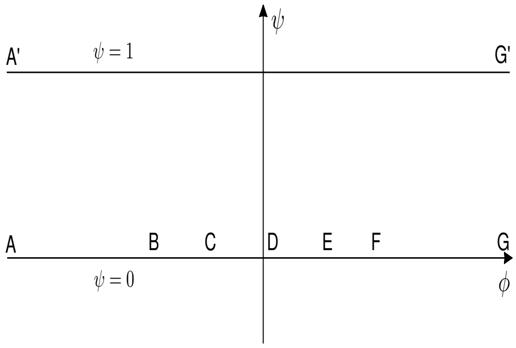

Fig. 2. Sketch of the flow in the complex potential f-plane $f = \phi + i\psi$.

Here $q = \sqrt{u^2 + v^2}$ is the flow speed, p is the fluid pressure and p_0 is the pressure at infinity. In dimensionless variables, Bernoulli's equation takes the form

$$q^2 + \frac{2}{\alpha}K + \frac{2}{Fr^2}(y - 1) = 1 \tag{6}$$

Here K is the curvature of the free surface, Fr and α are the Froude number and the Weber number defined by (1) and (2) respectively.

We introduce the function $\tau - i\theta$ as

$$\xi = u - iv = e^{\tau - i\theta} \tag{7}$$

Where θ is the angle between the speed vector and the horizontal.

The curvature K is defined by the relation

$$K = -e^{\tau}\left|\frac{\partial\theta}{\partial\phi}\right| \tag{8}$$

Substituting (8) into (6) and using (7), (6) becomes

$$e^{2\tau} - \frac{2}{\alpha}e^{\tau}\left|\frac{\partial\theta}{\partial\phi}\right| + \frac{2}{Fr^2}(y - 1) = 1 \qquad on \quad \psi = 1 \tag{9}$$

The kinematic conditions on the solid boundaries can be written as

$$\begin{cases} Im\xi = 0, \theta = 0 \quad on\ \psi = 0, \quad \phi_A < \phi < \phi_B \quad and \\ \phi_F < \psi < \phi_G \\ Im\xi = u\tan\gamma, \theta = |\gamma| \ on\ \psi = 0, \quad \phi_B < \phi < \phi_C, \\ \phi_C < \phi < \phi_D, \phi_D < \phi < \phi_E \ \ and\ \phi_E < \phi < \phi_F \end{cases} \tag{10}$$

We shall seek $(\tau - i\theta)$ as an analytic function of $f = \phi + i\psi$ in the strip $0 < \psi < 1$ (see figure 2), satisfying the conditions (9) and (10).

III. Numerical procedure

We define a new variable t by the relation

$$f = \frac{2}{\pi}\log\frac{1 + t}{1 - t} \tag{11}$$

This transformation maps the flow domain into the upper half of the unit disc in the complex $t-$plane so that the free surface on the circumference (see figure 3). The points of

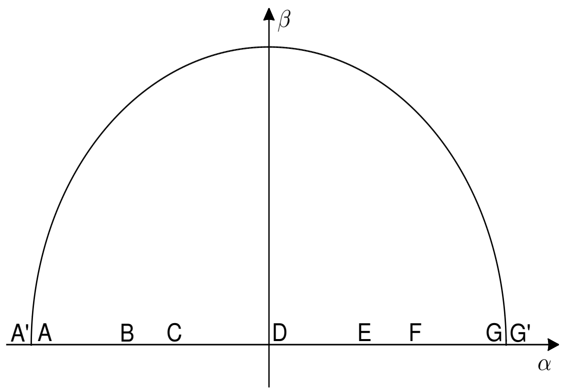

Fig. 3. The complex $t-$ plane.

the free surface in the $t-$plane are given by the relation:

$$t = e^{i\sigma}, 0 < \sigma < \pi \tag{12}$$

Since there are angled corners at t_B, t_C, t_D, t_E and t_F, ξ has singularities at these points. By using the symmetry of the problem with respect to the vertical $y-$ axis, we choose $t_B = -t_F$ and $t_C = -t_E$. The appropriate singularities are

$$\xi = \left(\frac{t - t_B}{1 - tt_B}\right)^{\frac{-\gamma}{\pi}} \qquad as \quad t \longrightarrow t_B \tag{13}$$

$$\xi = \left(\frac{t - t_C}{1 - tt_C}\right)^{\frac{2\gamma}{\pi}} \qquad as \quad t \longrightarrow t_C \tag{14}$$

$$\xi = t^{\frac{-2\gamma}{\pi}} \qquad as \quad t \longrightarrow t_D \tag{15}$$

