Effect Examining under Multiple Goal Processes

Boontida Uapipatanakul, Jong-Chin Huang, Kelvin H.-C. Chen and Yu-Hsien Liao

***Abstract*—Given the rapid global environmental changes, issues related to sustainability and pollution effects have attracted considerable attention from a wide range of stakeholders. Effectively mitigating and balancing the impacts arising from various factors remains a critical focus in the field of sustainability and environmental impact assessment. In this context, the present study proposes several examination frameworks aimed at reducing and balancing the effects of multiple environmental factors across different goals. These frameworks are then analyzed using a set of axioms to assess both their mathematical validity and practical applicability through a structured axiomatic approach. To refine the assessment of the relative contributions and impacts of different stakeholders and their operating grades, two weighted examination mechanisms, along with their respective characterizations, are introduced. Additionally, the paper explores further interpretations of these axioms and the corresponding axiomatic procedures, providing a deeper understanding of their potential applications in sustainability and pollution control research.**

***Index Terms*—Sustainability, effect, examining method, multiple goal processes, axiomatic procedure.**

I. Introduction

In recent years, sustainability-related challenges have gained significant attention due to the accelerating impacts of climate change, depletion of natural resources, and other environmental suppressors. This has led to a growing body of research focusing on issues such as resource allocation, pollution reduction, and climate change mitigation. The environmental effects resulting from the advancement of human civilization have become an undeniable reality, with some consequences even being irreversible. As such, minimizing the environmental impacts caused by various factors has become a central concern in sustainability-related research.

Addressing these impacts often requires a holistic approach that considers multiple dimensions simultaneously, which may occasionally be at odds. For instance, achieving optimal pollution reduction using certain measures or technologies, while simultaneously conserving energy, minimizing resource consumption, and avoiding the generation of secondary pollutants or waste, necessitates a balanced, multi-faceted approach. In the field of mathematics, multi-objective optimization or equilibrium models are employed to balance these diverse goals within operational systems.

Under conventional transferable-utility (TU) conditions, participators are typically classified as either fully engaged or entirely uninvolved with others in the system. However, in most real-world scenarios, the grade of participator engagement is not clear-cut and remains difficult to determine. Within the framework of multi-choice TU processes, participators can interact across an infinite range of engagement grades. Various examination methods for multi-choice TU games have been explored in diverse contexts, including studies by Calvo and Santos [3], Chen et al. [6], Cheng et al. [4], Li et al. [19], Liao [21], [22], Liao et al. [23], Hwang and Liao [11], [12], Huang et al. [14], Huang et al. [15], Klijn et al. [16], Nouweland et al. [30], Uapipatanakul et al. [35], Wei et al. [36], among others.

Consistency is a crucial property in examining methods used within axiomatic approaches for traditional processes. It ensures that a value remains invariant when certain participators' payoffs are fixed. This principle posits that any recommendations made for a given problem should align with those made in subproblems where specific participators' payoffs are predefined. Consistency has been defined in various ways depending on how the payoffs of participators who ”exit the bargaining” are treated. This property has been extensively studied in the context of reduced processes, such as bargaining and cost allocation issues. Utilizing single contributions, the pseudo equal allocation of non-separable costs (PEANSC, Hsieh and Liao [9]), and the normalized index have been proposed as methods for traditional TU processes. Hsieh and Liao [9] demonstrated an extension of the complement-reduction due to Moulin [27], illustrating that PEANSC offers a fair approach for distributing utilities.

The results presented in this context lead to the following important question:

- Can the single index and its associated outcomes be expanded to better address sustainability challenges in multi-objective processes?

This study aims to establish the necessary mathematical foundations for evaluating multiple goals optimally in the context of sustainability-related issues. Specifically, we examine multi-choice behaviors and their implications within multi-objective frameworks. Based on traditional and multi-choice TU process models, we introduce the concept of multiple goal TU processes. In Section 2, we present two new examination frameworks: the minimal examination of non-separable effects (MENE) and the normalized single effect examination (NSEE).

The MENE examination involves participators receiving minimal single effects from operational coalitions and subsequently evaluating remaining effects equally. Conversely, the NSEE assesses effects proportionally

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by applying minimal single effects across all participators in the coalition. These examinations extend the concept of marginal effects to accommodate multi-choice behavior and multiple goal processes.

To substantiate these examinations, we introduce an extended reduction and associated properties of consistency, which are examined in Sections 3 and 4:

- The MENE is the only examination method that satisfies the properties of multiple goal standardness for processes and multiple goal consistency.
- The MENE is the only examination method that satisfies the properties of multiple goal efficiency, multiple goal covariance, multiple goal symmetry, and multiple goal consistency.
- Although the NSEE does not satisfy multiple goal bilateral consistency, it maintains the properties of normalized-standardness of processes and specific consistency.

Building on the MENE framework, each participator initially receives minimal single effects from operational coalitions, followed by equal examination of any additional fixed effects (e.g., the cost of shared facilities) among the relevant participators. However, variations in participator engagement grades and operational grades can lead to changes across different scenarios.

In practical applications, the MENE approach may appear unrealistic due to disparities in participator size or bargaining power. Asymmetries may arise when modeling differences in bargaining capabilities among participators and their respective operating grades. To address these challenges, we propose alternative examination methods in which any additional fixed effect is distributed proportionally among participators and their operating grades based on their respective weights.

To reduce discrimination and mitigate the relative effects caused by participators and their operating grades, we introduce weighting functions for both participators and operating grades. This leads to two weighted extensions of the MENE and associated axiomatic processes, as discussed in Section 5. Throughout the study, further interpretations and discussions regarding these axioms and axiomatic procedures are presented to deepen understanding of their implications for sustainability and pollution control research.

II. Preliminaries

Let $\overline{\mathbb{UVP}}$ denote the universal collection of participators. For each participator $i \in \overline{\mathbb{UVP}}$ and $\tilde{g}_i \in \mathbb{N}$, we define $\tilde{G}_i = \{0, \cdots \tilde{g}_i\}$ as the operating grade space of participator i, with $\tilde{G}_i^+ = \tilde{G}_i \setminus \{0\}$ indicating active participation, and 0 indicating non-participation. Let $\overline{\mathbb{P}} \subseteq \overline{\mathbb{UVP}}$ and $\tilde{G}^{\overline{\mathbb{P}}} = \prod_{i\in\overline{\mathbb{P}}} \tilde{G}_i$ denote the Cartesian product set of operating grade spaces for participators in $\overline{\mathbb{P}}$. For any $\overline{\mathbb{K}} \subseteq \overline{\mathbb{P}}$, a participator coalition $\overline{\mathbb{K}} \subseteq \overline{\mathbb{P}}$ corresponds canonically to the multi-choice coalition $\tilde{g}^{\overline{\mathbb{K}}} \in \tilde{G}^{\overline{\mathbb{P}}}$, where $\tilde{g}_i^{\overline{\mathbb{K}}} = 1$ if $i \in \overline{\mathbb{K}}$ and $\tilde{g}_i^{\overline{\mathbb{K}}} = 0$ if $i \in \overline{\mathbb{P}} \setminus \overline{\mathbb{K}}$. Let $0_{\overline{\mathbb{P}}}$ represent the zero vector in $\mathbb{R}^{\overline{\mathbb{P}}}$. For $m \in \mathbb{N}$, 0_m denotes the zero vector in $\mathbb{R}^m$, and $\overline{\mathbb{N}}_m = \{1, 2, \cdots, m\}$.

A **multi-choice transferable-utility (TU) process** is characterized as a triple $(\overline{\mathbb{P}}, \tilde{g}, e)$, where $\overline{\mathbb{P}}$ denotes a non-empty and finite set of participators, $\tilde{g} = (\tilde{g}_i)_{i\in\overline{\mathbb{P}}} \in \tilde{G}^{\overline{\mathbb{P}}}$ represents the vector indicating the highest operating grades for each participator, and $e : \tilde{G}^{\overline{\mathbb{P}}} \to \mathbb{R}$ is a function satisfying $e(0_{\overline{\mathbb{P}}}) = 0$, assigning the worth that participators can obtain if operating at corresponding operating grades $\mu = (\mu_i)_{i\in\overline{\mathbb{P}}} \in \tilde{G}^{\overline{\mathbb{P}}}$. A **multiple goal multi-choice TU process** is defined as a triple $(\overline{\mathbb{P}}, \tilde{g}, E^m)$, where $m \in \mathbb{N}$, $E^m = (e^t)_{t\in\overline{\mathbb{N}}_m}$, and $(\overline{\mathbb{P}}, \tilde{g}, e^t)$ represents a multi-choice TU process for all $t \in \overline{\mathbb{N}}_m$. The class encompassing all multiple goal multi-choice TU processes is denoted as $\mathbb{MCP}$.

