Application of the Shooting Method in 2-Point Block BDF for Nonlinear Boundary Value Problems

Hira Soomro, Nooraini Zainuddin, Nor Ain Azeany Mohd Nasir, Hanita Daud, Rahimah Jusoh, Siti Ainor Mohd Yatim and Iskandar Shah Mohd Zawawi

***Abstract*— In this research, the boundary value problems (BVPs) are solved by applying a variable step two-point block backward differentiation formula (2BDF) in a shooting technique. An iterative procedure known as Newton-Raphson and the linear combination of the initial guesses are applied in this shooting approach to find appropriate initial conditions for a related initial value problem (IVP) of BVP. Initially, the nonlinear BVP is reduced to the form of first-order ordinary differential equation (ODE). Subsequently, the missing IVP is approximated using the shooting technique and then solved by the 2BDF. Comparisons to exact solutions demonstrate the outcome of the proposed method for solving nonlinear BVP. The results show that the application of shooting technique is appropriate for the 2BDF to solve nonlinear BVP.**

***Index Terms*— backward differentiation formula, block method, boundary value problems, shooting technique.**

I. Introduction

Boundary value problems (BVPs) occur in an extensive variety of scientific disciplines. Some of them pursue engineering, technology, and optimisation theory [1, 2].

It is commonly understood that not all BVPs can be solved analytically. As a result of that, numerical approaches are essential for providing an approximate solution to the BVP.

Manuscript received May 24, 2024; revised January 14, 2025.

This work was supported by the National Collaborative Research Fund between Universiti Teknologi PETRONAS and Universiti Malaysia Pahang with cost centre 015MC0-036.

Hira Soomro is a postdoctoral researcher of Department of Fundamental and Applied Sciences, Universiti Teknologi PETRONAS, 32610 Seri Iskandar, Perak, Malaysia. (e-mail: hira.soomro@utp.edu.my)

Nooraini Zainuddin is a lecturer at Department of Fundamental and Applied Sciences, Universiti Teknologi PETRONAS, 32610 Seri Iskandar, Perak, Malaysia. (e-mail: aini_zainuddin@utp.edu.my)

Nor Ain Azeany Mohd Nasir is a senior lecturer at Department of Mathematics, Centre for Defence Foundation Studies, Universiti Pertahanan Nasional Malaysia, Kem Sungai Besi 57000 Kuala Lumpur, Malaysia. (e-mail: norainazeany@upnm.edu.my).

Hanita Daud is an associate professor of Department of Fundamental and Applied Sciences, Universiti Teknologi PETRONAS, 32610 Seri Iskandar, Perak, Malaysia. (e-mail: hanita_daud@utp.edu.my).

Rahimah Jusoh is a senior lecturer of Centre for Mathematical Sciences, Universiti Malaysia Pahang, 26300 Gambang, Kuantan, Pahang, Malaysia. (e-mail: rahimahj@ump.edu.my).

Siti Ainor Mohd Yatim is a senior lecturer at School of Distance Education, Universiti Sains Malaysia, 11800 USM, Pulau Pinang, Malaysia. (e-mail: ainor@usm.my).

Iskandar Shah Mohd Zawawi is a senior lecturer at School of Mathematical Sciences, College of Computing, Informatics and Mathematics, Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia. (e-mail: iskandarshah@uitm.edu.my).

There are numerous approaches proposed in the literature to solve BVPs. Three primary numerical approaches have been presented by [3] for solving the BVPs of ordinary differential equations (ODEs) Dirichlet, Neumann, and Robin boundary conditions. Using the least squares approach in conjunction with a third-degree B-spline function, [4] created an approximate solution for third-order linear and nonlinear BVPs. [5] proposed a quadratic nonpolynomial spline approach for numerically solving third-order two-point BVPs. For the numerical solution of a third-order two-point BVP, [6] provided second and fourth-order convergent approaches based on quartic nonpolynomial spline function. On the other hand, [7] used a two-step direct method by shooting technique to solve second-order BVP. In order to solve it directly, the fourth-order two-point block approach also uses shooting technique.

[8, 9] proposed a second-order technique for solving a system of third-order two-point BVPs applying cubic and quartic polynomial spline functions, respectively, to obtain approximate solutions of such type of BVPs. However, at midpoints, [10] have developed a second order finite difference technique. Other approaches for the solution of the BVP are the automatic differentiation, block strategies, the modified Adomian decomposition technique, multi-step methods, and the spline technique [3, 6, 11-15].

The shooting method is the numerical approach that is commonly employed in the iterative algorithm to determine the suitable initial conditions for an associated initial value problem (IVP) that gives the solutions to the initial BVPs. In essence, the shooting method for BVPs involves redefining a problem as one of non-linear parameter estimation. In order to solve the new problem, initial conditions for an associated IVP that approximates the boundary conditions at the other endpoint must be identified. If the boundary conditions are not met with the desired precision, the procedure is repeated with an alternate set of initial conditions until the required level of accuracy is attained or an iteration limit is reached.

