Stress Centrality of Some Classes of Graphs

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***Abstract*—Centrality measures are used to rank the nodes of the network. Stress is one such important centrality measure of graphs applicable to the study of social and biological networks. We study the stress of a few standard classes of graphs along with a few graphs of diameter two. We have also identified the graphs with the maximum stress in two particular families of unicyclic graphs of order n.**

***Index Terms*—Unicyclic graph, Windmill graph, Rooted product graph, Half graph Subdivision graph.**

I. Introduction

LET $G = (V(G), E(G))$ be a finite simple graph with the set of vertex $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the set of edges $E(G) = \{e_1, e_2, \ldots, e_m\}$. If two vertices v_i and v_j are adjacent, we write $v_i \sim v_j$, and the edge between them is denoted by e_{ij}.

Let $P = (u = v_0, v_1, \ldots, v_k = v)$ be a u-v path of length k in G with *origin* $u = v_0$ and *terminus* $v = v_k$. The vertices v_i, $1 \le i \le k-1$ are called the *internal* vertices of the path P. The length of a shortest u-v path, denoted by $d(u, v)$ is called $distance$ between u and v. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is given by $\max\{d(u, v) : u, v \in V\}$.

A vertex centrality measure assigns a real number to each vertex of a graph, and it quantifies the importance or criticality of a vertex from a particular perspective. Different centrality measures describe the importance of a vertex from different perspectives. A few examples of vertex centrality measures are betweenness, closeness, degree, eigenvector centrality, and stress.

The centrality measures in a graph are used to rank the vertices, and the vertices with the highest rank are considered to be more important than the others. For further results on centrality measures, the readers are referred to [6], [13], [15] and [16]. Applications of centrality measures include identification of the most influential person in a social network, proteins that play a significant role in a biological process, and key infrastructure vertices in an urban network or internet, etc. For a brief survey of centrality measures with emphasis on applications in the study of biological networks, we refer to [4]. For graph-theoretic terminology we refer to Chartrand and Lesniak [17].

In the present paper, our focus is on the study of stress, which is a vertex centrality measure studied to some extent in [2], [3] and [14].

Definition 1.1: Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The $stress$ of a vertex v_i is the number of shortest paths in G having v_i as an internal vertex and is denoted by $st(v_i)$ or $st_G(v_i)$.

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Definition 1.2: The stress of a graph G is defined by $st(G) = \sum\limits_{i=1}^{n} st(v_i)$.

For recent work on stress centrality, the reader can refer to [7]–[9].
We write $\mathrm{st}(v)$ instead of $\mathrm{st}_G(v)$, whenever the graph under discussion is clear by the context.

Stress centrality of a few graphs of diameter two is discussed in Section 2. Section 3 deals with the stress of some standard classes of graphs, such as the windmill graph, the half graph, some classes of unicyclic graphs and rooted product graphs and Section 4 contains the results related to the maximum stress among the two families of unicyclic graphs of order n.

II. Diameter two graphs

In this section, we determine the stress of some diameter 2 graphs using Theorem 2.1.

Theorem 2.1: [7] If diam(G)=2 and $deg(w) \ge 2$, then

$$st(w) = \binom{deg(w)}{2} - m(w),$$

where $m(w)$ denotes the number of edges in the induced subgraph $G[N(w)]$.

Theorem 2.2: Let $G = W_{m,n} = \overline{K_m} + C_n$, be a generalized wheel graph on $m + n$ vertices. Then,

$$st(G) = \begin{cases} \dfrac{mn(m+n-4)}{2}; & n = 3 \\ \dfrac{mn(m+n-4)+2n}{2}; & n > 3 \end{cases}$$

Proof: Let $\{v_1, v_2, ..., v_n\}$ be the cycle vertices and $\{w_1, w_2, ..., w_m\}$ be the central vertices.
Then, for all $1 \le i \le m$,

$$st(w_i) = \binom{n}{2} - n.$$

Stress of cycle vertices is given by,

$$st(v_j) = \binom{deg(v_j)}{2} - m(v_j).$$

where $m(v_j)$ denotes the number of edges in the induced subgraph $G[N(v_j)]$.