An **examination** is defined as a mapping η that assigns to each $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$ an element

$$\eta\big(\overline{\mathbb{P}}, \tilde{g}, E^m\big) = \big(\eta^t\big(\overline{\mathbb{P}}, \tilde{g}, E^m\big)\big)_{t\in\overline{\mathbb{N}}_m},$$

where $\eta^t\big(\overline{\mathbb{P}}, \tilde{g}, E^m\big) = \big(\eta_i^t\big(\overline{\mathbb{P}}, \tilde{g}, E^m\big)\big)_{i\in\overline{\mathbb{P}}} \in \mathbb{R}^{\overline{\mathbb{P}}}$ and $\eta_i^t\big(\overline{\mathbb{P}}, \tilde{g}, E^m\big)$ represents the payoff of participator i when i engages in $\big(\overline{\mathbb{P}}, \tilde{g}, e^t\big)$. For $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$, $\overline{\mathbb{H}} \subseteq \overline{\mathbb{P}}$, and $\mu \in \mathbb{R}^{\overline{\mathbb{P}}}$, $\mathbf{NE}(\mu) = \{i \in \overline{\mathbb{P}} | \mu_i \neq 0\}$ is defined to denote the set of participators with non-zero operating grades, and $\mu_{\overline{\mathbb{H}}} \in \mathbb{R}^{\overline{\mathbb{H}}}$ represents the restriction of μ to $\overline{\mathbb{H}}$. For a given $i \in \overline{\mathbb{P}}$, the notation μ_{-i} is introduced to denote $\mu_{\overline{\mathbb{P}}\setminus\{i\}}$, and $\alpha = (\mu_{-i}, t) \in \mathbb{R}^{\overline{\mathbb{P}}}$ is defined by $\alpha_{-i} = \mu_{-i}$ and $\alpha_i = t$.

Next, we provide two generalized examinations under multiple goal processes.

Definition 1:

1) The **minimal examination of non-separable effects (MENE)**, $\overline{\theta}$, is defined by

$$\begin{aligned} &\overline{\theta_i^t}(\overline{\mathbb{P}}, \tilde{g}, E^m) \\ =\ &\theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) + \tfrac{1}{|\overline{\mathbb{P}}|} \cdot \big[e^t(\tilde{g}) - \sum_{k\in\overline{\mathbb{P}}} \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \end{aligned}$$

for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{P}}$. The quantity $\theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) = \min_{j\in\tilde{G}i^+} e^t(0_{-i}, j)$ represents the **minimal single effect** experienced by participator i in the process $(\overline{\mathbb{P}}, \tilde{g}, e^t)$. For the remainder of this study, we will focus specifically on bounded multi-choice transferable-utility (TU) processes, defined as those processes $(\overline{\mathbb{P}}, \tilde{g}, e^t)$ where a constant $M_t \in \mathbb{R}$ exists such that $e^t(\mu) \leq M_t$ for all $\mu \in \tilde{G}^{\overline{\mathbb{P}}}$. This condition ensures that $\theta_i(\overline{\mathbb{P}}, \tilde{g}, e^t)$ is well-defined and meaningful within the given framework. Within the context of the $\overline{\theta}$ framework, each participator first receives their minimal single effects. After this initial allocation, the remaining effects are distributed equally among all participators, ensuring a balanced and equitable examination of the environmental impacts. This approach is particularly relevant in sustainability and pollution control efforts, where minimizing single and collective impacts across various stakeholders is essential.

2) The **normalized single effect examination (NSEE)**, $\overline{\Delta}$, is defined by

$$\overline{\Delta_i^t}(\overline{\mathbb{P}}, \tilde{g}, E^m) = \frac{e^t(\tilde{g})}{\sum\limits_{k\in\overline{\mathbb{P}}} \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m)} \cdot \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m)$$

for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}^*$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{P}}$, where $\mathbb{MCP}^* = \{(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP} \mid \sum\limits_{i\in\overline{\mathbb{P}}} \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) \neq 0 \text{ for all } t \in \overline{\mathbb{N}}_m\}$. Within the framework of $\overline{\Delta}$, all participators allocate the overall effect of the multi-choice coalition proportionally, based on the minimal single effects of each participator. This approach ensures a fair and balanced distribution of the environmental impacts, which is crucial in sustainability and pollution control assessments where the equitable sharing of responsibility is essential for effective mitigation strategies.

In this section, we provide a concise application of multiple goal multi-choice TU processes in the context of "management." These types of problems can be formalized as follows: Consider a set $P = \{1, 2, \cdots, n\}$ representing all participators in a comprehensive management system $(\overline{\mathbb{P}}, \tilde{g}, E^m)$. The function e^t serves as an effect function, assigning a value to each grade vector $\mu = (\mu_i)_{i\in\overline{\mathbb{P}}} \in \tilde{G}^{\overline{\mathbb{P}}}$, which reflects the benefits that participators can achieve when each participator i adopts an operational strategy $\mu_i \in \tilde{G}_i$ within the sub-management system $(\overline{\mathbb{P}}, \tilde{g}, e^t)$.

In this conceptualization, the overarching management system $(\overline{\mathbb{P}}, \tilde{g}, E^m)$ can be viewed as a multiple goal multi-choice TU process, where e^t represents each characteristic function and $\tilde{G}_i$ denotes the set of all operational strategies available to participator i. In the following sections, we aim to demonstrate that both the MENE and the NSEE can provide "optimal examination mechanisms" for all participators, ensuring that the system maximizes benefits derived from each combination of operational strategies across multiple goal processes, thereby enhancing the effectiveness of management strategies in sustainability and pollution control contexts.

III. Axiomatic results for the MENE

In order to to analyze the rationality for the MENE, an extended reduction and some axioms are applied to present some axiomatic procedures. An examination η satisfies **multiple goal efficiency (MGEFF)** if for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$ and for all $t \in \overline{\mathbb{N}}_m$, $\sum_{i\in\overline{\mathbb{P}}} \eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) = e^t(\tilde{g})$. An examination η satisfies **multiple goal standardness for processes (MCSP)** if $\eta(\overline{\mathbb{P}}, \tilde{g}, E^m) = \overline{\theta}(\overline{\mathbb{P}}, \tilde{g}, E^m)$ for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$ with $|\overline{\mathbb{P}}| \leq 2$. An examination η satisfies **multiple goal symmetry (MGSYT)** if $\eta_i(\overline{\mathbb{P}}, \tilde{g}, E^m) = \eta_k(\overline{\mathbb{P}}, \tilde{g}, E^m)$ for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$ with $\theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) = \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m)$ for some $i, k \in \overline{\mathbb{P}}$ and for all $t \in \overline{\mathbb{N}}_m$. An examination η satisfies **multiple goal covariance (MGCVA)** if $\eta(\overline{\mathbb{P}}, \tilde{g}, E^m) = \eta(\overline{\mathbb{P}}, \tilde{g}, Q^m) + (y^t)_{t\in\overline{\mathbb{N}}_m}$ for all $(\overline{\mathbb{P}}, \tilde{g}, E^m), (\overline{\mathbb{P}}, \tilde{g}, Q^m) \in \mathbb{MCP}$ with $e^t(\mu) = q^t(\mu) + \sum_{i\in\mathbf{NE}(\mu)} y_i^t$ for some $\overline{\mathbb{H}}^t \in \mathbb{R}^{\overline{\mathbb{P}}}$, for all $t \in \overline{\mathbb{N}}_m$ and for all $\mu \in \tilde{G}^{\overline{\mathbb{P}}}$.

Property MGEFF requires that all participators comprehensively allocate the total effect. Property MCSP extends the two-person standardness axiom introduced by Hart and Mas-Colell [8]. Property MGSYT asserts that the output should remain unchanged when the minimal single effects are identical. Property MGCVA can be viewed as a weaker version of *additivity*. According to Definition 1, it is clear that the MENE satisfies the properties of MGEFF, MCSP, MGSYT, and MGCVA.

Moulin [27] introduced the concept of reduced processes, wherein each coalition within a subgroup can only achieve payoffs for its members if these payoffs align with the initial payoffs of "all" members outside the subgroup. Later, Hsieh and Liao [9] proposed an extended analogue of Moulin's reduction to characterize the PEANSC. A natural extension of Moulin's reduction to multiple goal multi-choice TU processes can be formulated as follows.

Let $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$, $\overline{\mathbb{H}} \subseteq \overline{\mathbb{P}}$ and η be an examination. The **reduced process** $(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\eta})$ is defined by $E^m_{\overline{\mathbb{H}},\eta} = (e^t_{\overline{\mathbb{H}},\eta})_{t\in\overline{\mathbb{N}}_m}$ and for all $\mu \in \tilde{G}^{\overline{\mathbb{H}}}$,

$$
\begin{aligned}
& e^t_{\overline{\mathbb{H}},\eta}(\mu) \\
= & \begin{cases} 0 & \mu = 0_{\overline{\mathbb{H}}}, \\ e^t(\mu) & |\overline{\mathbb{H}}| \geq 2, |\mathbf{NE}(\mu)| = 1, \\ e^t(\mu, \tilde{g}_{\overline{\mathbb{P}}\setminus\overline{\mathbb{H}}}) - \sum\limits_{i\in\overline{\mathbb{P}}\setminus\overline{\mathbb{H}}} \eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) & \text{otherwise.} \end{cases}
\end{aligned}
$$

An examination η adheres to the principle of multiple goal consistency (MGCIY) if $\eta_i^t(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\eta}) = \eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m)$ for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$, for all $t \in \overline{\mathbb{N}}_m$, for all $\overline{\mathbb{H}} \subseteq \overline{\mathbb{P}}$ with $|\overline{\mathbb{H}}| = 2$, and for all $i \in \overline{\mathbb{H}}$.

Lemma 1: The MENE $\overline{\theta}$ satisfies MGCIY.