In this paper, the block backward differentiation formula is utilized in conjunction with the shooting technique method for solving ODEs with mixed boundary conditions. The variable step two-point block backward differentiation formula (2BDF) was used for solving the initial value problems (IVP) of first-order ODEs [16]. Therefore, in this study, an appropriate shooting technique is applied to 2BDF for solving BVP. The 2BDF is implemented in a fixed ratio of step size adjustment, which is a constant ratio ($r = 1$), increased by 1.9 ($r = 10/19$), and decreased by 0.5 ($r = 2.0$). The 2BDF method is given as:

$r = 1$:

$$y_{n+1} = -\frac{1}{3}y_{n-1} + 2y_n - \frac{2}{3}y_{n+2} + 2hy'_{n+1},$$
$$y_{n+2} = \frac{2}{11}y_{n-1} - \frac{9}{11}y_n + \frac{18}{11}y_{n+1} + \frac{6}{11}hy'_{n+2}. \quad (1)$$

$r = 2$:

$$y_{n+1} = -\frac{1}{8}y_{n-1} + \frac{9}{4}y_n - \frac{9}{8}y_{n+2} + 3hy'_{n+1},$$
$$y_{n+2} = \frac{1}{21}y_{n-1} - \frac{4}{7}y_n + \frac{32}{21}y_{n+1} + \frac{4}{7}hy'_{n+2}. \quad (2)$$

$r = \frac{10}{19}$:

$$y_{n+1} = -\frac{361}{480}y_{n-1} + \frac{841}{380}y_n - \frac{841}{1824}y_{n+2} + \frac{29}{19}hy'_{n+1},$$
$$y_{n+2} = \frac{6859}{13195}y_{n-1} - \frac{576}{455}y_n + \frac{4608}{2639}y_{n+1} + \frac{48}{91}hy'_{n+2}. \quad (3)$$

II. Methodology

To solve the BVP, the shooting technique is applied to the 2BDF method. This technique will transform the BVP to an IVP. Missing initial value in IVP is guessed and updated by using the linear combination of errors at the boundary conditions.

At first, s_0 which is the missing initial value is estimated by using the gradient between the boundary conditions. We discuss the proposed shooting technique in terms of second-order ODEs for the following BVP,

$$y''(x) = f(x, y(x), y'(x)), y(a) = A, y(b) = B, x \in [a, b] \quad (4)$$

Reducing (4) in the equivalent first-order ODE, generates the following:

$$y'_1(x) = y_2(x),$$
$$y'_2(x) = f(x, y_1(x), y_2(x)),$$

with the boundary conditions $y_1(a) = A, y_1(b) = B$. The missing initial value, $y_2(a)$ is predicted as

$$y_2(a) = s_0 = \frac{(y_1(b) - y_1(a))}{(b - a)} \quad (5)$$

Subsequently, the 2BDF method as given by Eqs. (1-3) is applied, and the error at $y_1(b) = B$ is computed and is denoted as B_0. The second guess for the missing initial value is updated by using the Newton-Raphson formula as below:

$$y_2(a) = s_1 = s_0 - \frac{y_2(b)}{y'_2(b)} \quad (6)$$

2BDF is reapplied to obtain the second error of the approximation, B_1. Two points,(s_0, B_0) and (s_1, B_1) are used for the iteration of the missing initial value $y_2(a) = s_n,\ n = 2,3, \ldots$. The approximation of s_n, is predicted with the linear combinations of (s_{n-1}, B_{n-1}) and (s_{n-2}, B_{n-2}). The iteration of s_2 is terminated when the error B_n is less than the tolerance, TOL. The algorithm to implement the proposed shooting technique in the 2BDF method is as follows:

Step 1: Set TOL and calculate the missing initial value s_0 by using Eqn (5).

Step 2: Apply 2BDF and record the B_0. B_0 is the error at the boundary condition when the IVP s_0 is used.

Step 3: The missing initial value is now updated, denoted as s_1 using Newton-Raphson method (eqn (6).

Step 4: Apply 2BDF and record the B_1, the error when initial value s_1 is applied.

Step 5: The missing initial values, $s_n, n = 2,3, \ldots$ is updated by using linear combination of the previous two points, which are (s_{n-2}, B_{n-2}) and (s_{n-1}, B_{n-1}). The following equation is used:

$$s_{n-2} = s_{n-1} - \frac{B_{n-1}}{mm}$$
$$\therefore mm = \frac{B_{n-2} - B_{n-1}}{s_{n-2} - s_{n-1}}$$

Step 6: Apply 2BDF and record the B_n.