We represent the complex velocity ξ by the expansion

$$\xi = \left(\frac{t^2 - t_B^2}{1 - t^2 t_B^2}\right)^{\frac{-\gamma}{\pi}} \left(\frac{t^2 - t_C^2}{1 - t^2 t_C^2}\right)^{\frac{2\gamma}{\pi}} t^{\frac{-2\gamma}{\pi}} \exp\left(\Omega(t)\right) \tag{16}$$

Where the function $\Omega(t)$ is bounded and continuous on the unit circle and analytic in the interior. The conditions (10) show that $\Omega(t)$ can be expanded in the form of a Taylor expansion in even powers of t. Hence, we write

$$\xi = \left(\frac{t^2 - t_B^2}{1 - t^2 t_B^2}\right)^{\frac{-\gamma}{\pi}} \left(\frac{t^2 - t_C^2}{1 - t^2 t_C^2}\right)^{\frac{2\gamma}{\pi}} t^{\frac{-2\gamma}{\pi}} \exp\left(\sum_{k=0}^{k=\infty} a_k t^{2k}\right) \tag{17}$$

The kinematic conditions (10) are satisfied by requiring the coefficients a_k to be real. It can be checked that (16) satisfies $(13)-(15)$. Therefore we expect the series in (17) to converge for $|t| \leq 1$. The coefficients a_k must be determined to satisfy the boundary condition (9) on the free surface $A'G'$. According to (12) and (17), we have:

$$\begin{cases} \theta(\sigma) = \frac{\gamma}{\pi}\left(\arctan\frac{\sin 2\sigma}{\cos 2\sigma - t_B^2} - \arctan\frac{t_B^2 \sin 2\sigma}{t_B^2 \cos 2\sigma - 1}\right) + \frac{2\gamma}{\pi}\sigma \\ +\frac{2\gamma}{\pi}\left(\arctan\frac{t_C^2 \sin 2\sigma}{t_C^2 \cos 2\sigma - 1} - \arctan\frac{\sin 2\sigma}{\cos 2\sigma - t_C^2}\right) - \\ \sum_{k=1}^{k=\infty} a_k \sin(2k-1)\sigma \\ \tau(\sigma) = \sum_{k=1}^{k=\infty} a_k \cos(2(k-1)\sigma) \end{cases} \tag{18}$$

It is convenient to eliminate y from (9) by differentiating (9) with respect to σ. Using $(11), (12)$ and the identity

$$\frac{\partial x}{\partial \phi} + i\frac{\partial y}{\partial \phi} = e^{-\tau + i\theta} \tag{19}$$

we obtain after some algebra

$$\begin{gathered} \left(-\sum_{k=1}^{k=\infty} 2(k-1) a_k \sin 2(k-1)\sigma\right) e^{2\tau(\sigma)} - \frac{\pi}{\alpha} e^{\tau(\sigma)} \times \\ \left[\left(\frac{\partial \tau}{\partial \sigma}\sin\sigma + \cos\sigma\right)\left|\frac{\partial\theta(\sigma)}{\partial\sigma}\right| + \sin\sigma\frac{\partial}{\partial\sigma}\left(\left|\frac{\partial\theta(\sigma)}{\partial\sigma}\right|\right)\right] \\ -\frac{1}{\pi\sin\sigma Fr^2} e^{-\tau(\sigma)} \sin\theta(\sigma) = 0 \end{gathered} \tag{20}$$

Here $\tau(\sigma)$ and $\theta(\sigma)$ denote the values of τ and θ on the free surface $A'G'$.

We solve the problem numerically by truncating the infinite series in (17) after N terms. We find the N coefficients a_k by collocation. Thus we introduce the N mesh points

$$\sigma_I = \frac{\pi}{N}\left(I - \frac{1}{2}\right), I = 1, ..., N \tag{21}$$

Using (16) we obtain $[\tau(\sigma)], [\theta(\sigma)], \left[\frac{\partial \tau}{\partial \sigma}\right]$ and $\left[\frac{\partial \theta}{\partial \sigma}\right]$ in terms of coefficients a_k. Thus, we obtain N non-linear algebraic equations of N unknowns $(a_k, k = 1, ..., N)$. We solve this system by Newton's method.

We calculate the height W of the triangle by integrating (19) from $t = t_C$ to $t = t_B$ and using (11)

$$W = \sin\gamma \int_{t_C}^{t_B} \frac{1}{\xi}\frac{df}{dt} dt \tag{22}$$

The shape of the free surface is obtained by integrating numerically the relation (19), we obtain

$$\begin{cases} \frac{\partial x}{\partial \sigma} = \frac{-2}{\pi \sin\sigma} e^{-\tau(\sigma)} \cos(\theta(\sigma)) \\ \frac{\partial y}{\partial \sigma} = \frac{-2}{\pi \sin\sigma} e^{-\tau(\sigma)} \sin(\theta(\sigma)) \end{cases} \tag{23}$$

Most of the calculations were performed with $N = 50$.