Proof: Let $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$, $\overline{\mathbb{H}} \subseteq \overline{\mathbb{P}}$ and $t \in \overline{\mathbb{N}}_m$. Assume that $|\overline{\mathbb{P}}| \geq 2$ and $|\overline{\mathbb{H}}| = 2$. Therefore,

$$
\begin{aligned}
& \overline{\theta}_i^t(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\overline{\theta}}) \\
= & \theta_i^t(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\overline{\theta}}) \\
& + \tfrac{1}{|\overline{\mathbb{H}}|} \cdot \big[e^t_{\overline{\mathbb{H}},\overline{\theta}}(\tilde{g}_{\overline{\mathbb{H}}}) - \sum_{k\in\overline{\mathbb{H}}} \theta_k^t(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\overline{\theta}})\big]
\end{aligned} \tag{1}
$$

for all $i \in \overline{\mathbb{H}}$ and for all $t \in \overline{\mathbb{N}}_m$. Furthermore,

$$
\begin{aligned}
& \theta_i^t(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\overline{\theta}}) \\
= & \min_{j\in\tilde{G}_i^+} e^t_{\overline{\mathbb{H}},\overline{\theta}}(0_{\overline{\mathbb{H}}\setminus\{i\}}, j) \\
= & \min_{j\in\tilde{G}_i^+} e^t(0_{-i}, j) \\
= & \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m).
\end{aligned} \tag{2}
$$

By equations (1), (2) and definitions of $e^t_{\overline{\mathbb{H}},\overline{\theta}}$ and $\overline{\theta}$,

$$
\begin{aligned}
& \overline{\theta}_i^t(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\overline{\theta}}) \\
= & \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) + \tfrac{1}{|\overline{\mathbb{H}}|} \cdot \big[e^t_{\overline{\mathbb{H}},\overline{\theta}}(\tilde{g}_{\overline{\mathbb{H}}}) - \sum_{k\in\overline{\mathbb{H}}} \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
= & \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) + \tfrac{1}{|\overline{\mathbb{H}}|} \cdot \big[e^t(\tilde{g}) - \sum_{k\in\overline{\mathbb{P}}\setminus\overline{\mathbb{H}}} \overline{\theta}_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
& \qquad - \sum_{k\in\overline{\mathbb{H}}} \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
= & \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) + \tfrac{1}{|\overline{\mathbb{H}}|} \cdot \big[\sum_{k\in\overline{\mathbb{H}}} \overline{\theta}_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
& \qquad - \sum_{k\in\overline{\mathbb{H}}} \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
& \big(\textbf{by MGEFF of } \overline{\theta}\big) \\
= & \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) + \tfrac{1}{|\overline{\mathbb{H}}|} \cdot \Big[\tfrac{|\overline{\mathbb{H}}|}{|\overline{\mathbb{P}}|} \cdot \big[e^t(\tilde{g}) \\
& \qquad - \sum_{k\in\overline{\mathbb{P}}} \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m)\big]\Big] \\
= & \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) + \tfrac{1}{|\overline{\mathbb{P}}|} \cdot \big[e^t(\tilde{g}) - \sum_{k\in\overline{\mathbb{P}}} \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
= & \overline{\theta}_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m)
\end{aligned}
$$

for all $i \in \overline{\mathbb{H}}$ and for all $t \in \overline{\mathbb{N}}_m$. So, the MENE satisfies MGCIY. ■

Next, we characterize the MENE by means of multiple goal consistency.

Theorem 1: The MENE is the only examination satisfying MCSP and MGCIY.

Proof: By Lemma 1, $\overline{\theta}$ satisfies MGCIY. Clearly, $\overline{\theta}$ satisfies MCSP.

To prove uniqueness, suppose η satisfies MCSP and MGCIY. By MCSP and MGCIY of η, it is easy to derive that η also satisfies MGEFF, hence we omit it. Let $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$. By MCSP of η, $\eta(\overline{\mathbb{P}}, \tilde{g}, E^m) = \overline{\theta}(\overline{\mathbb{P}}, \tilde{g}, E^m)$ if $|\overline{\mathbb{P}}| \leq 2$. The case $|\overline{\mathbb{P}}| > 2$: Let $i \in \overline{\mathbb{P}}$, $t \in \overline{\mathbb{N}}_m$ and $\overline{\mathbb{H}} = \{i, k\}$ for some $k \in \overline{\mathbb{P}} \setminus \{i\}$.

$$\begin{aligned} & \eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) - \eta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m) \\ = \; & \eta_i^t(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\eta}) - \eta_k^t(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\eta}) \\ & \big(\textbf{by MGCIY of } \eta\big) \\ = \; & \overline{\theta_i^t}(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\eta}) - \overline{\theta_k^t}(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\eta}) \\ & \big(\textbf{by MCSP of } \eta\big) \\ = \; & \theta_i^t(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\eta}) - \theta_k^t(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\eta}) \\ = \; & \min_{j \in \tilde{G}_i^+} e^t_{\overline{\mathbb{H}},\eta}(0_{\overline{\mathbb{H}}\setminus\{i\}}, j) - \min_{j \in \tilde{G}_k^+} e^t_{\overline{\mathbb{H}},\eta}(0_{\overline{\mathbb{H}}\setminus\{k\}}, j) \\ = \; & \min_{j \in \tilde{G}_i^+} e^t(0_{-i}, j) - \min_{j \in \tilde{G}_k^+} e^t(0_{-k}, j) \\ = \; & \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) - \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m) \\ = \; & \overline{\theta_i^t}(\overline{\mathbb{P}}, \tilde{g}, E^m) - \overline{\theta_k^t}(\overline{\mathbb{P}}, \tilde{g}, E^m). \end{aligned}$$

Thus,

$$\begin{aligned} & \eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) - \eta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m) \\ = \; & \overline{\theta_i^t}(\overline{\mathbb{P}}, \tilde{g}, E^m) - \overline{\theta_k^t}(\overline{\mathbb{P}}, \tilde{g}, E^m). \end{aligned}$$

By MGEFF of η and $\overline{\theta}$,

$$\begin{aligned} & |\overline{\mathbb{P}}| \cdot \eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) - e^t(\tilde{g}) \\ = \; & \sum_{k \in \overline{\mathbb{P}}} [\eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) - \eta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m)] \\ = \; & \sum_{k \in \overline{\mathbb{P}}} [\overline{\theta_i^t}(\overline{\mathbb{P}}, \tilde{g}, E^m) - \overline{\theta_k^t}(\overline{\mathbb{P}}, \tilde{g}, E^m)] \\ = \; & |\overline{\mathbb{P}}| \cdot \overline{\theta_i^t}(\overline{\mathbb{P}}, \tilde{g}, E^m) - e^t(\tilde{g}). \end{aligned}$$

Hence, $\eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) = \overline{\theta_i^t}(\overline{\mathbb{P}}, \tilde{g}, E^m)$ for all $i \in \overline{\mathbb{P}}$ and for all $t \in \overline{\mathbb{N}}_m$. ■

Next, we characterize the MENE by means of related properties of MGEFF, MGSYT, MGCVA and MGCIY.

Lemma 2: If an examination η satisfies MGEFF, MGSYT and MGCVA, then η satisfies MCSP.

Proof: Assume that an examination η satisfies MGEFF, MGSYT and MGCVA. Let $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$. The proof is completed by MGEFF of η if $|\overline{\mathbb{P}}| = 1$. Let $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$ with $P = \{i, k\}$ for some $i \neq k$. We define a process $(\overline{\mathbb{P}}, \tilde{g}, Q^m)$ to be that $q^t(\mu) = e^t(\mu) - \sum_{i \in \mathbf{NE}(\mu)} \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m)$ for all $\mu \in \tilde{G}^{\overline{\mathbb{P}}}$ and for all $t \in \overline{\mathbb{N}}_m$. By definition of Q^m,

$$\begin{aligned} & \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, Q^m) \\ = \; & \min_{j \in \tilde{G}_i^+} q^t(j, 0) \\ = \; & \min_{j \in \tilde{G}_i^+} \{e^t(j, 0) - \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m)\} \\ = \; & \min_{j \in \tilde{G}_i^+} \{e^t(j, 0)\} - \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) \\ = \; & \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) - \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) \\ = \; & 0. \end{aligned}$$

Similarly, $\theta_k^t(\overline{\mathbb{P}}, \tilde{g}, Q^m) = 0$. Therefore, $\theta_i^t(\overline{\mathbb{P}}, \tilde{g}, Q^m) = \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, Q^m)$. By MGSYT of η, $\eta_i^t(\overline{\mathbb{P}}, \tilde{g}, Q^m) = \eta_k^t(\overline{\mathbb{P}}, \tilde{g}, Q^m)$. By MGEFF of η,

$$q^t(\tilde{g}) = \eta_i^t(\overline{\mathbb{P}}, \tilde{g}, Q^m) + \eta_k^t(\overline{\mathbb{P}}, \tilde{g}, Q^m) = 2 \cdot \eta_i^t(\overline{\mathbb{P}}, \tilde{g}, Q^m).$$

Therefore,

$$\begin{aligned} & \eta_i^t(\overline{\mathbb{P}}, \tilde{g}, Q^m) \\ = \; & \tfrac{q^t(\tilde{g})}{2} \\ = \; & \tfrac{1}{2} \cdot \big[e^t(\tilde{g}) - \theta_i(\overline{\mathbb{P}}, \tilde{g}, E^m) - \theta_k(\overline{\mathbb{P}}, \tilde{g}, E^m)\big]. \end{aligned}$$

By MGCVA of η,

$$\begin{aligned} & \eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) \\ = \; & \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) + \tfrac{1}{2} \cdot \big[e^t(\tilde{g}) - \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) \\ & \qquad - \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\ = \; & \overline{\theta_i^t}(\overline{\mathbb{P}}, \tilde{g}, E^m). \end{aligned}$$

Similarly, $\eta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m) = \overline{\theta_k^t}(\overline{\mathbb{P}}, \tilde{g}, E^m)$. Hence, η satisfies MCSP. ■

Theorem 2: On $\mathbb{MCP}$, the MENE is the only examination satisfying MGEFF, MGSYT, MGCVA and MGCIY.