Step 7: While $B_n \geq 1. \times 10^{-3}$ and $n \leq 10$; repeat steps 5-6.

Step 8: Print the numerical result.

Step 9: End

III. Results and Discussion

In this section, five numerical examples are considered to evaluate the performance of the shooting 2BDF over the exact solution. For Problems 1-3, only one initial condition is missing. Thus, the shooting technique is applied to locate this missing initial condition. The graph of the plotting with the exact solutions and the absolute error are given in Fig. 2-4. Meanwhile, for Problems 4 and 5, two missing initial conditions are guessed using the shooting technique. The plots for Problems 4 and 5 include the solution and its derivative. The tolerance used is $1.\times 10^{-3}$.

Problem 1: The source of BVP is [17].

$$\varepsilon y'' + y' - 1 - 2x = 0, y(0) = 0, y(1) = 1, x \in [0,1], \varepsilon = 0.001$$

Exact: $y(x) = x(x + 1 - 2\varepsilon) + (2\varepsilon - 1)\frac{(1-\exp(-\frac{x}{\varepsilon}))}{(1-\exp(-\frac{1}{\varepsilon}))}$

Initial guess $s_0 = y'(0) = 1$

TABLE 1: APPROXIMATE SOLUTION FOR PROBLEM 1

x	y	Exact
0.00075	-0.529563	-0.529554
0.00933	-0.988520	-0.988511
0.01066	-0.987230	-0.987221
0.01259	-0.985279	-0.985270
0.01625	-0.981517	-0.981508
0.02322	-0.974288	-0.974279
0.04838	-0.947382	-0.947373
0.13918	-0.839724	-0.839715
0.46700	-0.313850	-0.313841
1.00000	0.999991	1.000000

Table 1 compares the computed values y to the exact values for various x, along with the associated absolute errors shown in Fig. 1. The results demonstrate that the numerical method provides an accurate approximation of the exact solution across the entire range of x, with only minor deviations. The method seems robust and effective for solving this problem.

Problem 2: This BVP is obtained from [3].

$$y'' = y + x^2 - 2, y(0) = 0, y(1) = 1, x \in [0,1]$$

Exact: $y(x) = \frac{(e^2x^2 - x^2 + 2e^{1-x} - 2e^{x+1})}{(1-e^2)}$

Initial guess $s_0 = y'(0) = 1$

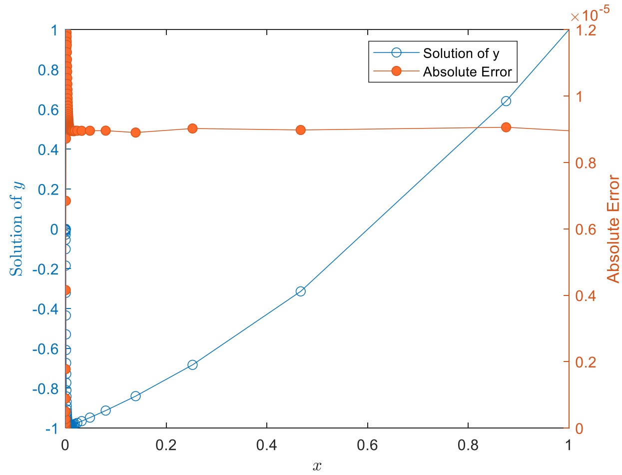


Fig. 1. Plotting of solution and absolute error for Problem 1

TABLE 2. APPROXIMATE SOLUTION FOR PROBLEM 2

x	y	Exact
0.01109	0.018754	0.018753
0.02108	0.035443	0.035442
0.0400	0.066614	0.066611
0.07615	0.123928	0.123923
0.14469	0.226186	0.226176
0.27493	0.398245	0.398226
0.40517	0.544428	0.544400
0.65263	0.765320	0.765287
0.90009	0.937040	0.937021
1.00000	1.000026	1.000000

Results of Problem 2 are tabulated in Table 2 where the computed y values and the precise values for various x positions are compared. The outcomes show that the procedure is reliable and with minimal errors. From Fig. 2, it can be seen that the absolute errors stay small across all x values, demonstrating that the approach is substantially accurate.

