Combination of the Fourth-Order Runge-Kutta and an Explicit Finite Difference Method for an Advection-Diffusion Equation

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***Abstract*— Groundwater pollution monitoring is critical for preserving drinking water quality. If a landfill is to be established, the potential impact on groundwater quality must be assessed, which can be done with a mathematical model. This work proposes a long-term assessment of groundwater quality using a heterogeneous soil model. Two numerical models are presented by using one-dimensional advection-diffusion equation. The standard forward-time and Centered-space finite differences method is employed to estimate the concentration of Contaminants in the groundwater within the nearby region. The concentration is also estimated with the fourth-order Runge–Kutta method. A comparison is conducted between an FTCS and the approximate solutions. Both numerical methods produce an accurate approximate solution. However, the fourth-order Runge–Kutta approach achieves a higher accuracy than the conventional method.**

***Index Terms*—One-dimensional advection-diffusion equation, explicit finite difference method, fourth-order Runge–Kutta method.**

I. Introduction

LEACHATE generation from landfills poses substantial environmental risks, particularly to watercourses as well as groundwater. These dangers can be reduced by creating and constructing them on geologically impermeable materials [1]. The arrangement of solid waste at a specific landfill and the primary factors influencing the properties and makeup of the leachates are the local rainfall conditions. In both India and Indonesia, the handling of leachate collection from these locations impacts the groundwater and, therefore, is a crucial factor in reducing the negative impacts on groundwater quality.

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The composition of the solid waste at a given landfill site and the local rainfall conditions are the main determinants of the characteristics and composition of the leachates. In both India and Indonesia, the handling of leachate collection from these locations impacts the groundwater and, therefore, is a crucial factor in reducing the negative impacts on groundwater quality. The handling of leachates from landfills differs among nations, some of which employ specific technologies and procedures for managing them in contrast, in others, they are not properly handled.

In [2], a comparative analysis was provided about groundwater pollution caused by leachate from landfill areas in India and Indonesia. It was discovered that inadequate landfill design and the lack of a liner permit leachates to seep into the soil and pollute groundwater. The elements influencing groundwater pollution from the leachates were also analyzed, revealing that both countries' landfill sites were not well-managed. Additionally, it was revealed that proper methods for disposing of solid waste were not followed.

Many developing countries are struggling with municipal solid waste management due to the rise in solid waste generation driven by urbanization and industrialization. A large amount of this waste includes harmful substances that gradually infiltrate the soil and groundwater, leading to long-lasting environmental dangers. There are hazards associated with landfill leachate production to human health and the environment by polluting soil, surface water, and groundwater. Key areas of research focus on designing effective landfill liner systems, assessing the extent of contaminants leaching into groundwater, and understanding the potential health and environmental dangers [4].

Monitoring groundwater quality is crucial for assessing the potential risk and impact of contamination events. Municipal solid waste contains various materials such as glass, metal, paper, rags, plastics, ash, and flammable substances [5]. Leachate from landfills presents serious health and environmental hazards because of its ability to pollute groundwater and water on the surface [6]. In certain places, groundwater near landfills is at a higher risk of contamination; numerous studies have examined the negative consequences of leachate from landfills on both Groundwater and surface water [7–9]. A point source, such as a landfill, able to release large amounts of pollutants into due to leachate, the groundwater flow from its bottom [10]. The disposal of water and pollution are inherently connected. The unregulated disposal of waste causes various environmental hazards, such as the contamination of water, soil, and air, along with health threats. The pollution of groundwater due to leachate produced by waste disposal areas is a significant health issue to a multitude of researchers and experts globally. A Leachate can be any liquid. that passes via solid waste and eliminates suspended solids, solutes, or other harmful substances from the substance it contains traversed. The utilization of groundwater tainted by leachate is often cited as a risk to human health [11].

One-dimensional advection–diffusion equations were proposed in [12] as an appropriate mathematical model for an accurate evaluation of groundwater quality. A new fourth-order method based on the Saulyev technique was used in a groundwater quality assessment model. The environment suffers when landfills contaminate groundwater. In this research, a model was proposed for the long-term assessment of groundwater quality in varied soil, comprising a 1D-(ADE) transient. Two mathematical models were presented. The conventional centered-space forward-time finite difference approach was employed to estimate the concentration of groundwater contaminants in an area adjacent to a landfill. A novel fourth-order finite difference approach utilizing the Saulyev method was used to estimate the solution. The estimated solutions were contrasted with the perfect exact solution. Both numerical techniques produced comparable outcomes; however, the new fourth-order method using the Saulyev technique provided more accurate solutions than the traditional method. In [13], a long-term numerical model for assessing groundwater quality was introduced, utilizing a modified fourth-order finite difference method combined with a Saulyev scheme. The prediction of groundwater quality over the long term was utilized to present the environmental impact evaluation of landfill site initiatives. 1D- transient advection-diffusion equation was employed to represent groundwater contaminant levels.