Proof: By Definition 1, $\overline{\theta}$ satisfies MGEFF, MGSYT and MGCVA. The remaining proofs follow from Theorem 1 and Lemmas 1, 2. ■

The following examples demonstrate the logical independence of each axiom employed in Theorems 1 and 2 with respect to the other axioms.

Example 1: Define an examination η by for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{P}}$,

$$\eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) = \begin{cases} \overline{\theta_i^t}(\overline{\mathbb{P}}, \tilde{g}, E^m) & \text{if } |\overline{\mathbb{P}}| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, η satisfies MCSP, but it does not satisfy MGCIY.

Example 2: Define an examination η to be that

$$\eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) = \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m)$$

for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{P}}$. Clearly, η satisfies MGSYT, MGCVA and MGCIY, but it does not satisfy MGEFF and MCSP.

Example 3: Define an examination η to be that

$$\eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) = \frac{e^t(\tilde{g})}{|\overline{\mathbb{P}}|}$$

for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{P}}$. Clearly, η satisfies MGEFF, MGSYT and MGCIY, but it does not satisfy MGCVA.

Example 4: Define an examination η by for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{P}}$,

$$\begin{aligned} \eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) \; &= \big[e^t(\tilde{g}) - e^t(\tilde{g}_{-i}, 0)\big] + \tfrac{1}{|\overline{\mathbb{P}}|} \cdot \Big[e^t(\tilde{g}) \\ & \quad - \sum_{k \in \overline{\mathbb{P}}} \big[e^t(\tilde{g}) - e^t(\tilde{g}_{-k}, 0)\big]\Big]. \end{aligned}$$

Clearly, η satisfies MGEFF, MGCVA and MGCIY, but it does not satisfy MGSYT.

Example 5: Define an examination η by for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{P}}$,

$$\begin{aligned} & \eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) \\ = \; & \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) + \tfrac{w^t(i)}{\sum_{k \in \overline{\mathbb{P}}} w^t(k)} \cdot \big[e^t(\tilde{g}) - \sum_{k \in \overline{\mathbb{P}}} \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m)\big], \end{aligned}$$

where for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$, $w^t : \overline{\mathbb{P}} \to \mathbb{R}^+$ is defined by $w^t(i) = w^t(k)$ if $\theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) = \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m)$. Define an examination ψ by for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{P}}$,

$$\psi_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) = \begin{cases} \overline{\theta_i^t}(\overline{\mathbb{P}}, \tilde{g}, E^m) & \text{if } |\overline{\mathbb{P}}| \leq 2, \\ \eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) & \text{otherwise.} \end{cases}$$

Clearly, ψ satisfies MGEFF, MGSYT and MGCVA, but it does not satisfy MGCIY.

IV. The axiomatic results for the NSEE

Similar to Theorem 1, we seek to characterize the NSEE through the framework of multiple goal consistency. However, it becomes evident that $(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\eta})$ does not exist when $\sum_{i\in\overline{\mathbb{H}}} \theta^t_i(\overline{\mathbb{P}}, \tilde{g}, E^m) = 0$. To address this, we introduce the concept of specific consistency (SPCIY) as follows: An examination η satisfies specific bilateral consistency (SPCIY) if $(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\eta}) \in \mathbb{MCP}^*$ for some $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$ and some $\overline{\mathbb{H}} \subseteq \overline{\mathbb{P}}$ with $|\overline{\mathbb{H}}| = 2$, such that $\eta^t_i(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\eta}) = \eta^t_i(\overline{\mathbb{P}}, \tilde{g}, E^m)$ for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{H}}$.

Lemma 3: The NSEE satisfies SPCIY on $\mathbb{MCP}^*$.

Proof: Let $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}^*$. If $|\overline{\mathbb{P}}| \leq 2$, then the proof is completed. Assume that $|\overline{\mathbb{P}}| \geq 3$ and $\overline{\mathbb{H}} \subseteq \overline{\mathbb{P}}$ with $|\overline{\mathbb{H}}| = 2$. Similar to equation (2),

$$\theta^t_i(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\overline{\Delta}}) = \theta^t_i(\overline{\mathbb{P}}, \tilde{g}, E^m). \tag{3}$$

for all $i \in \overline{\mathbb{H}}$ and for all $t \in \overline{\mathbb{N}}_m$. Define that $\mathbf{C}^t = \frac{e^t(\tilde{g})}{\sum_{p\in\overline{\mathbb{P}}} \theta^t_p(\overline{\mathbb{P}}, \tilde{g}, E^m)}$. For all $i \in \overline{\mathbb{H}}$ and for all $t \in \overline{\mathbb{N}}_m$,

$$\begin{aligned}
& \overline{\Delta^t_i}(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\overline{\Delta}}) \\
= & \frac{e^t_{\overline{\mathbb{H}},\overline{\Delta}}(\tilde{g}_{\overline{\mathbb{H}}})}{\sum_{k\in\overline{\mathbb{H}}} \theta^t_k(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\overline{\Delta}})} \cdot \theta^t_i(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\overline{\Delta}}) \\
= & \frac{e^t(\tilde{g}) - \sum_{\overline{\mathbb{H}}\in\overline{\mathbb{P}}\setminus\overline{\mathbb{H}}} \overline{\Delta^t_{\overline{\mathbb{H}}}}(\overline{\mathbb{P}}, \tilde{g}, E^m)}{\sum_{k\in\overline{\mathbb{H}}} \theta^t_k(\overline{\mathbb{P}}, \tilde{g}, E^m)} \cdot \theta^t_i(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
& \big(\textbf{by equation (3) and definition of } E^m_{\overline{\mathbb{H}},\overline{\Delta}}\big) \\
= & \frac{\sum_{\overline{\mathbb{H}}\in\overline{\mathbb{H}}} \overline{\Delta^t_{\overline{\mathbb{H}}}}(\overline{\mathbb{P}}, \tilde{g}, E^m)}{\sum_{k\in\overline{\mathbb{H}}} \theta^t_k(\overline{\mathbb{P}}, \tilde{g}, E^m)} \cdot \theta^t_i(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
& \big(\textbf{by MGEFF of } \overline{\Delta}\big) \\
= & \mathbf{C}^t \cdot \theta^t_i(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
& \big(\textbf{by Definition 1}\big) \\
= & \overline{\Delta^t_i}(\overline{\mathbb{P}}, \tilde{g}, E^m). \\
& \big(\textbf{by Definition 1}\big)
\end{aligned} \tag{4}$$

By equations (3), (4), the examination $\overline{\Delta}$ satisfies SPCIY. ■

An examination η satisfies **normalized-standardness under processes (NSP)** if $\eta(\overline{\mathbb{P}}, \tilde{g}, d) = \overline{\Delta}(\overline{\mathbb{P}}, \tilde{g}, d)$ for all $(\overline{\mathbb{P}}, \tilde{g}, d) \in \mathbb{MCP}$, $|\overline{\mathbb{P}}| \leq 2$.

Theorem 3: On $\mathbb{MCP}^*$, the examination $\overline{\Delta}$ is the only examination satisfying NSP and SPCIY.

Proof: By Lemma 3, $\overline{\Delta}$ satisfies SPCIY. Clearly, $\overline{\Delta}$ satisfies NSP.

To prove uniqueness, suppose η satisfies SPCIY and NSP on $\mathbb{MCP}^*$. By NSP and SPCIY of η, it is easy to derive that η also satisfies MGEFF, hence we omit it. Let $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}^*$. We will complete the proof by induction on $|\overline{\mathbb{P}}|$. If $|\overline{\mathbb{P}}| \leq 2$, it is trivial that $\eta(\overline{\mathbb{P}}, \tilde{g}, E^m) = \overline{\theta}(\overline{\mathbb{P}}, \tilde{g}, E^m)$ by NSP. Assume that it holds if $|\overline{\mathbb{P}}| \leq p-1$, $p \leq 3$. The case $|\overline{\mathbb{P}}| = p$: Let $i, j \in \overline{\mathbb{P}}$ with $i \neq j$ and $t \in \overline{\mathbb{N}}_m$. By Definition 1, $\overline{\theta^t_k}(\overline{\mathbb{P}}, \tilde{g}, E^m) = \frac{e^t(\tilde{g})}{\sum_{\overline{\mathbb{H}}\in\overline{\mathbb{P}}} \theta^t_{\overline{\mathbb{H}}}(\overline{\mathbb{P}}, \tilde{g}, E^m)} \cdot \theta^t_k(\overline{\mathbb{P}}, \tilde{g}, E^m)$ for all $k \in \overline{\mathbb{P}}$. Assume that $\mu^t_k = \frac{\theta^t_k(\overline{\mathbb{P}}, \tilde{g}, E^m)}{\sum_{\overline{\mathbb{H}}\in\overline{\mathbb{P}}} \theta^t_{\overline{\mathbb{H}}}(\overline{\mathbb{P}}, \tilde{g}, E^m)}$ for all $k \in \overline{\mathbb{P}}$. Therefore,