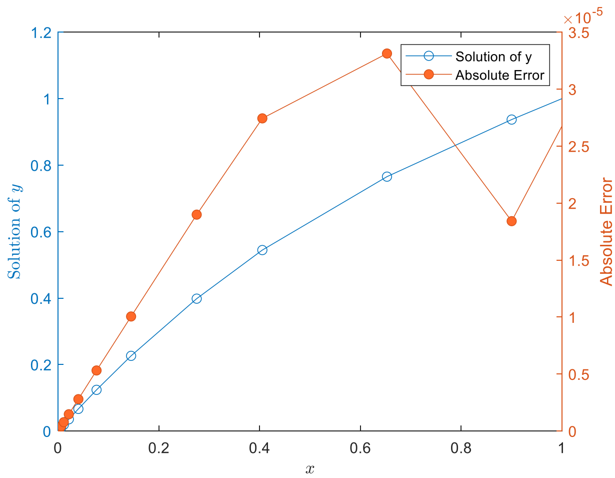


Fig. 2. Plotting of solution and absolute error for Problem 2

Problem 3: This BVP is obtained from [3].

$$y'' = y^3 - yy', y(1) = \frac{1}{2}, y(2) = \frac{1}{3}, x \in [1,2]$$

Exact: $y(x) = \frac{1}{x+1}$

Initial guess $s_0 = y'(1) = -\frac{1}{6}$

TABLE 3: APPROXIMATE SOLUTION FOR PROBLEM 3

x	y	Exact
1.00433	0.498919	0.4989197
1.00824	0.497946	0.497946
1.01568	0.496109	0.496109
1.02982	0.492655	0.492654
1.05667	0.486223	0.486222
1.10769	0.474454	0.474452
1.20463	0.453594	0.453589
1.38882	0.418625	0.418616
1.73877	0.365145	0.365126
2.00000	0.333363	0.333333

Table 3 highlights the computed values for y at various x. According to the findings, the procedure reliably and closely approximates the precise values. Fig. 3 shows the absolute errors that correspond to the differences between the computed y values and the exact values for various x positions. The absolute error progressively rises as x gets larger, however still satisfied the required error tolerance of below $1.\times 10^{-3}$.

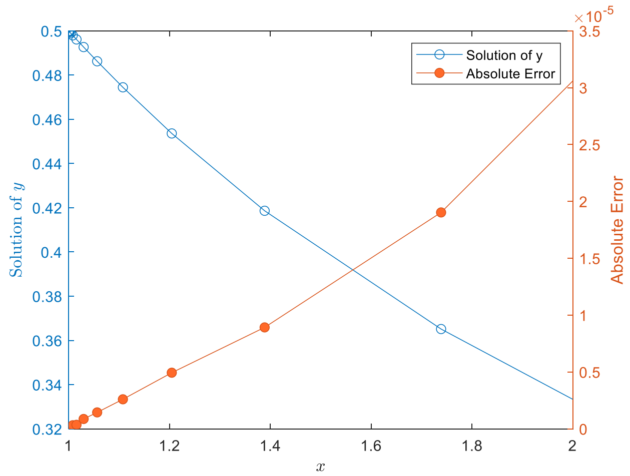


Fig. 3. Plotting of solution and absolute error for Problem 3

The convergence of the boundary condition for Problem 1-3 is given in Table 4. It is clearly seen that a low number of iterations are needed to satisfy the stopping condition of error tolerance below $1.\times 10^{-3}$.

TABLE 4: CONVERGENCE OF PROBLEMS 1 – 3

Problem	Iteration	Convergence Test (-Log(Absolute Error))
1	1	5.047941
2	1	4.572431
3	1	2.69307
	2	4.514382

Two initial values are missing in Problems 4 and 5. These values are approximated by utilizing the gradient formula applied to the points of lower derivatives.

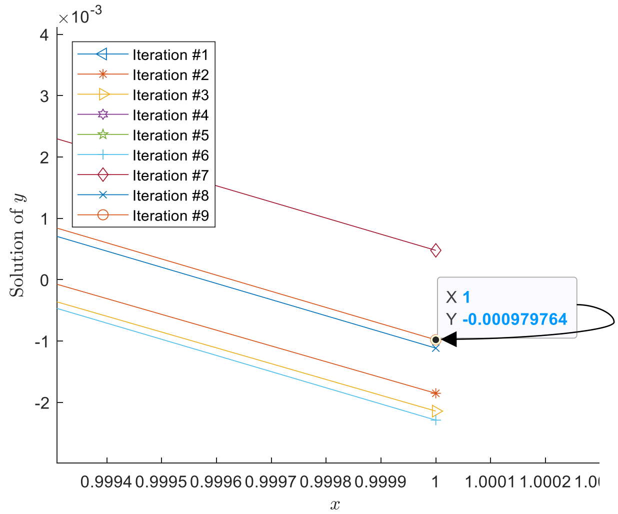
Problem 4: The source of BVP is [18].