Models for simulations of groundwater quality using numerical methods assessment over extended time frames were suggested. The traditional explicit FTCS and the typical fourth-order finite difference techniques were applied. The conventional fourth-order finite difference method using a Saulyev scheme, along with a revised fourth-order finite difference method with a Saulyev scheme, were suggested. Estimated solution were compared with the exact solution in a specific case. The proposed modified new fourth-order finite difference technique gives accurate approximate solutions. The suggested numerical simulation can be utilized for various kinds of soil physics.

Numerical methods were employed to solve 1D-advection-dispersion-reaction equation for non-uniform flow in [14]. In [15], An appropriate model was developed to replicate the transportation of contaminants in a medium, demonstrating spatial fourth-order precision and temporal second-order precision. In [16], An implicit method was applied to address a hydrodynamic model, along with the backward-time centered-space scheme was used to solve a dispersion model in [17]. Mathematical frameworks depicting groundwater flow and solute transfer in homogeneous and heterogeneous porous media were developed in [18]. The groundwater model employed both implicit and explicit traditional finite difference techniques along with alternating path approaches. The finite difference techniques produced precise outcomes, with the fastest finite difference method discovered being (FTCS) followed by (ADEM), (ADIM), and finally (BTCS).

The aquifer simulation showed the head influenced by water, which then revealed the water table. For groundwater movement, the recognized technique of minimal variation was utilized. The varied framework measurements of the aquifer's size, depressions, and sources created the intricate geometry in the model. In [19], it was demonstrated that alternating path techniques for groundwater simulation can effectively represent real-world scenarios.

Employing the category explicit technique, Evans and Abdullah [20] created the (AGE) approach, that could be distinctly tackled and utilized for further concerns. Inspired by these ideas, the authors of [21] utilized the integral conversion on a (CDE) and suggested a plan that was a truncation error of order and maintains unconditional stability. Zheng [22] suggested a category of explicit iterative methods for alternating groups, that are convergent, unconditionally stable, and precise to the second degree in both space and time. The authors of [23] performed a comparative analysis on the stability boundaries of fourth-order compact and first-order upwind methods for discretization a 1D unsteady (CDE. These methods were applied alongside Runge–Kutta temporal discretizations of up to sixth order [24]. Conducted a thorough evaluation of two-tier three-point finite difference technique of sequence two in duration and four in area. The reliability and the oscillatory nature of these approaches were examined and the evaluation was enhanced by numerical trials

In order to recalled, papers are worried using linearly implicit Runge–Kutta technique to the numerical calculation of systems of (FDM) that emerge from the spatial discretization of (ARD) partial differential equations. The main difficulty in managing these systems resides and the reality that explicit time integrators usually exhibit inefficiency, as the system grows more rigid when the spatial grid is refined. However, selecting a strictly accurate integrator requires the unspoken solution of nonlinear equations, that may be challenging, particularly when used in conjunction with spectral methods [25] and the usual method to resolving the nonlinear equations that emerge in the approach of implicit methods involves employing a modified Newton iteration alongside banded approximations of the Jacobian obtained via finite differences. However, when tackling spectral spatial discretization of equations for (ARD), countless setbacks in the Newton method may occur [26].

The goal of this research is to generate numerical answers of 1D-ADE employing a sixth-order compact difference method in spatial dimensions and a fourth-order Runge–Kutta method in temporal dimensions. It is regarded as a more dependable option compared to current methods for such applications [27]. A model for the dispersion of groundwater pollutants is presented. A conventional finite difference approach, along with a blend of the fourth-order Runge–Kutta and an explicit finite difference method, is employed to resolve the advection–diffusion equation

II. GOVERNING EQUATION

A. A. Distribution of Groundwater Contamination in Uneven Soil

In a model of groundwater quality, the primary is a partial differential equation in one dimension that describes advection–diffusion [28],

$$\frac{\partial C(x,t)}{\partial t}=\frac{\partial}{\partial x}\left(D\left(x,t\right)\frac{\partial C(x,t)}{\partial x}-u(x,t)C(x,t)\right), \tag{1}$$

for all $(x,t)\in[0,L]\times[0,T]$,

where $C(x,t)$ indicates the quantity of pollutants in groundwater on the premises. x presently in the longitudinal orientation t,D is the pollutant's dispersion coefficient, u represents a consistent flow speed, L is the distance of the examined region from the source of the pollutant to the endpoint, and T represents the duration of the simulation. The unevenness of the soil leads to differences in the velocity of groundwater flow Kumar et al. [28] suggested a modification of a growing nature. They also presumed that the velocity and dispersion parameters are defined functions. $f_1(x,t)$ and $f_2(x,t)$.. Eq. (1) can be rewritten as [28]

$$\frac{\partial C(x,t)}{\partial t}=\frac{\partial}{\partial x}\left(D_0 f_1\left(x,t\right)\frac{\partial C(x,t)}{\partial x}-u_0 f_2(x,t)C(x,t)\right). \tag{2}$$