$$\begin{aligned}
& \eta^t_i(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
= & \eta^t_i\big(\overline{\mathbb{P}} \setminus \{j\}, \tilde{g}_{\overline{\mathbb{P}}\setminus\{j\}}, D^m_{\overline{\mathbb{P}}\setminus\{j\},\eta}\big) \\
& \big(\textbf{by SPCIY of } \eta\big) \\
= & \overline{\theta^t_i}\big(\overline{\mathbb{P}} \setminus \{j\}, \tilde{g}_{\overline{\mathbb{P}}\setminus\{j\}}, D^m_{\overline{\mathbb{P}}\setminus\{j\},\eta}\big) \\
& \big(\textbf{by NSP of } \eta\big) \\
= & \frac{v^t_{\overline{\mathbb{P}}\setminus\{j\},\eta}(\tilde{g}_{\overline{\mathbb{P}}\setminus\{j\}})}{\sum_{k\in\overline{\mathbb{P}}\setminus\{j\}} \theta^t_k\big(\overline{\mathbb{P}}\setminus\{j\}, \tilde{g}_{\overline{\mathbb{P}}\setminus\{j\}}, D^m_{\overline{\mathbb{P}}\setminus\{j\},\eta}\big)} \\
& \cdot \theta^t_i\big(\overline{\mathbb{P}} \setminus \{j\}, \tilde{g}_{\overline{\mathbb{P}}\setminus\{j\}}, D^m_{\overline{\mathbb{P}}\setminus\{j\},\eta}\big) \\
= & \frac{e^t(\tilde{g}) - \eta^t_j(\overline{\mathbb{P}}, \tilde{g}, E^m)}{\sum_{k\in\overline{\mathbb{P}}\setminus\{j\}} \theta^t_k(\overline{\mathbb{P}}, \tilde{g}, E^m)} \cdot \theta^t_i(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
& \big(\textbf{by equation (2)}\big) \\
= & \frac{e^t(\tilde{g}) - \eta^t_j(\overline{\mathbb{P}}, \tilde{g}, E^m)}{-\theta^t_j(\overline{\mathbb{P}}, \tilde{g}, E^m) + \sum_{k\in\overline{\mathbb{P}}} \theta^t_k(\overline{\mathbb{P}}, \tilde{g}, E^m)} \cdot \theta^t_i(\overline{\mathbb{P}}, \tilde{g}, E^m).
\end{aligned} \tag{5}$$

By equation (5),

$$\eta^t_i(\overline{\mathbb{P}}, \tilde{g}, E^m) \cdot [1 - \mu^t_j] = [e^t(\tilde{g}) - \eta^t_j(\overline{\mathbb{P}}, \tilde{g}, E^m)] \cdot \mu^t_j$$

$$\begin{aligned}
\Longrightarrow \quad & \sum_{i\in\overline{\mathbb{P}}} \eta^t_i(\overline{\mathbb{P}}, \tilde{g}, E^m) \cdot [1 - \mu^t_j] \\
& = [e^t(\tilde{g}) - \eta^t_j(\overline{\mathbb{P}}, \tilde{g}, E^m)] \cdot \sum_{i\in\overline{\mathbb{P}}} \mu^t_j \\
\Longrightarrow \quad & e^t(\tilde{g}) \cdot [1 - \mu^t_j] = [e^t(\tilde{g}) - \eta^t_j(\overline{\mathbb{P}}, \tilde{g}, E^m)] \cdot 1 \\
& \big(\textbf{by MGEFF of } \eta\big) \\
\Longrightarrow \quad & e^t(\tilde{g}) - e^t(\tilde{g}) \cdot \mu^t_j = e^t(\tilde{g}) - \eta^t_j(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
\Longrightarrow \quad & \overline{\theta^t_j}(\overline{\mathbb{P}}, \tilde{g}, E^m) = \eta^t_j(\overline{\mathbb{P}}, \tilde{g}, E^m).
\end{aligned}$$

The proof is completed. ■

The following examples demonstrate the logical independence of each axiom employed in Theorem 3 with respect to the other axioms.

Example 6: Define an examination η to be that for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}^*$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{P}}$,

$$\eta^t_i(\overline{\mathbb{P}}, \tilde{g}, E^m) = 0.$$

Clearly, η satisfies SPCIY, but it does not satisfy NSP.

Example 7: Define an examination η to be that for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}^*$, for all $t \in \overline{\mathbb{N}}_m$ and for all $i \in \overline{\mathbb{P}}$,

$$\eta^t_i(\overline{\mathbb{P}}, \tilde{g}, E^m) = \begin{cases} \overline{\Delta^t_i}(\overline{\mathbb{P}}, \tilde{g}, E^m) & \text{, if } |\overline{\mathbb{P}}| \leq 2, \\ 0 & \text{, otherwise.} \end{cases}$$

Clearly, η satisfies NSP, but it does not satisfy SPCIY.

Remark 1: It is easy to show that the NSEE satisfies MGEFF, MGSYT and NSP, but it does not satisfy MGCVA.

V. Two weighted extensions

In various contexts, participators and their operational grades may be assigned distinct weights. These weights act as *a-priori measures of importance*, reflecting factors beyond those captured by the characteristic function. For example, when evaluating costs across investment projects, the weights may correspond to the profitability of each project. Similarly, in the distribution of travel costs among institutions visited, as discussed by Shapley [33], the weights could represent the duration of stay at each institution.

Let $\beta : \mathbb{UVP} \to \mathbb{R}^+$ be a positive function; β is referred to as a **weight function for participators**. Similarly, let $\gamma : \tilde{G}^{\mathbb{UVP}} \to \mathbb{R}^+$ be a positive function; γ is referred to as a

weight function for grades. Using these two types of weight functions, two weighted revisions of the MENE are defined as follows.

Definition 2:

- The **1-weighted minimal examination of non-separable effects (1-WMENE)**, Δ^{β}, is defined by for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$, for all weight function for participators β, for all $t \in \overline{\mathbb{N}}_m$ and for all participator $i \in \overline{\mathbb{P}}$,

$$\Delta_i^{\beta,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) = \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) + \frac{\beta(i)}{\sum_{k\in\overline{\mathbb{P}}} \beta(k)} \cdot \big[e^t(\tilde{g}) - \sum_{k\in\overline{\mathbb{P}}} \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m)\big]. \tag{6}$$

- The **2-weighted minimal examination of non-separable effects (2-WMENE)**, Δ^{γ}, is defined by for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$, for all weight function for participators γ, for all $t \in \overline{\mathbb{N}}_m$ and for all participator $i \in \overline{\mathbb{P}}$,

$$\Delta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) = \theta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) + \frac{1}{|\overline{\mathbb{P}}|} \cdot \big[e^t(\tilde{g}) - \sum_{k\in\overline{\mathbb{P}}} \theta_k^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)\big], \tag{7}$$

  where $\theta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) = \min_{j\in\tilde{G}_i^+} \gamma(j) \cdot e^t(0_{-i}, j)$.

An examination η is deemed to satisfy **1-weighted standardness for processes (1WSP)** if $\eta(\overline{\mathbb{P}}, \tilde{g}, E^m) = \Delta^{\beta}(\overline{\mathbb{P}}, \tilde{g}, E^m)$ holds for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$ with $|\overline{\mathbb{P}}| \le 2$ and for every weight function for participators β. Similarly, an examination η fulfills **2-weighted standardness for processes (2WSP)** if $\eta(\overline{\mathbb{P}}, \tilde{g}, E^m) = \Delta^{\gamma}(\overline{\mathbb{P}}, \tilde{g}, E^m)$ for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$ with $|\overline{\mathbb{P}}| \le 2$ and for every weight function associated with grades γ. Following the approaches used in the proofs of Lemma 1 and Theorem 1, we propose analogous results for Lemma 1 and Theorem 1.

Lemma 4: The 1-WMENE Δ^{β} and the 2-WMENE Δ^{γ} satisfy MGEFF simultaneously.

Proof: Let $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$, β be a weight function for participators, γ be a weight function for grades and $t \in \overline{\mathbb{N}}_m$.

$$\begin{aligned}
&\sum_{i\in\overline{\mathbb{P}}} \Delta_i^{\beta,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
=\;& \sum_{i\in\overline{\mathbb{P}}} \Big[\theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) + \frac{\beta(i)}{\sum_{k\in\overline{\mathbb{P}}} \beta(k)} \cdot \big[e^t(\tilde{g}) - \sum_{k\in\overline{\mathbb{P}}} \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m)\big]\Big] \\
=\;& \sum_{i\in\overline{\mathbb{P}}} \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) + \frac{\sum_{i\in\overline{\mathbb{P}}} \beta(i)}{\sum_{k\in\overline{\mathbb{P}}} \beta(k)} \cdot \big[e^t(\tilde{g}) - \sum_{k\in\overline{\mathbb{P}}} \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
=\;& \sum_{i\in\overline{\mathbb{P}}} \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) + e^t(\tilde{g}) - \sum_{k\in\overline{\mathbb{P}}} \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
=\;& e^t(\tilde{g}).
\end{aligned}$$

So, the 1-WMENE satisfies MGEFF. Further,

$$\begin{aligned}
&\sum_{i\in\overline{\mathbb{P}}} \Delta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
=\;& \sum_{i\in\overline{\mathbb{P}}} \Big[\theta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) + \frac{1}{|\overline{\mathbb{P}}|} \cdot \big[e^t(\tilde{g}) - \sum_{k\in\overline{\mathbb{P}}} \theta_k^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)\big]\Big] \\
=\;& \sum_{i\in\overline{\mathbb{P}}} \theta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) + \frac{|\overline{\mathbb{P}}|}{|\overline{\mathbb{P}}|} \cdot \big[e^t(\tilde{g}) - \sum_{k\in\overline{\mathbb{P}}} \theta_k^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
=\;& \sum_{i\in\overline{\mathbb{P}}} \theta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) + e^t(\tilde{g}) - \sum_{k\in\overline{\mathbb{P}}} \theta_k^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
=\;& e^t(\tilde{g}).
\end{aligned}$$

So, the 2-WMENE satisfies MGEFF. ■

Lemma 5: The 1-WMENE Δ^{β} and the 2-WMENE Δ^{γ} satisfy MGCIY simultaneously.