$$y^{(4)} = y + y'' + e^x(x-3),$$

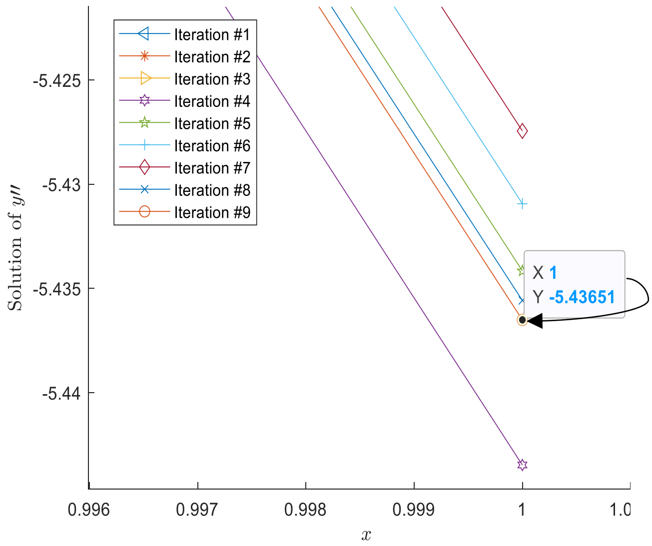
$$y(0) = 1, y(1) = 0, y''(0) = -1, y''(1) = -2e, x \in [0,1]$$

Exact: $y(x) = (1 - x)e^x$
Initial guess $s_{0,1} = y'(0) = -1$
Initial guess $s_{0,2} = y'''(0) = -2e + 1$

Problem 4 uses up to nine iterations to satisfy the error tolerance of $1. \times 10^{-3}$. Fig. 4a and 4b give the convergence of these nine iterations at the boundary conditions. The final solutions of 2BDF are $y(1) = -0.000979764$, and $y''(1) = -5.43651$, while the exact solutions are $y(1) = 0$ and $y''(1) = -2e$. These produce absolute errors of 9.8×10^{-4} and 5.4×10^{-5} respectively.



(a)



(b)

Fig. 4. Iteration of Problem 4

The absolute maximum errors for the computation of $y(1)$ and $y''(1)$ are illustrated in Fig. 5. At ninth iterations, both numerical values of $y(1)$ and $y''(1)$ satisfy the conditions of absolute error of less than $1. \times 10^{-3}$.

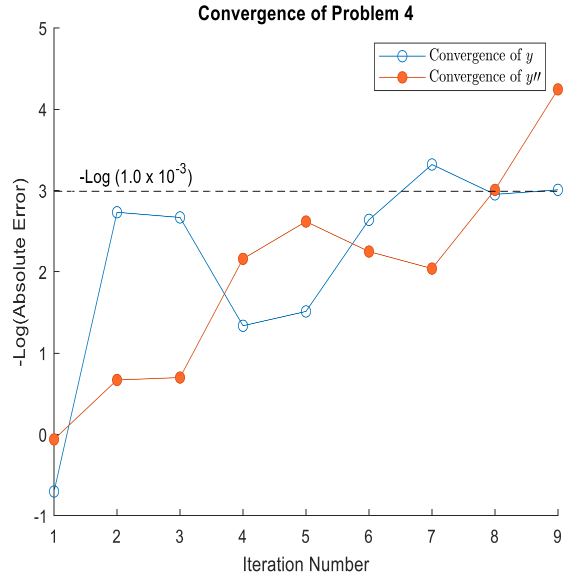


Fig. 5. Convergence for Problem 4

The solution at the ninth iteration with its absolute error for problem 4 is plotted in Fig. 6.

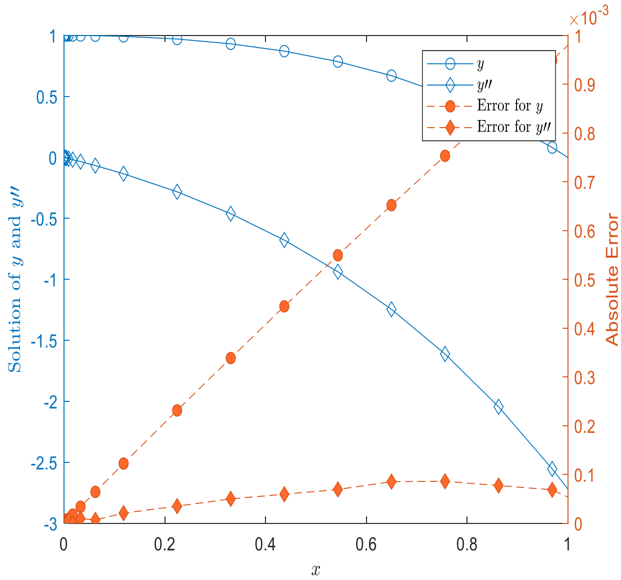


Fig.6. Plotting absolute error for problem 4

From the absolute error at the ninth iteration, the solution produces absolute errors which are less than $1. \times 10^{-3}$.