IV. Discussion of the results

The numerical scheme described in section 3 was used to compute solutions for various values of the Weber number α and Froude number Fr. We found that the coefficients a_k decrease rapidly as N increases. For example, $a_1 = 1.66 \times 10^{-2}, a_{10} = -9.88 \times 10^{-6}, a_{30} = -3.77 \times 10^{-6}$ and $a_{50} = -7.04 \times 10^{-7}$ for $\gamma = \frac{\pi}{6}, \alpha = 20$ and $Fr \longrightarrow \infty$.

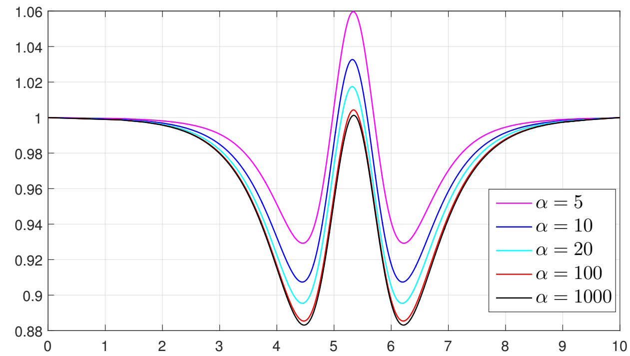

Fig. 4. The free surface profiles for $\gamma = \frac{\pi}{6}, Fr = \infty$ and various values of Weber number α.

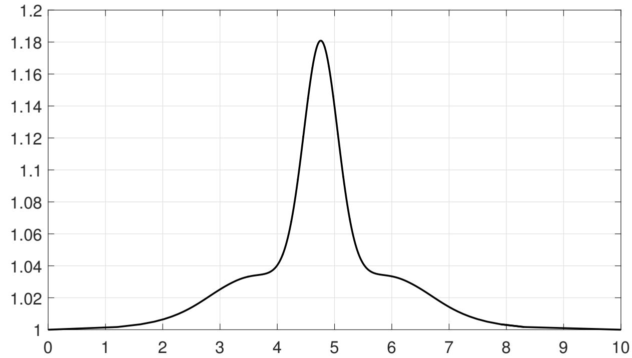

Fig. 5. The free surface profile for $\gamma = \frac{\pi}{6}, Fr = \infty$ and $\alpha_0 = 0.97$.

A. Solution with surface tension and without gravity.

When gravity is neglected and surface tension is included in the free surface condition, the numerical computations show that there is a solution for each value of α and γ. This results shows that there is a critical Weber number α_0 for each value of γ such that for $\alpha = \alpha_0$ the solution approach limiting configuration with a free surface flow over an obstacle. For example, the free surface profile for a critical value $\alpha_0 = 0.97$ and $\gamma = \frac{\pi}{6}$ is shown in figure 5. Most of the results are obtained with $N = 50, W = 0.18, \gamma = \frac{\pi}{6}$. The effect of the Weber number $\alpha > \alpha_0$ on free surface profile is shown in figure 4. It should be noted that the free surface elevation increases when the Weber number α decreases which when $\alpha \longrightarrow \alpha_0$, this elevation reaches its maximum. Figure 6 shows the effect of the angle γ on the shape of free surface for $\alpha = 200$. The influence of the size of triangle W is shown in figure 7.

B. Solution with gravity and without surface tension.

When surface tension is neglected $(\alpha \longrightarrow \infty)$ and gravity is taken into account, numerical solutions exist for different values of Fr and γ. Most of the results are obtained with $N = 50, W = 0.65, \gamma = \frac{\pi}{4}$. We noticed that the coefficients a_k of the series (17) were found to decrease very rapidly. For example, $a_5 = -6.88 \times 10^{-5}, a_{10} = -5.17 \times 10^{-9}, a_{30} = -1.63 \times 10^{-11}$ and $a_{50} = -1.57 \times 10^{-13}$ for $\gamma = \frac{\pi}{3}$ and $Fr = 2$. Figure 8 shows the effect of Froude number on the free surface profile of supercritical flow $(Fr > 1)$ for $\gamma = \frac{\pi}{6}$. For subcritical flow $(Fr < 1)$, the effect of Froude number

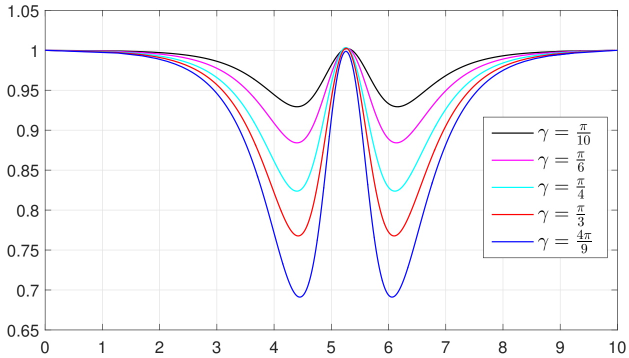

Fig. 6. The free surface profiles for $Fr = \infty, \alpha = 200$ and various values of the angle γ.