Eq. (2) can be written in the following form:

$$\frac{\partial C(x,t)}{\partial t}=\frac{\partial}{\partial x}\left(D_0\frac{\partial f_1(x,t)}{\partial x}-u_0 f_2(x,t)\right)\frac{\partial C(x,t)}{\partial x}$$
$$+D_0 f_1(x,t)\frac{\partial^2 C(x,t)}{\partial x^2}-u_0\frac{\partial f_2(x,t)}{\partial x}C(x,t). \tag{3}$$

In the formula presented above, D_0 and u_0 are constants with dimensions are determined by the expressions $f_1(x,t)$ and $f_2(x,t)$. The unevenness of the soil results in changes in the flow velocity. Kumar et al. [28] have explored a variation involving the greater spread of groundwater pollutants in uneven soil. It is likewise presumed that the velocity and the dispersion parameter are proportional squared. Thus, Eq. (2) becomes

$$f_1(x,t)=\left(1+ax\right)^2,\quad\text{and}\quad f_2(x,t)=1+ax, \tag{4}$$

where the parameter a with dimension of $(length)^{-1}$ accounts for the soil inhomogeneity. in Eq. (3) becomes

$$\frac{\partial C(x,t)}{\partial t}=\left[\left(1+ax\right)\left(2aD_0-u_0\right)\right]\frac{\partial C(x,t)}{\partial x}$$
$$+D_0(1+ax)^2\frac{\partial^2 C(x,t)}{\partial x^2}-u_0 aC(x,t), \tag{5}$$

B. Initial and boundary conditions

There was no pollution or degree of groundwater contamination in the soil's initial state or boundary. shows the initial condition listed below:

$$C(x,0)=r(x),\ 0\le x\le L,\ t=0. \tag{6}$$

where $r(x)$ is initially assessed groundwater contaminant function. Because of a constant influx, a groundwater contaminant is released at the source, while the gradient in concentration at the endpoint is determined by the mean rate of change of contaminants in groundwater concentration in their vicinity, leading to the boundary conditions listed below:

$$C(0,t)=C_0,\quad t>0, \tag{7}$$

$$\frac{\partial C(x,t)}{\partial x}=C_s,\quad x=L,\quad t\ge 0. \tag{8}$$

where C_0 is a considering the average concentration of groundwater pollutants at the landfill under consideration and C_s is rate of change of the pollutant concentration around the far field monitoring station.

III. NUMERICAL TECHNIQUES

A. The traditional forward time central space method (FTCS)

Putting both a forward and a central difference scheme into practice to Eq. (2) [28], Each term's discretization is produced as follows:

$$C(x,t)\cong C_i^n, \tag{9}$$

$$\frac{\partial C(x,t)}{\partial t}\cong\frac{C_i^{n+1}-C_i^n}{\Delta t}, \tag{10}$$

$$\frac{\partial C(x,t)}{\partial x}\cong\frac{C_{1+1}^n-C_{i-1}^n}{2\Delta x}, \tag{11}$$

$$\frac{\partial^2 C(x,t)}{\partial x^2}\cong\frac{C_{i+1}^n-2C_i^n+C_{i-1}^n}{(\Delta x)^2}. \tag{12}$$

Substituting Eqs. (9)–(12) into Eq. (3), we find that

$$\frac{C_i^{n+1}-C_i^n}{\Delta t}=\left[(1+ai\Delta x)(2aD_0-u_0)\right]\frac{C_{i+1}^n-C_{i-1}^n}{2\Delta x}$$
$$+D_0(1+ai\Delta x)^2\frac{C_{i+1}^n-2C_i^n+C_{i-1}^n}{(\Delta x)^2}-u_0 aC_i^n, \tag{13}$$

$$=\left[\frac{D_0\left(1+ai\Delta x\right)^2\Delta t}{(\Delta x)^2}-\frac{(1+ai\Delta x)(2aD_0-u_0)\Delta t}{2\Delta x}\right]C_{i-1}^n$$
$$+\left[1-\frac{2D_0\left(1+ai\Delta x\right)^2\Delta t}{(\Delta x)^2}-u_0 a\Delta t\right]C_i^n$$
$$+\left[\frac{(1+ai\Delta x)(2aD_0-u_0)\Delta t}{2\Delta x}+\frac{D_0\left(1+ai\Delta x\right)^2\Delta t}{(\Delta x)^2}\right]C_{i+1}^n. \tag{14}$$