TABLE 5: APPROXIMATE SOLUTION FOR PROBLEM 4

x	y	y Exact	y''	y'' Exact
0.00250	9.999E-01	9.999E-01	-1.0050	-1.0050E+00
0.00476	9.999E-01	9.9998E-01	-1.0095	-1.0095E+00
0.00905	9.999E-01	9.9995E-01	-1.0182	-1.0182E+00
0.11814	9.923E-01	9.9244E-01	-1.2583	-1.2583E+00
0.43716	8.709E-01	8.7144E-01	-2.2251	-2.2251E+00
0.54350	7.855E-01	7.8609E-01	-2.6579	-2.6579E+00
0.64984	6.699E-01	6.7062E-01	-3.1597	-3.1598E+00
0.75618	5.185E-01	5.1935E-01	-3.7408	-3.7409E+00
0.86253	3.248E-01	3.2568E-01	-4.4125	-4.4126E+00
0.96887	8.107E-02	8.2024E-02	-5.1878	-5.1879E+00
1.00000	-9.797E-04	0.000E+00	-5.4365	-5.4365E+00

Table 5 shows a comparison of numerical and exact values of y and y'' for a given set of x-values. The values of y and y'' computed numerically are highly consistent with the exact values, as seen in the minimal differences between the corresponding columns. The method used for calculating the values of y and y'' is highly effective and accurate across the range of x-values, as indicated by the low absolute errors visualized in Fig. 6. This suggests a strong agreement between the numerical and exact solutions, confirming the reliability of the computational approach.

Problem 5: The source of BVP is [18].

$$y^{(4)} = e^{-x}y^2$$

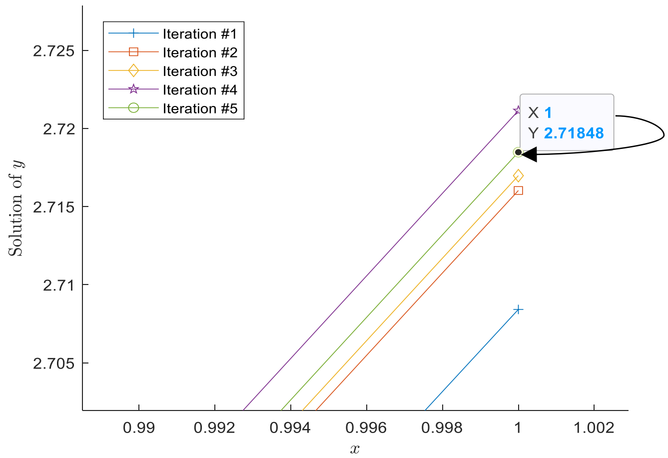
$$y(0) = 1, y(1) = e, y''(0) = 1, y''(1) = e, x \in [0,1]$$

Exact: $y(x) = e^x$

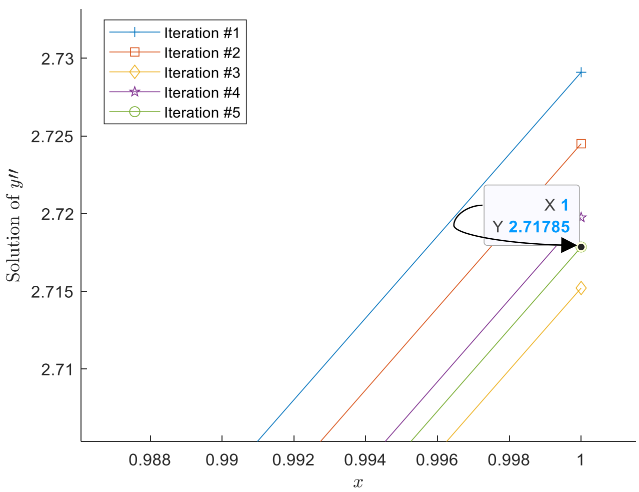
Initial guess $s_{0,1} = y'(0) = e - 1$

Initial guess $s_{0,2} = y'''(0) = e - 1$

Problem 5 uses up to five iterations to satisfy the error of convergence of $1.\times 10^{-3}$. Fig. 7 gives the convergence of these five iterations at the boundary conditions, $y(1) = e$ (Fig. 7(a)) and $y''(1) = e$ (Fig. 7(b)). The final solutions produced by 2BDF are $y(1) = 2.71848$, and $y''(1) = 2.71834$, while the exact solutions are $y(1) = y''(1) = e$. These produce absolute errors of 1.98×10^{-4} and 5.82×10^{-5} respectively.



(a)



(b)

Fig. 7. Iteration of Problem 5

The absolute maximum error for the computation of $y(1)$ and $y''(1)$ are shown in Fig. 8. At the fifth iteration, both numerical values of $y(1)$ and $y''(1)$ satisfy the conditions of absolute error of less than $1. \times 10^{-3}$.