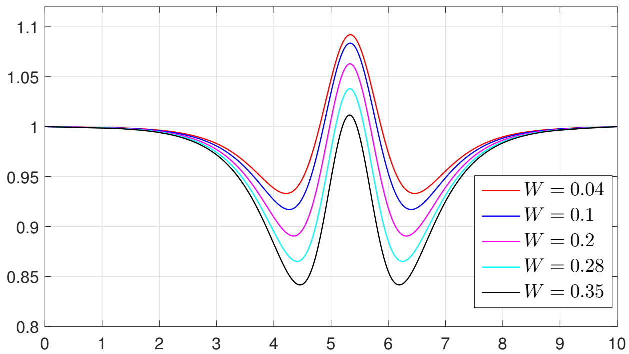

Fig. 7. The free surface profiles for $Fr = \infty, \alpha = 100, \gamma = \frac{\pi}{4}$ and various values of the height W.

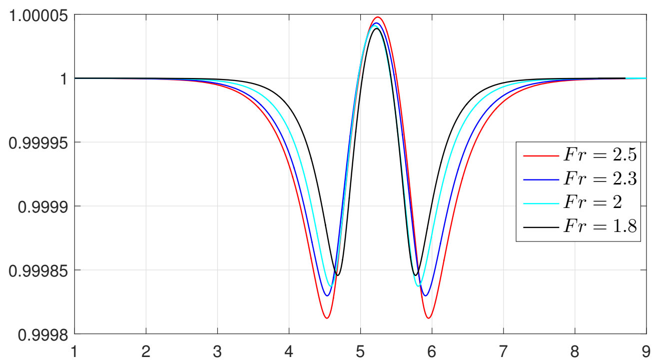

Fig. 8. The free surface profiles of supercritical flow for $\alpha = \infty, \gamma = \frac{\pi}{6}, W = 0.65$ and various values of the Froude number.

is shown in figure 9 for $\gamma = \frac{\pi}{3}$. For large Froude number is shown in figure 10. It can be seen that the elevation of the free surfaces decreases as Froude number Fr decreases.

V. Conclusion

We have presented numerical solutions for a free surface flow over a successive triangular obstacles. The fluid is subjected to the combined effects of gravity and surface tension. When surface tension is neglected there are solutions corresponding to two different values of the angle γ and the Froude number. It is shown that the effect of Froude number on the free surface profile of supercritical flow $(Fr > 1)$ and subcritical flow $(Fr < 1)$. When gravity is neglected

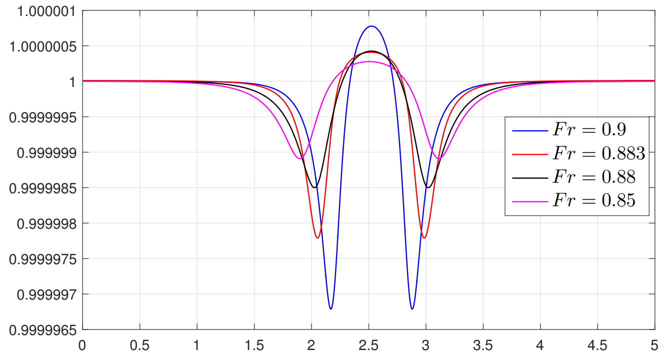

Fig. 9. The free surface profiles of sub-critical flow for $\alpha = \infty, \gamma = \frac{\pi}{3}, W = 0.07$ and various values of the Froude number.

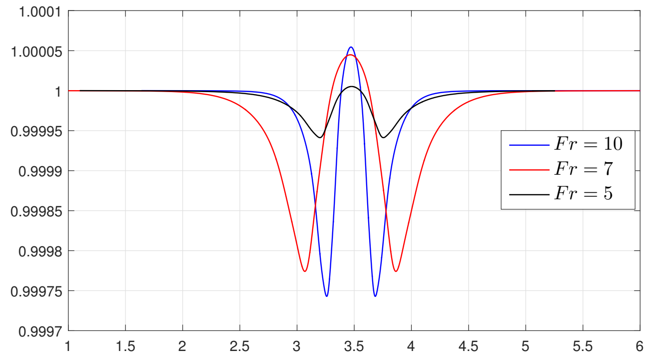

Fig. 10. The free surface profiles of supercritical flow for $\gamma = \frac{\pi}{4}, \alpha = \infty, W = 0.65$ and various values of Froude number Fr.

and surface tension is included in the free surface condition, It was found that for given values of γ and size of obstacle, there is a unique solution. It should be noted that the Weber number α increases as the elevation of the free surfaces decreases.

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