Thus $C_i^{n+1} = (E_i - K_i)C_{i-1}^n + (1-2E_i - \vartheta)C_i^n$

$$+(E_i + K_i)C_{i+1}^n, \tag{15}$$

where

$$E_i = \frac{D_0\left(1+ai\Delta x\right)^2 \Delta t}{(\Delta x)^2}, \tag{16}$$

$$K_i = \frac{(1+ai\Delta x)(2aD_0 - u_0)\Delta t}{2\Delta x}, \tag{17}$$

$$\vartheta = u_0 a \Delta t. \tag{18}$$

The difference equation's truncation error Eq. (16) is $O(\Delta t, \Delta x^2)$. Making use of sufficiently small values of Δx and Δt, Until the accuracy attained falls within the error tolerance, the truncation error can be decreased [29,30]. The first circumstance Eq. (3) for Eq. (15) can be written as follows in finite difference form:

$$C_i^0 = 0,\ x \geq 0,\ t = 0. \tag{19}$$

The boundary condition Eq. (7) can be written in finite difference form as

$$C_0^n = C_0,\ x = 0,\ t = 0. \tag{20}$$

If the forward space method is used in Eq. (8) to the appropriate boundary condition,

$$C_N^n = C_{N-1}^n + \Delta x C_s,\ x = L,\ t \geq 0. \tag{21}$$

B. Runge-Kutta MethodThe Compact Finite Difference Method

Methods for compact finite differences are widely used. within the community of fluid dynamics due to their high accuracy and the benefits of using stencils [30]. These techniques can effectively increase accuracy without requiring any additional in the size of the stencil, whereas traditional high-order finite difference methods use larger stencil sizes that complicate the treatment of boundaries. CD schemes have also been demonstrated to be more precise and computationally efficient. The use of smaller stencil sizes in CD methods is helpful when dealing with nonperiodic boundary conditions. A one-dimensional mesh is considered, consisting of N points: x_1, x_2, ..., x_{i-1}, x_i, $x_{i+1}, \ldots, x_N$ with mesh size $\Delta x = x_{i+1} - x_i$. The unknown function's first-order derivatives can be expressed as follows at interior nodes [29]:

$$\alpha C'_{i-1} + C'_i + \alpha C'_{i+1} = b\frac{C_{i+2} + C_{i-2}}{4\Delta x} + a\frac{C_{i+1} + C_{i-1}}{2\Delta x}. \tag{22}$$

leading to an α-family of fourth-order tridiagonal schemes with

$$a = \frac{2}{3}\left(\alpha + 2\right),\ b = \frac{1}{3}\left(4\alpha - 1\right). \tag{23}$$

A sixth-order tridiagonal scheme is obtained by $b = \dfrac{1}{3}$,

$$C'_{i-1} + 3C'_i + C'_{i+1} = \frac{1}{12\Delta x}\left(C_{i+2} + 28C_{i+1} - 28C_{i-1} + C_{i-2}\right). \tag{24}$$

The RK4 scheme is used to obtain the temporal integration in the present study. Applying the CD6 technique to Eq. (1) gives rise to the following differential equation in time:

$$\frac{dC_i}{dt} = LC_i, \tag{25}$$

$$\frac{dC_i}{dt} = L(C_i) = f(t_i, c_i), \tag{26}$$

with initial conditions and next to step terms given by $C(t_n) = C_n$ and $C(t_{n+1}) = C_{n+1}$, respectively. The RK4 formula is [29]

$$y_{n+1} = y_n + hk, \tag{27}$$

where $h = 1, h = \Delta t$ and $k = LC$, we are employing RK4 applied to a new equation in this work.

$$C_{n+1} = C_n + hLC. \tag{28}$$

We have RK4 scheme to the flowing:

$$C = \frac{1}{6}\left(C_n + 2C_1 + 2C_2 + C_3\right). \tag{29}$$

The spatial terms express to the following operation:

$$C_n = hf(t_i, c_i), \tag{30}$$

$$C_n = hLC_i. \tag{31}$$

where L shows a linear differential operator in space. The CD6 approximates the spatial and temporal terms. and the RK4 schemes, respectively. Eq. (25) is solved using the RK4 scheme through the following operations [30]:

solved equation C_1

$$C_1 = hf(t_i + \frac{h}{2}, C_i + \frac{C_n}{2}), \tag{32}$$

$$C_1 = hL(C_i + \frac{C_n}{2}), \tag{33}$$

$$C_1 = hLC_i + \frac{hLC_n}{2}, \tag{34}$$

$$C_1 = C_n + \frac{hLC_n}{2}, \tag{35}$$

$$C_1 = C_n + \frac{1}{2}\Delta t LC_n. \tag{36}$$

solved equation C_2

$$C_2 = hf(t_i + \frac{h}{2}, C_i + \frac{C_1}{2}), \tag{37}$$

$$C_2 = hL(C_i + \frac{C_1}{2}), \tag{38}$$

$$C_2 = hLC_i + \frac{hLC_1}{2}, \tag{39}$$

$$C_2 = C_n + \frac{hLC_1}{2}, \tag{40}$$

$$C_2 = C_n + \frac{1}{2}\Delta t L C_1. \tag{41}$$

solved equation C_3

$$C_3 = hf(t_i + h, C_i + C_2), \tag{42}$$

$$C_3 = hL(C_i + C_2), \tag{43}$$

$$C_3 = hLC_i + hLC_2, \tag{44}$$

$$C_3 = C_n + \Delta t L C_2. \tag{45}$$

Substitution Eq. (29) in to Eq. (28) we obtain that

$$C_{n+1} = C_n + hLC, \tag{46}$$

$$C_{n+1} = C_n + \Delta t L \frac{1}{6}(C_n + 2C_1 + 2C_2 + C_3), \tag{47}$$

$$C_{n+1} = C_n + \frac{1}{6}\Delta t(LC_n + 2LC_1 + 2LC_2 + LC_3), \tag{48}$$

when annotation $C_* = C^{(*)}$ thus we get a new equation follow that

$$C^{n+1} = C^n + \frac{1}{6}\Delta t(L(C^{(n)}) + 2L(C^{(1)}) + 2L(C^{(2)}) + L(C^{(3)})). \tag{49}$$

The fourth-order Runge-Kutta equation

where $C(x,t)$ is the amount of pollutants that are dispersing in groundwater. at the location x throughout the longitudinal path at that moment t, D is a pollutant method dispersion coefficient and u is a consistent flow rate, L is the distance measured from the source of the pollutant to the final location and T is the simulation time rate of chance. Variability results from the soil's inhomogeneity in the velocity of groundwater flow. In Kumar et al [28], They suggest an increasing nature variation. Additionally, they presume that the parameter of dispersion and the The velocity parameters are provided, $f_1(x,t)$ and $f_2(x,t)$. Eq.(1) can be rewritten as:[28], substitution Eq.(1) in to Eq.(26) we obtain that

$$\frac{dC_i}{dt} = LC_i. \tag{25}$$

By utilizing a forward difference plan and a scheme for central differences to Eq. (2) [28], Each term's discretization is produced as follows:

$$C^n(x,t) \cong C_i^n \tag{50}$$

$$\frac{\partial C(x,t)}{\partial t} \cong \frac{C_i^{n+1} - C_i^n}{\Delta t}, \tag{10}$$

$$\frac{\partial C(x,t)}{\partial x} \cong \frac{C_{1+1}^n - C_{i-1}^n}{2\Delta x}, \tag{11}$$

$$\frac{\partial^2 C(x,t)}{\partial x^2} \cong \frac{C_{i+1}^n - 2C_i^n + C_{i-1}^n}{(\Delta x)^2}. \tag{12}$$

Substituting Eqs. (10)–(12) and Eq. (47) into Eq. (46), we obtain

$$C_i^{n+1} = C_i^n + \frac{1}{6}\Delta t\left(\frac{\partial}{\partial x}\left(D(x,t)\frac{\partial C(x,t)}{\partial x} - u(x,t)C(x,t)\right)\right), \tag{51}$$

$$= C_i^n + \frac{1}{6}\Delta t\left(-u_0\left(\frac{C_{1+1}^n - C_{i-1}^n}{2\Delta x}\right) + D_0\left(\frac{C_{i+1}^n - 2C_i^n + C_{i-1}^n}{(\Delta x)^2}\right)\right)$$

$$= C_i^n + \left(\frac{-\Delta t u_0 C_{i+1}^n + \Delta t u_0 C_{i-1}^n}{12\Delta x} + \frac{\Delta t D_0 C_{i+1}^n - \Delta t D_0 2C_i^n + \Delta t D_0 C_{i-1}^n}{6(\Delta x)^2}\right)$$

$$= C_i^n + \left(\frac{-\Delta t u_0 C_{i+1}^n + \Delta t u_0 C_{i-1}^n}{12\Delta x} + \frac{2\Delta t D_0 C_{i+1}^n - 4\Delta t D_0 C_i^n + 2\Delta t D_0 C_{i-1}^n}{12(\Delta x)^2}\right)$$

$$= C_i^n + \left(\frac{-\Delta t u_0 \Delta x C_{i+1}^n + \Delta t u_0 \Delta x C_{i-1}^n}{12(\Delta x)^2} + \frac{2\Delta t D_0 C_{i+1}^n - 4\Delta t D_0 C_i^n + 2\Delta t D_0 C_{i-1}^n}{12(\Delta x)^2}\right)$$

$$= C_i^n - \frac{\Delta t u_0 \Delta x C_{i+1}^n}{12(\Delta x)^2} + \frac{\Delta t u_0 \Delta x C_{i-1}^n}{12(\Delta x)^2} + \frac{2\Delta t D_0 C_{i+1}^n}{12(\Delta x)^2} - \frac{4\Delta t D_0 C_i^n}{12(\Delta x)^2} + \frac{2\Delta t D_0 C_{i-1}^n}{12(\Delta x)^2}$$