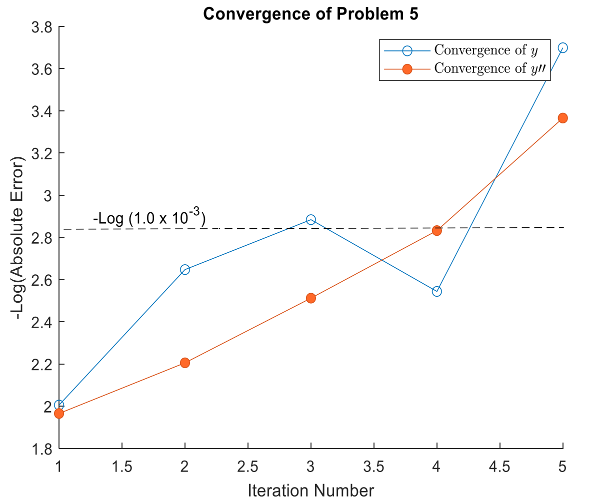


Fig. 8. Convergence Results for Problem 5

The solution at the fifth iteration with its absolute error for problem 5 is plotted in Fig. 9.

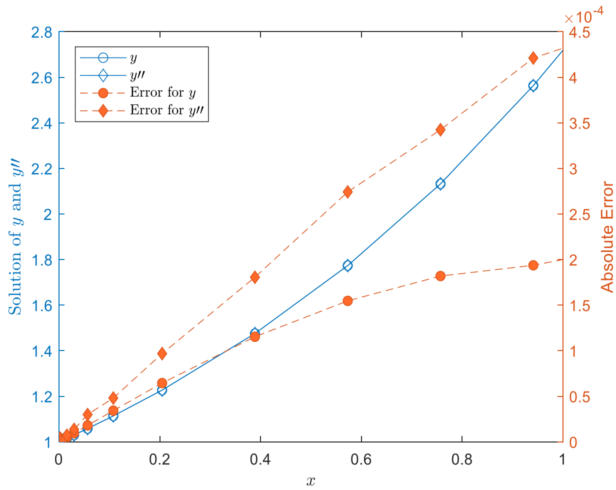


Fig. 9. Plotting of absolute error for problem 5

From the absolute error at the fifth iteration, the solution produces absolute errors which are less than 4.5×10^{-4}.

TABLE 6: APPROXIMATE SOLUTION FOR PROBLEM 5

x	y	y Exact	y''	y'' Exact
0.00031	1.000316	1.000316	1.00032	1.00031
0.00061	1.000616	1.000616	1.00062	1.00061
0.01568	1.015812	1.015807	1.0158	1.01580
0.10767	1.113722	1.113688	1.11364	1.11368
0.20460	1.227101	1.227036	1.22694	1.227036
0.38876	1.475265	1.475150	1.47497	1.475150
0.57291	1.773588	1.773434	1.77316	1.773434
0.75707	2.132214	2.132032	2.13169	2.132032
0.94123	2.563334	2.563141	2.56272	2.563141
1.00000	2.718481	2.718281	2.71785	2.718281

Table 6 demonstrates that the numerical method provides accurate approximations for both the function y and its first derivative y', particularly at initial steps. However, Fig. 9 shows as x increases, the absolute errors for both y and y'' increase, with the error in the second derivative being more pronounced at higher x values. Despite the growing errors, the approximations remain reasonably close to the exact values, particularly for the function y.

For Problems 1–3, there is one unknown for the initial conditions. Meanwhile, for problems 4 and 5, two unknowns are needed. Problems 1 and 2 only require one iteration to satisfy the stopping condition, while Problem 3 needs 2 iterations. Problem 4 requires nine iterations to satisfy the stopping condition, and only five iterations are needed to satisfy the stopping condition for Problem 5. This indicates that the proposed shooting technique is suitable for the 2BDF method.

Tables 1- 3 and 5- 6 represent the numerical results for five chosen BVPs to illustrate the performance of the variable step two-point block BDF with the embedded shooting technique. For each problem, it contains the value at x with corresponding approximate solution, y_{approx} and exact solution, y_{exact}. Whereas Figs 1-3, 6 and 9 included the absolute error between the approximate solution and the exact solution, $|y_{exact} - y_{approx}|$.

On each problem, approximate solutions are approximate to the exact ones, and the absolute errors remain very small. This proves that both variable step 2BDF and shooting technique are reliable and accurate combinations for the solution of nonlinear BVPs as the errors are below the tolerance of $1.\times 10^{-3}$. The small magnitude of the errors indicates that the proposed approach can handle different types of BVPs rather effectively, making the approach quite robust and applicable in practice for scientific and engineering applications.