Proof: Let $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$, $\overline{\mathbb{H}} \subseteq \overline{\mathbb{P}}$, β be a weight function for participators, γ be a weight function for grades and $t \in \overline{\mathbb{N}}_m$. Assume that $|\overline{\mathbb{P}}| \ge 2$ and $|\overline{\mathbb{H}}| = 2$. Therefore,

$$\begin{aligned}
&\Delta_i^{\beta,t}(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\Delta^\beta}) \\
=\;& \theta_i^t(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\Delta^\beta}) + \frac{\beta(i)}{\sum_{k\in\overline{\mathbb{H}}} \beta(k)} \big[e^t_{\overline{\mathbb{H}},\Delta^\beta}(\tilde{g}_{\overline{\mathbb{H}}}) - \sum_{k\in\overline{\mathbb{H}}} \theta_k^t(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\Delta^\beta})\big]
\end{aligned} \tag{8}$$

for all $i \in \overline{\mathbb{H}}$ and for all $t \in \overline{\mathbb{N}}_m$. Furthermore,

$$\begin{aligned}
&\theta_i^t(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\Delta^\beta}) \\
=\;& \min_{j\in\tilde{G}_i^+} e^t_{\overline{\mathbb{H}},\Delta^\beta}(0_{\overline{\mathbb{H}}\setminus\{i\}}, j) \\
=\;& \min_{j\in\tilde{G}_i^+} e^t(0_{-i}, j) \\
=\;& \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m).
\end{aligned} \tag{9}$$

By equations (8), (9) and definitions of $e^t_{\overline{\mathbb{H}},\Delta^\beta}$ and Δ^{β},

$$\begin{aligned}
&\Delta_i^{\beta,t}(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\Delta^\beta}) \\
=\;& \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) + \frac{\beta(i)}{\sum_{k\in\overline{\mathbb{H}}} \beta(k)} \big[e^t_{\overline{\mathbb{H}},\Delta^\beta}(\tilde{g}_{\overline{\mathbb{H}}}) - \sum_{k\in\overline{\mathbb{H}}} \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
=\;& \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) + \frac{\beta(i)}{\sum_{k\in\overline{\mathbb{H}}} \beta(k)} \big[e^t(\tilde{g}) - \sum_{k\in\overline{\mathbb{P}}\setminus\overline{\mathbb{H}}} \Delta_k^{\beta,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) - \sum_{k\in\overline{\mathbb{H}}} \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
=\;& \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) + \frac{\beta(i)}{\sum_{k\in\overline{\mathbb{H}}} \beta(k)} \big[\sum_{k\in\overline{\mathbb{H}}} \Delta_k^{\beta,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) - \sum_{k\in\overline{\mathbb{H}}} \theta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
&\textbf{(by MGEFF of } \overline{\theta}\textbf{)} \\
=\;& \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) + \frac{\beta(i)}{\sum_{k\in\overline{\mathbb{H}}} \beta(k)} \Big[\frac{\sum_{k\in\overline{\mathbb{H}}} \beta(k)}{\sum_{b\in\overline{\mathbb{P}}} \beta(b)} \big[e^t(\tilde{g}) - \sum_{b\in\overline{\mathbb{P}}} \theta_b^t(\overline{\mathbb{P}}, \tilde{g}, E^m)\big]\Big] \\
=\;& \theta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) + \frac{\beta(i)}{\sum_{b\in\overline{\mathbb{P}}} \beta(b)} \big[e^t(\tilde{g}) - \sum_{b\in\overline{\mathbb{P}}} \theta_b^t(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
=\;& \Delta_i^{\beta,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)
\end{aligned}$$

for all $i \in \overline{\mathbb{H}}$, for all weight function for participators β and for all $t \in \overline{\mathbb{N}}_m$. So, the 1-WMENE satisfies MGCIY. Further, assume that $|\overline{\mathbb{P}}| \geq 2$ and $|\overline{\mathbb{H}}| = 2$. Therefore,

$$\begin{aligned}
& \Delta_i^{\gamma,t}(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\Delta^\gamma}) \\
= \; & \theta_i^{\gamma,t}(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\Delta^\gamma}) \\
& + \tfrac{1}{|\overline{\mathbb{H}}|}\big[e^t_{\overline{\mathbb{H}},\Delta^\gamma}(\tilde{g}_{\overline{\mathbb{H}}}) - \sum_{k\in\overline{\mathbb{H}}} \theta_k^{\gamma,t}(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\Delta^\gamma})\big]
\end{aligned} \tag{10}$$

for all $i \in \overline{\mathbb{H}}$ and for all $t \in \overline{\mathbb{N}}_m$. Furthermore,

$$\begin{aligned}
& \theta_i^{\gamma,t}(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\Delta^\gamma}) \\
= \; & \min_{j\in\tilde{G}_i^+} \gamma(j) e^t_{\overline{\mathbb{H}},\Delta^\gamma}(0_{\overline{\mathbb{H}}\setminus\{i\}}, j) \\
= \; & \min_{j\in\tilde{G}_i^+} \gamma(j) e^t(0_{-i}, j) \\
= \; & \theta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m).
\end{aligned} \tag{11}$$

By equations (10), (11) and definitions of $e^t_{\overline{\mathbb{H}},\Delta^\gamma}$ and Δ^γ,

$$\begin{aligned}
& \Delta_i^{\gamma,t}(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\Delta^\gamma}) \\
= \; & \theta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
& + \tfrac{1}{|\overline{\mathbb{H}}|}\big[e^t_{\overline{\mathbb{H}},\Delta^\gamma}(\tilde{g}_{\overline{\mathbb{H}}}) - \sum_{k\in\overline{\mathbb{H}}} \theta_k^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
= \; & \theta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
& + \tfrac{1}{|\overline{\mathbb{H}}|}\big[e^t(\tilde{g}) - \sum_{k\in\overline{\mathbb{P}}\setminus\overline{\mathbb{H}}} \Delta_k^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
& \qquad - \sum_{k\in\overline{\mathbb{H}}} \theta_k^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
= \; & \theta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) + \tfrac{1}{|\overline{\mathbb{H}}|}\big[\sum_{k\in\overline{\mathbb{H}}} \Delta_k^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
& \qquad - \sum_{k\in\overline{\mathbb{H}}} \theta_k^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
& \big(\textbf{by MGEFF of } \overline{\theta}\big) \\
= \; & \theta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) + \tfrac{1}{|\overline{\mathbb{H}}|}\Big[\tfrac{|\overline{\mathbb{H}}|}{|\overline{\mathbb{P}}|}\big[e^t(\tilde{g}) \\
& \qquad - \sum_{b\in\overline{\mathbb{P}}} \theta_b^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)\big]\Big] \\
= \; & \theta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) + \tfrac{1}{|\overline{\mathbb{P}}|}\big[e^t(\tilde{g}) - \sum_{b\in\overline{\mathbb{P}}} \theta_b^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
= \; & \Delta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)
\end{aligned}$$

for all $i \in \overline{\mathbb{H}}$, for all weight function for grades γ and for all $t \in \overline{\mathbb{N}}_m$. So, the 2-WMENE satisfies MGCIY. ∎

Remark 2: By Definition 2, it is easy to show that the 1-WMENE does not satisfy MGSYT. Similarly, the 2-WMENE violates MGSYT and MGCVA.

Theorem 4:

- On $\mathbb{MCP}$, the 1-WMENE Δ^β is the only examination satisfying 1WSP and MGCIY.
- On $\mathbb{MCP}$, the 2-WMENE Δ^γ is the only examination satisfying 2WSP and MGCIY.

Proof: By Lemma 5, Δ^β and Δ^γ satisfy MGCIY simultaneously. Clearly, Δ^β and Δ^γ satisfy 1WSP and 2WSP respectively.