$$= \frac{12(\Delta x)^2}{12(\Delta x)^2}C_i^n - \frac{\Delta t u_0 \Delta x C_{i+1}^n}{12(\Delta x)^2} + \frac{\Delta t u_0 \Delta x C_{i-1}^n}{12(\Delta x)^2} + \frac{2\Delta t D_0 C_{i+1}^n}{12(\Delta x)^2} - \frac{4\Delta t D_0 C_i^n}{12(\Delta x)^2} + \frac{2\Delta t D_0 C_{i-1}^n}{12(\Delta x)^2}$$

$$C_i^{n+1} = \left(\frac{2\Delta t D_0 C_{i-1}^n}{12(\Delta x)^2} + \frac{\Delta t u_0 \Delta x C_{i-1}^n}{12(\Delta x)^2}\right) + \left(\frac{12(\Delta x)^2 C_i^n}{12(\Delta x)^2} - \frac{4\Delta t D_0 C_i^n}{12(\Delta x)^2}\right) + \left(\frac{\Delta t u_0 \Delta x C_{i+1}^n}{12(\Delta x)^2} + \frac{2\Delta t D_0 C_{i+1}^n}{12(\Delta x)^2}\right), \tag{52}$$

and we arrive at the fourth-order Runge–Kutta equation,

$$C_i^{n+1} = \left(\frac{2\Delta t D_0 + \Delta t u_0 \Delta x}{12(\Delta x)^2}\right)C_{i-1}^n + \left(\frac{12(\Delta x)^2 - 4\Delta t D_0}{12(\Delta x)^2}\right)C_i^n + \left(\frac{\Delta t u_0 \Delta x + 2\Delta t D_0}{12(\Delta x)^2}\right)C_{i+1}^n. \tag{53}$$

To derive a rough solution of Eq. (1) utilizing CD6-RK4 for the boundary and initial conditions, the area [0, L] is initially divided into discrete parts so that $0 = x_1 < x_2 < ... < x_N = L$, where N represents the total count of grid points

IV. Numerical Experiments

Imagine that the concentration of groundwater pollutants C underneath a landfill and its surrounding area is being evaluated. The studied area is aligned over a longitudinal span, measuring a total of 1.0 km. The studied area is aligned over a longitudinal span, measuring a total of 1.0 km in length $C_0 = 1.0$ kg/l, $D_0 = 0.71$ km²/year, $u_0 = 0.60$ km/year, and $a = 1$ km^{-1}. Within the numerical test, both time and space are divided into discrete units by $\Delta x = 0.1$ km, and $\Delta t = 0.0001$ year. The concentration of groundwater is estimated by applying (FTCS) and the fourth-order Runge-Kutta technique, respectively. An analytical solution for an ideal (ADE) is presented in [27],

$$\tilde{C}(x,t)=\frac{C_0}{2}\left(\begin{array}{l}(1+ax)^{-1}\,erfc\left(\dfrac{\ln(1+ax)}{2a\sqrt{D_0t}}-\beta_0\sqrt{t}\right)\\+(1+ax)^{\delta}\,erfc\left(\dfrac{\ln(1+ax)}{2a\sqrt{D_0t}}+\beta_0\sqrt{t}\right)\end{array}\right). \tag{54}$$

where

$$\omega_0=\left(au_0-a^2D_0\right), \tag{55}$$

$$\beta_0=\sqrt{\frac{\omega_0^2}{4a^2D_0}+au_0}=\frac{u_0+aD_0}{2\sqrt{D_0}}, \tag{56}$$

$$\delta=\frac{u_0}{aD_0}. \tag{57}$$

If we utilize a conventional FCTS approach, as outlined by the equations. (15)–(18), we derive the estimated concentration of groundwater pollutants in the area studied for up to 1.3 years, as illustrated in Figs. 1 and 2 and Table I. If we utilize the fourth-order Runge–Kutta technique, as outlined by Eqs. (25) samt (49)–(54), We derive the estimated concentration of groundwater pollutants across the longitudinal region depicted in the figures. 3, 4, and Table II. The precision of the classical FCTS and fourth-order Runge–Kutta techniques is illustrated in Figs. 2 and 4, accordingly. The precision of each estimate is evaluated using the analytical solution; the absolute error is presented in Tables III and IV