IV. Conclusion

This work presents the development of a variable-step, two-point block BDF to the shooting technique for the solution of BVPs. The methodology is commenced by reducing a given nonlinear BVP into a first-order ODE form, approximating the missing IVP through the implementation of a shooting technique, then solving it using the 2BDF. The results have consistently shown that the proposed method, which involves comprehensive comparisons with exact solutions, furnishes an accurate and reliable solution for nonlinear BVPs.

Future studies could be oriented towards extending this method to more complex BVPs, possibly with prospects for the application of higher-dimensional problems and integration with parallel computing frameworks that further scale up its feasibility for many scientific and engineering applications.

References

[1] J. Sooknum and N. Pochai, A Mathematical Model for Evaluating the Risk of Airborne Infection Among Bus Passengers Using Ventilation Systems," *IAENG International Journal of Applied Mathematics*, vol. 54, no. 2, pp 169 - 176, 2024.

[2] G. Buzuzi, M. Magodora, M. T. Kudinha, W. M. Manamela, and M. H. Mambo, "Steady MHD Williamson Nanofluid Flow Past an Inclined Stretching Sheet in the Presence of Heat Generation, Chemical Reaction and Aligned Magnetic Field," *IAENG International Journal of Applied Mathematics*, vol. 54, no. 6, pp 1125 - 1135, 2024.

[3] H. Ramos and M. A. Rufai, "A third-derivative two-step block Falkner-type method for solving general second-order boundary-value systems," *Mathematics and Computers in Simulation*, vol. 165, pp. 139-155, 2019.

[4] G. Loghmani and M. Ahmadinia, "Numerical solution of third-order boundary value problems,", pp. 291-295, 2006.

[5] T. S. El-Danaf, "Quartic nonpolynomial spline solutions for third order two-point boundary value problem," *World Academy of Science, Engineering and Technology*, vol. 45, pp. 453-456, 2008.

[6] F. Abd El-Salam, A. El-Sabbagh, and Z. Zaki, "The numerical solution of linear third order boundary value problems using nonpolynomial spline technique," *Journal of American Science*, vol. 6, no. 12, pp. 303-309, 2010.

[7] P. S. Phang, Z. A. Majid, and M. Suleiman, "Solving nonlinear two point boundary value problem using two step direct method (menyelesaikan masalah nilai sempadan dua titik tak linear menggunakan kaedah langsung dua langkah)," *Journal of Quality Measurement and Analysis*, vol. 7, no. 1, pp. 129-140, 2011.

[8] E. A. Al-Said and M. A. Noor, "Cubic splines method for a system of third-order boundary value problems," *Applied Mathematics and Computation*, vol. 142, no. 2-3, pp. 195-204, 2003.

[9] M. A. Noor and E. E. Al-Said, "Quartic splines solutions of third-order obstacle problems," *Applied Mathematics and Computation*, vol. 153, no. 2, pp. 307-316, 2004.

[10] E. A. Al-Said and M. A. Noor, "Numerical solutions of third-order system of boundary value problems," *Applied mathematics and computation*, vol. 190, no. 1, pp. 332-338, 2007.

[11] G. Akram and H. Rehman, "Homotopy perturbation method with reproducing kernel method for third order non linear boundary value problems," *J. Basic. Appl. Sci. Res*, vol. 4, no. 1, pp. 60-67, 2014.

[12] P. Pue-On and N. Viriyapong, "Modified Adomian decomposition method for solving particular third-order ordinary differential equations," *Applied Mathematical Sciences*, vol. 6, no. 30, pp. 1463-1469, 2012.

[13] A. Asaithambi, "Numerical solution of a third-order nonlinear boundary-value problem by automatic differentiation," *International Journal of Computer Mathematics*, vol. 88, no. 7, pp. 1484-1496, 2011.

[14] D. Awoyemi, "A P-stable linear multistep method for solving general third order ordinary differential equations," *International Journal of Computer Mathematics*, vol. 80, no. 8, pp. 985-991, 2003.

[15] M. A. Rufai and H. Ramos, "Numerical solution of third-order boundary value problems by using a two-step hybrid block method with a fourth derivative," *Computational and Mathematical Methods*, vol. 3, no. 6, p. e1166, 2021.

[16] S. Yatim, Z. Ibrahim, K. Othman, and M. Suleiman, "A numerical algorithm for solving stiff ordinary differential equations," *Mathematical Problems in Engineering*, vol. 2013, no. 1, p. 989381, 2013.

[17] H. J. Reinhardt, G. Hsiao, L. Abrahamsson, H. Keller, and O. Kreiss, "Singular perturbations of difference methods for linear ordinary differential equations," *Applicable Analysis*, vol. 10, no. 1, pp. 53-70, 1980.

[18] A. A. Moghadam, A. R. Soheili, and A. S. Bagherzadeh, "Numerical solution of fourth-order BVPs by using Lidstone-collocation method," *Applied Mathematics and Computation*, vol. 425, p. 127055, 2022.