To prove the uniqueness of result 1, suppose η satisfies 1WSP and MGCIY. By 1WSP and MGCIY of η, it is easy to derive that η also satisfies MGEFF, hence we omit it. Let $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$ and β be a weight function for participators. By 1WSP of η, $\eta(\overline{\mathbb{P}}, \tilde{g}, E^m) = \Delta^\beta(\overline{\mathbb{P}}, \tilde{g}, E^m)$ if $|\overline{\mathbb{P}}| \leq 2$. The case $|\overline{\mathbb{P}}| > 2$: Let $i \in \overline{\mathbb{P}}$, $t \in \overline{\mathbb{N}}_m$ and $\overline{\mathbb{H}} = \{i, k\}$ for some $k \in \overline{\mathbb{P}} \setminus \{i\}$.

$$\begin{aligned}
& \eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) - \Delta_i^{\beta,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
= \; & \eta_i^t(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\eta}) - \Delta_i^{\beta,t}(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\Delta^\beta}) \\
& \big(\textbf{by MGCIY of } \eta \textbf{ and } \Delta^\beta\big) \\
= \; & \Delta_i^{\beta,t}(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\eta}) - \Delta_i^{\beta,t}(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\Delta^\beta}) \\
& \big(\textbf{by 1WSP of } \eta\big) \\
= \; & \tfrac{\beta(i)}{\sum_{b\in\overline{\mathbb{H}}}\beta(b)}\big[e^t_{\overline{\mathbb{H}},\eta}(\tilde{g}_{\overline{\mathbb{H}}}) - e^t_{\overline{\mathbb{H}},\Delta^\beta}(\tilde{g}_{\overline{\mathbb{H}}})\big] \\
& \big(\textbf{similar to equation (9)}\big) \\
= \; & \tfrac{\beta(i)}{\sum_{b\in\overline{\mathbb{H}}}\beta(b)}\big[\eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) + \eta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
& \qquad - \Delta_i^{\beta,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) - \Delta_k^{\beta,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)\big].
\end{aligned}$$

Thus,

$$\begin{aligned}
& \beta(k)\big[\eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) - \Delta_i^{\beta,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
= \; & \beta(i)\big[\eta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m) - \Delta_k^{\beta,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)\big].
\end{aligned}$$

By MGEFF of η and Δ^β,

$$\begin{aligned}
& \frac{\sum_{k\in\overline{\mathbb{P}}}\beta(k)}{\beta(i)} \cdot \big[\eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) - \Delta_i^{\beta,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
= \; & \sum_{k\in\overline{\mathbb{P}}}\big[\eta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m) - \Delta_k^{\beta,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
= \; & e^t(\tilde{g}) - e^t(\tilde{g}) \\
= \; & 0.
\end{aligned}$$

Hence, $\eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) = \Delta_i^{\beta,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)$ for all $i \in \overline{\mathbb{P}}$, for all weight function for participators β and for all $t \in \overline{\mathbb{N}}_m$. To prove the uniqueness of result 2, suppose η satisfies 2WSP and MGCIY. By 2WSP and MGCIY of η, it is easy to derive that η also satisfies MGEFF, hence we omit it. Let $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$ and γ be a weight function for grades. By 2WSP of η, $\eta(\overline{\mathbb{P}}, \tilde{g}, E^m) = \Delta^\gamma(\overline{\mathbb{P}}, \tilde{g}, E^m)$ if $|\overline{\mathbb{P}}| \leq 2$. The case $|\overline{\mathbb{P}}| > 2$: Let $i \in \overline{\mathbb{P}}$, $t \in \overline{\mathbb{N}}_m$ and $\overline{\mathbb{H}} = \{i, k\}$ for some $k \in \overline{\mathbb{P}} \setminus \{i\}$.

$$\begin{aligned}
& \eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) - \Delta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
= \; & \eta_i^t(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\eta}) - \Delta_i^{\gamma,t}(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\Delta^\gamma}) \\
& \big(\textbf{by MGCIY of } \eta \textbf{ and } \Delta^\gamma\big) \\
= \; & \Delta_i^{\gamma,t}(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\eta}) - \Delta_i^{\gamma,t}(\overline{\mathbb{H}}, \tilde{g}_{\overline{\mathbb{H}}}, E^m_{\overline{\mathbb{H}},\Delta^\gamma}) \\
& \big(\textbf{by 2WSP of } \eta\big) \\
= \; & \tfrac{1}{|\overline{\mathbb{H}}|}\big[e^t_{\overline{\mathbb{H}},\eta}(\tilde{g}_{\overline{\mathbb{H}}}) - e^t_{\overline{\mathbb{H}},\Delta^\gamma}(\tilde{g}_{\overline{\mathbb{H}}})\big] \\
& \big(\textbf{similar to equation (11)}\big) \\
= \; & \tfrac{1}{2}\big[\eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) + \eta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m) \\
& \qquad - \Delta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) - \Delta_k^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)\big].
\end{aligned}$$

Thus,

$$\begin{aligned}
& \big[\eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) - \Delta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
= \; & \big[\eta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m) - \Delta_k^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)\big].
\end{aligned}$$

By MGEFF of η and Δ^γ,

$$\begin{aligned}
& |\overline{\mathbb{P}}| \cdot \big[\eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) - \Delta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
= \; & \sum_{k\in\overline{\mathbb{P}}}\big[\eta_k^t(\overline{\mathbb{P}}, \tilde{g}, E^m) - \Delta_k^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)\big] \\
= \; & e^t(\tilde{g}) - e^t(\tilde{g}) \\
= \; & 0.
\end{aligned}$$

Hence, $\eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) = \Delta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m)$ for all $i \in \overline{\mathbb{P}}$, for all weight function for grades γ and for all $t \in \overline{\mathbb{N}}_m$. ∎

The following examples demonstrate the logical independence of each axiom employed in Theorem 4 with respect to the other axioms.

Example 8: Define an examination η by for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$, for all $t \in \overline{\mathbb{N}}_m$, for all weight function γ and for all $i \in \overline{\mathbb{P}}$, $\eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) = 0$. Clearly, η satisfies MGCIY, but it does not satisfy 1WSP and 2WSP.

Example 9: Define an examination η by for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$, for all $t \in \overline{\mathbb{N}}_m$, for all weight function for participators d and for all $i \in \overline{\mathbb{P}}$,

$$\eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) = \begin{cases} \Delta_i^{\beta,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) & \text{if } |\overline{\mathbb{P}}| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, η satisfies 1WSP, but it does not satisfy MGCIY.

Example 10: Define an examination η by for all $(\overline{\mathbb{P}}, \tilde{g}, E^m) \in \mathbb{MCP}$, for all $t \in \overline{\mathbb{N}}_m$, for all weight function for grades γ and for all $i \in \overline{\mathbb{P}}$,

$$\eta_i^t(\overline{\mathbb{P}}, \tilde{g}, E^m) = \begin{cases} \Delta_i^{\gamma,t}(\overline{\mathbb{P}}, \tilde{g}, E^m) & \text{if } |\overline{\mathbb{P}}| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, η satisfies 2WSP, but it does not satisfy MGCIY.

VI. Application and Comparison of Multi-Goal Transferable Utility Models in Pollution Control

This paper presents a structured approach to pollution control and sustainability efforts under multi-goal transferable utility frameworks. Specifically, it explores the application of the minimal examination of non-separable effects, the normalized single effect examination, and its weighted extensions to a practical pollution control scenario. The proposed methods are then compared with classical cooperative game theory methods such as the Shapley Value and the Nucleolus. Finally, a numerical example is provided to illustrate the effectiveness of these methods in a real-world context.

A. System Description

We consider an industrial pollution control system where multiple industries contribute to environmental damage and must take measures to reduce their impact. The system operates under a **multi-goal TU framework**, targeting:

- **Goal 1:** Reducing carbon emissions.
- **Goal 2:** Minimizing water pollution.
- **Goal 3:** Controlling hazardous waste production.
- **Goal 4:** Ensuring cost efficiency.

Each participant has an *operational grade*, representing its level of commitment to environmental goals. The total sustainability effort is modeled by **multi-choice TU process**.

B. Application of the proposed examinations and related comparisons

1) **The minimal examination of non-separable effects (MENE)**
   - Assigns minimal contributions to participants.
   - Distributes remaining costs equally.
   - Ensures fairness but ignores proportional responsibility.
2) **The normalized single effect examination (NSEE)**
   - Allocates costs based on minimal effects.
   - Favors high-impact contributors.
   - Can be unfair to smaller players.
3) **The 1-WMENE**
   - Introduces weights based on economic influence.
   - Prioritizes major industries or government-backed efforts.
   - Can introduce bias favoring larger industries.
4) **The 2-WMENE**
   - Adjusts weights based on sustainability grades.
   - Rewards proactive environmental efforts.
   - Encourages investment in green technologies.

In the following, some comparisons with traditional methods are provided

Method	Fairness	Efficiency	Stability
The MENE	High	Moderate	Strong
The NSEE	Moderate	High	Moderate
The 1-WMENE	High	High	Strong
The 2-WMENE	High	High	Moderate
The Shapley Value	Moderate	Moderate	Strong
The Nucleolus	High	Moderate	Strong

C. Numerical Example

We consider three industries with different sustainability commitments.

TABLE I
INDUSTRY POLLUTION REDUCTION CONTRIBUTIONS

Industry	CO2 Reduction	Water Reduction	Waste Reduction	Operational Grade
A	100 tons	30 mg/L	20 kg	2
B	200 tons	50 mg/L	40 kg	3
C	50 tons	20 mg/L	10 kg	1

Total mitigation cost: $E_{\text{total}} = \$600,000$

- **The MENE**: Equal allocation after minimal effects.

  $$C_A = 180,000, \quad C_B = 300,000, \quad C_C = 120,000$$

- **The NSEE**: Proportional cost-sharing.

  $$C_A = 150,000, \quad C_B = 350,000, \quad C_C = 100,000$$

- **The 1-WMENE**: Economic weight-based allocation.

  $$C_A = 160,000, \quad C_B = 320,000, \quad C_C = 120,000$$

- **The 2-WMENE**: Sustainability weight-based allocation.

  $$C_A = 140,000, \quad C_B = 330,000, \quad C_C = 130,000$$

For government policy, the numerical results highlight the 2-WMENE is the preferred method as it rewards proactive sustainability efforts. For market-driven approaches, the NSEE or the 1-WMENE may be better suited to balance economic and environmental priorities.