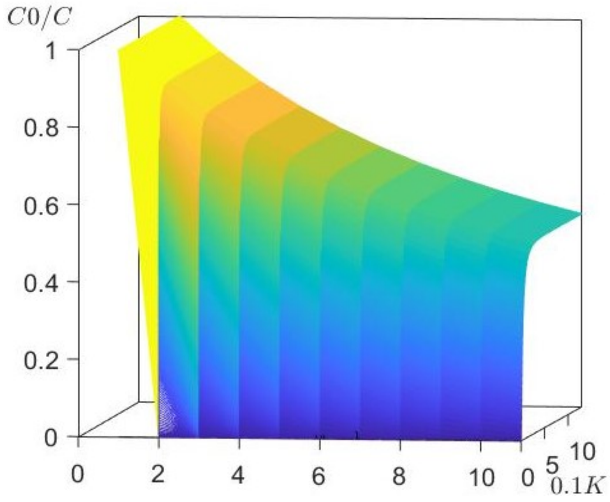

Fig 1. Estimated groundwater contaminants derived through the FTCS technique

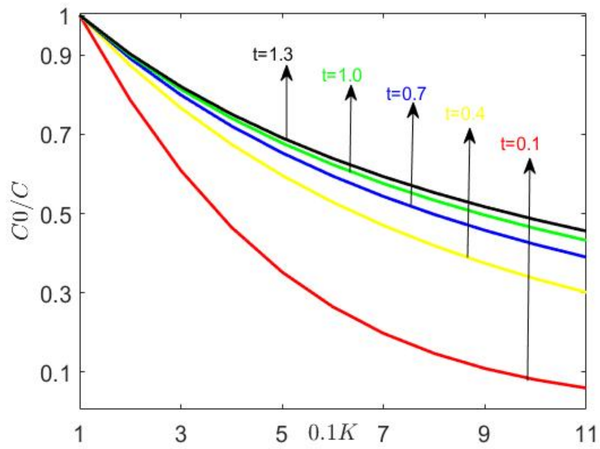

Fig 2. Groundwater contaminants collected via the FTCS method at 0.1, 0.4, 0.7, 1.0, and 1.3 years. Asterisks denote FTCS solutions, while curved lines indicate analytical solutions

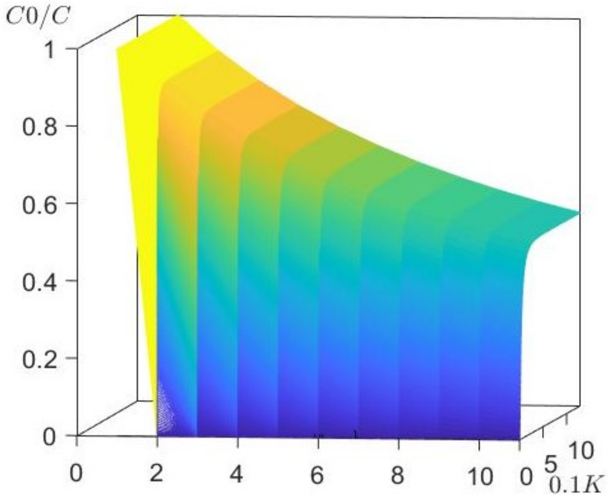

Fig 3. Estimate groundwater contaminants using the fourth-order Runge–Kutta method.

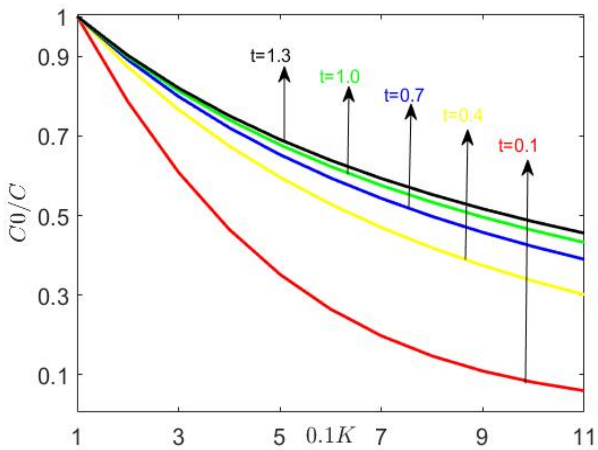

Fig 4. Concentrations of groundwater pollutants at 0.1, 0.4, 0.7, 1.0, and 1.3 years. Asterisks denote the fourth-order Runge-Kutta solutions, while curved lines indicate analytical solutions.

TABLE I ESTIMATE GROUNDWATER POLLUTANT LEVELS UTILIZING FTCS WITHIN A SPECIFIED REGION RANGING FROM 0.1 AND 1.3 YEARS

x \ t	0.0	0.1	0.2	0.3	0.4	0.5
0.1	1.00	0.7831	0.6034	0.5789	0.3476	0.2601
0.4	1.00	0.8719	0.7635	0.6712	0.5920	0.5238
0.7	1.00	0.8900	0.7973	0.7183	0.6503	0.5912
1.0	1.00	0.8976	0.8117	0.7386	0.6757	0.6209
1.3	1.00	0.9017	0.8193	0.7494	0.6892	0.6369

x \ t	0.6	0.7	0.8	0.9	1.0
0.1	0.1914	0.1414	0.1042	0.0766	0.0562
0.4	0.4649	0.4136	0.3690	0.3299	0.2957
0.7	0.5395	0.4940	0.4537	0.4178	0.3858
1.0	0.5729	0.5306	0.4929	0.4592	0.4290
1.3	0.5910	0.5505	0.5145	0.4822	0.4532