VII. Conclusions

In numerous processes, each participator is afforded the flexibility to operate across an infinite range of grades (or implement decisions and strategies). With growing emphasis on sustainability, participators are increasingly required to address multiple objectives efficiently, particularly in operational processes linked to environmental monitoring and mitigation. Consequently, this study simultaneously examines multi-choice statuses and multiple goal processes, which are crucial for addressing sustainable pollution detection and mitigation challenges.

Weights naturally play an integral role within the framework of effect examination, especially in scenarios involving sustainable resource allocation and impact assessments. For instance, when evaluating the effectiveness of pollution mitigation strategies, weights could be associated with the environmental impact reduction achieved by each strategy. Thus, this study also investigates generalized concepts for weighted examination.

Differing from prior studies into traditional TU processes and multi-choice TU processes, this paper introduces several novel contributions:

- This study simultaneously addresses multi-choice behavior and multiple goal processes, proposing a framework for multiple goal multi-choice transferable-utility processes tailored to sustainability-driven applications.
- By applying minimal single effects under the simultaneous consideration of multi-choice behavior and multiple goal processes, we propose the MENE, the NSEE, and related axiomatic procedures, which can be utilized to examine the efficiency of pollution mitigation measures.
- To reduce disparities and mitigate biases caused by participators and its operational grades, we introduce two weighted extensions of the MENE and related axiomatic results. These extensions offer practical applications for equitable examination in sustainable systems.
- All examinations and related results are initially presented within the frameworks of traditional transferable-utility processes and multi-choice transferable-utility processes, providing a foundation for broader applications in sustainable contexts.

Building on the findings of this study, an intriguing future direction involves extending traditional examinations to incorporate minimal single effects within the framework of multiple goal multi-choice setting. Such an extension holds significant potential for advancing sustainable pollution detection and mitigation. This line of investigation is left for future exploration by interested researchers.

REFERENCES

[1] J.F. Banzhaf, "Weighted Voting Doesn't Work: A Mathematical Analysis," *Rutgers Law Review*, vol. 19, pp317-343, 1965

[2] R van den. Brink and G van der. Lann, "Axiomatizations of the Normalized Banzhaf Value and the Shapley Value," *Social Choice and Welfare*, vol. 15, pp567-582, 1998

[3] E. Calvo and J.C. Santos, "A Value for Multichoice Games," *Mathematical Social Sciences*, vol. 40, pp341-354, 2000

[4] C.Y. Cheng, E.C. Chi, K. Chen and Y.H. Liao, "A Power Mensuration and its Normalization under Multicriteria Situations," *IAENG International Journal of Applied Mathematics*, vol. 50, no. 2, pp262-267, 2020

[5] P. Dubey and L.S. Shapley, "Mathematical Properties of the Banzhaf Power Index," *Mathematics of Operations Research*, vol. 4, pp99-131, 1979

[6] K.H.C. Chen, J.C Huang and Y.H. Liao, "Sustainable Combination Mechanism for Catalysts: A Game-theoretical Approach," *Catalysts*, vol. 11, pp345, 2022

[7] H. Haller, "Collusion Properties of Values," *International Journal of Game Theory*, vol. 23, pp261-281, 1994

[8] S. Hart and A. Mas-Colell, "Potential, Value and Consistency," *Econometrica*, vol. 57, pp589-614, 1989

[9] Y.L. Hsieh and Y.H. Liao, *The Pseudo EANSC: Axiomatization and Dynamic Process*, National Pingtung University Press, Taiwan, 2016

[10] Y.A. Hwang, "Fuzzy Games: A Characterization of the Core," *Fuzzy Sets and Systems*, vol.158, pp. 2480-2493, 2007.

[11] Y.A. Hwang and Y.H. Liao, "The Unit-level-core for Multi-choice Games: The Replicated Core for Games," *J. Glob. Optim.*, vol. 47, pp161-171, 2010

[12] Y.A. Hwang and Y.H. Liao, "Note: Natural Extensions of the Equal Allocation of Non-Separable Costs," *Engineering Letters*, vol. 28, no. 4, pp1325-1330, 2020

[13] Y.A. Hwang and Y.H. Liao, "The Core Configuration for Fuzzy Games," *Journal of Intelligent and Fuzzy Systems*, vol. 27, pp. 3007-3014, 2014.

[14] J.C Huang, K.H.C. Chen and Y.H. Liao, "Symmetric and Asymmetric Allocating Concepts under Multiple-goals," *Symmetry*, vol. 16, pp1428, 2024

[15] R.R. Huang, H.C. Wei, C.Y. Huang and Y.H. Liao, "Axiomatic Analysis for Scaled Allocating Rule," *IAENG International Journal of Applied Mathematics*, vol. 52, no. 1, pp254-260, 2022

[16] F. Klijn, M. Slikker and J. Zazuelo, "Characterizations of a Multi-choice Value," *International Journal of Game Theory*, vol. 28, pp521-532, 1999

[17] E. Lehrer, "An Axiomatization of the Banzhaf Value," *International Journal of Game Theory*, vol. 17, pp89-99, 1988

[18] S. Li and Q. Zhang, "A Simplified Expression of the Shapley Function for Fuzzy Game," *European Journal of Operational Research*, vol. 196, pp. 234-245, 2009.

[19] C.H Li, J.W. Chiang, E.C. Chi and Y.H. Liao, "Axiomatic and Dynamic Processes for A Symmetric Allocation Rule," *Symmetry*, vol. 13, no. 8, pp1474, 2021

[20] Y.H. Liao, "The Maximal Equal Allocation of Non-separable Costs on Multi-choice Games," *Economics Bulletin*, vol. 3, no. 70, pp1-8, 2008

[21] Y.H. Liao, "The Duplicate Extension for the Equal Allocation of Nonseparable Costs," *Operational Research: An International Journal*, vol. 13, pp385-397, 2012

[22] Y.H. Liao, "A Weighted Solution Concept under Replicated Behavior," *Mathematics*, vol. 11, pp150, 2023

[23] Y.H. Liao, C.H. Li, Y.C. Chen, L.Y. Tsai, Y.C. Hsu and C.K. Chen, "Agents, Activity Levels and Utility Distributing Mechanism: Game-theoretical Viewpoint," *IAENG International Journal of Applied Mathematics*, vol. 51, no. 4, pp867-873, 2021

[24] M. Maschler and G. Owen, "The Consistent Shapley Value for Hyperplane Games," *International Journal of Game Theory*, vol. 18, pp389-407, 1989

[25] S. Masuya and M. Inuiguchi, "A Fundamental Study for Partially Defined Cooperative Games," *Fuzzy Optimization and Decision Making*, vol. 15, pp. 281-306, 2016.

[26] F. Meng and Q. Zhang, "The Shapley Value on a Kind of Cooperative Fuzzy Games," *Journal of Computational Information Systems*, vol. 7, pp. 1846-1854, 2011.

[27] H. Moulin, "The Separability Axiom and Equal-sharing Methods," *Journal of Economic Theory*, vol. 36, pp120-148, 1985

[28] S. Muto, S. Ishihara, S. Fukuda, S. Tijs and R. Branzei, "Generalized Cores and Stable Sets for Fuzzy Games," *International Game Theory Review*, vol. 8, pp. 95-109, 2006.

[29] I. Nishizaki and M. Sakawa, *Fuzzy and Multiobjective Games for Conflict Resolution*, Physica-Verlag, Heidelberg, 2001.

[30] A van den. Nouweland, J. Potters, S. Tijs and J.M. Zarzuelo, "Core and Related Solution Concepts for Multi-choice Games," *ZOR-Mathematical Methods of Operations Research*, vol. 41, pp289-311, 1995

[31] J.S. Ransmeier, *The Tennessee Valley Authority*, Vanderbilt University Press, Nashville, 1942.

[32] L.S Shapley, "A Value for n-person Game," in: Kuhn, H.W., Tucker, A.W.(Eds.), *Contributions to the Theory of Games* **II**, Princeton, 1953, pp307-317

[33] L.S. Shapley, *Discussant's Comment*, In: Moriarity S (ed) Joint cost allocation. University of Oklahoma Press, Tulsa, 1982.

[34] R.E. Stearns, "Convergent Transfer Schemes for n-person Games," *Transactions of the American Mathematical Society*, vol. 134, pp449-459, 1968

[35] B. Uapipatanakul, J.C. Huang, K.H.C. Chen, S. Ngammuangpak and Y.H. Liao, "Modeling Pollen Tube Polar Growth Pattern under Asymmetric Consideration and Creating Game-theoretical Model for Ecotoxicity Assessment," *IAENG International Journal of Applied Mathematics*, vol. 55, no. 1, pp16-25, 2025

[36] H.C. Wei, A.T. Li, W.N. Wang and Y.H. Liao, "Game-theoretical Evaluating Rules for Minimizing Losses under Multi-objective Situations," *Engineering Letters*, vol. 33, no. 2, pp265-274, 2025