$C(x,t)$

TABLE II APPROXIMATE GROUNDWATER POLLUTANT CONCENTRATION FROM THE FOURTH-ORDER RUNGE–KUTTA EQUATION ALONG A CONSIDERED AREA BETWEEN 0.1 AND 1.3 YEARS

$C(x,t)$

x \ t	0.0	0.1	0.2	0.3	0.4	0.5
0.1	1.00	0.7898	0.6161	0.5609	0.3676	0.2719
0.4	1.00	0.8923	0.7662	0.7002	0.5988	0.5553
0.7	1.00	0.8962	0.7982	0.7199	0.6586	0.6012
1.0	1.00	0.9278	0.8381	0.7562	0.7152	0.7010
1.3	1.00	0.9168	0.8570	0.7698	0.7302	0.7374

x \ t	0.6	0.7	0.8	0.9	1.0
0.1	0.2093	0.1406	0.1158	0.0880	0.0664
0.4	0.4764	0.4653	0.3897	0.3547	0.2888
0.7	0.5320	0.4986	0.4590	0.4381	0.3978
1.0	0.5833	0.5315	0.4967	0.4678	0.4607
1.3	0.5907	0.5612	0.5392	0.4839	0.4546

TABLE III ABSOLUTE ERROR OF THE TRADITIONAL FORWARD TIME CENTRAL SPACE METHOD (FTCS) APPROXIMATION

$e(x,t)$

x \ t	0.0	0.1	0.2	0.3	0.4	0.5
0.1	0.00	0.0027	0.0048	0.0061	0.0066	0.0066
0.4	0.00	0.0011	0.0022	0.0030	0.0037	0.0043
0.7	0.00	0.0007	0.0014	0.0019	0.0024	0.0028
1.0	0.00	0.0005	0.0009	0.0014	0.0017	0.0020
1.3	0.00	0.0003	0.0007	0.0010	0.0013	0.0015

x \ t	0.6	0.7	0.8	0.9	1.0
0.1	0.0062	0.0056	0.0049	0.0042	0.0036
0.4	0.0047	0.0050	0.0052	0.0053	0.0054
0.7	0.0032	0.0035	0.0037	0.0039	0.0041
1.0	0.0023	0.0025	0.0027	0.0029	0.0030
1.3	0.0017	0.0019	0.0020	0.0022	0.0023

TABLE IV ABSOLUTE ERROR OF THE FOURTH-ORDER RUNGE–KUTTA EQUATION APPROXIMATION

$e(x,t)$

x \ t	0.0	0.1	0.2	0.3	0.4	0.5
0.1	0.00	0.0022	0.0100	0.0025	0.0039	0.0031
0.4	0.00	0.0007	0.0010	0.0024	0.0019	0.0018
0.7	0.00	0.0005	0.0008	0.0009	0.0013	0.0013
1.0	0.00	0.0003	0.0010	0.0007	0.0008	0.0008
1.3	0.00	0.0002	0.0012	0.0005	0.0007	0.0007

x \ t	0.6	0.7	0.8	0.9	1.0
0.1	0.0028	0.0026	0.0023	0.0019	0.0016
0.4	0.0023	0.0023	0.0024	0.0024	0.0025
0.7	0.0015	0.0016	0.0017	0.0018	0.0019
1.0	0.0011	0.0012	0.0012	0.0013	0.0014
1.3	0.0008	0.0008	0.0009	0.0011	0.0017

V. DISCUSSION

An explicit finite difference method and the fourth-order Runge–Kutta method produce approximate groundwater pollutant concentrations that agree closely in an ideal case, as shown in Figs. 2 and 4, respectively. In both cases, the measurement of groundwater pollutants was simulated for approximately 1.3 years, a considerable amount of time, as shown in Table I–II and Figs. 1 and 3. The fourth-order Runge–Kutta method provides better approximate solutions than the explicit finite difference method, as shown by the absolute error values in Tables III and IV. The proposed numerical techniques provide an accurate approximate solution.

VI. CONCLUSION

The prolonged behavior of groundwater pollution was modeled in uneven soil. A revised model of groundwater quality was utilized for an extended duration. The concentration of pollutants was estimated using a numerical method, and the concentration of groundwater contaminants at their monitoring sites was taken as the model's initial and boundary conditions to approximate the model solution, a finite difference method, namely, an explicit finite difference method and the fourth-order Runge–Kutta method, were used. The fourth-order Runge–Kutta technique offers a more accurate estimation compared to the conventional explicit finite difference method. The suggested model can be utilized to give alerts regarding the upcoming trends of groundwater pollution

The suggested numerical methods deliver a precise approximate answer and avoid causing excessive numerical dispersion. Additionally, this study may be advantageous in the field of mathematics education. It may serve to instruct Grade 12 calculus concepts through a project-based learning method, aiding students in enhancing their abilities to enable them to utilize mathematics in practical situations

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