Case 1: When $n = 3$.

$$st(v_j) = \binom{m+2}{2} - 2m.$$

Hence,

$$st(G) = \frac{mn(m+n-4)}{2}.$$

Case 2: When $n > 3$.

$$st(v_j) = \binom{m+2}{2} - (2m+1).$$

Hence,

$$st(G) = \frac{mn(m+n-4)+2n}{2}.$$

■

A split graph is a graph which admits a partition of its vertex set into two parts, say V_1 and V_2, so that the vertices of V_1 induce a co-clique, while the vertices of V_2 induce a clique. All other edges, the cross edges, join a vertex in V_1 with a vertex in V_2. A threshold graph is a split graph where the subsets of vertices V_1 and V_2 can be further partitioned into h cells $V_1 = V_{1,1} \cup V_{1,2} \cup \cdots \cup V_{1,h}$ and $V_2 = V_{2,1} \cup V_{2,2} \cup \cdots \cup V_{2,h}$ satisfying the following nesting property: For each vertex $u \in V_{1,i}$, $1 \leq i \leq h$, $N_G(u) = V_{2,1} \cup ... \cup V_{2,h-i+1}$. If $|V_{1,i}| = m_i$ and $|V_{2,i}| = n_i$, then we write $G = NSG(m_1, m_2, ..., m_h; n_1, n_2, ..., n_h)$.

Theorem 2.3: Let $G = NSG(m_1, m_2; n_1, n_2)$ be a graph on n vertices. Then,

$$st(G) = \frac{n_1(m_1+m_2-1)(m_1+m_2)+2n_1n_2m_2+n_2m_1(m_1-1)}{2}.$$

Proof: The stress of vertices present in $V_{1,1}$ and $V_{1,2}$ are 0 as they do not lie in the shortest paths between any vertices.

1) The stress of a vertex present in $V_{2,1}$ due to shortest paths between every two vertex in $V_{1,1} \cup V_{1,2}$ is given by,

$$\sum_{i=1}^{m_1+m_2-1} i.$$

The vertex in $V_{2,1}$ also lies in the shortest paths between vertices in $V_{1,2}$ to vertices in $V_{2,2}$ and this gives, m_2n_2. Therefore, the stress on n_1 such vertices of $V_{2,1}$ is,

$$n_1 \sum_{i=1}^{m_1+m_2-1} i + m_2n_1n_2.$$

2) The stress of n_2 vertices in $V_{2,2}$ due to shortest paths between every two vertices of $V_{1,1}$ is given by,

$$\frac{n_2m_1(m_1-1)}{2}.$$

Therefore,

$$st(G) = \frac{n_1(m_1+m_2-1)(m_1+m_2)+2n_1n_2m_2+n_2m_1(m_1-1)}{2}.$$

■

Theorem 2.4: Let $G = NSG(1, 1, ..., 1; 1, 1, ..., 1)$ be a graph on $2n$ vertices with $|V_1| = |V_2| = n$. Then,

$$st(G) = \frac{n(n^2-1)}{3}.$$

Proof: Let the vertex set of G be partitioned into $V_1 = \{v_1, v_2, ..., v_n\}$ and $V_2 = \{w_1, w_2, ..., w_n\}$ with $|V_1| = |V_2| = n$. The stress of vertices in V_1 is 0 as they do not lie in the shortest paths between any vertices. The vertex w_n is adjacent to every vertex in V_1 and every vertex in V_2. Stress of w_n due to shortest paths between every two vertices in V_1 is,

$$\frac{n(n-1)}{2}.$$

Stress of w_n due to shortest paths between vertices in V_1 to vertices in V_2 is given by,

$$\frac{n(n-1)}{2}.$$

Therefore,

$$st(w_n) = n(n-1).$$

This is true for all $w_x \in V_2$ with corresponding values of $x,\ 1 \leq x \leq n.$
Hence,

$$st(G) = \sum_{x=1}^{n} x(x-1).$$

Therefore,

$$st(G) = \frac{n(n^2-1)}{3}.$$

■

III. Standard class of graphs

Stress of a few standard graphs such as windmill graph K_n^l, half graphs, $DNG(m_1, m_2; n_1, n_2)$ and three classes of unicyclic graphs P_n^l, cycle-star graph $CS_{k,n-k}$ and rooted product graphs $C_n(P_m)$ are discussed in this section.

First we discuss the stress of a few classes of unicyclic graphs.

Definition 3.1: Let $P_n^l, l < n$ denote the unicyclic graph obtained by connecting a vertex of C_l with an end vertex of P_{n-l} as shown in Figure 1.

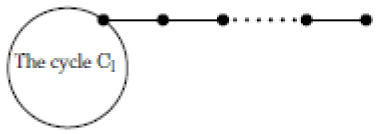

Fig. 1: Graph P_n^l

Theorem 3.1: Let $G = P_n^l, 3 \leq l < n$, be a unicyclic graph. Then, $st(G) =$

$$\begin{cases} \frac{4n(n^2-1)+l(5l^2-6l-6nl+4)}{24}; & \text{l is even} \\ \frac{2n(2n^2-6n+1)+l(5l^2-12l-6nl+12n+7)}{24}; & \text{l is odd.} \end{cases} \quad (1)$$

Proof:

Case 1: When l is even.

The stress of a cycle vertex associated with shortest paths between two cycle vertices is

$$\frac{l(l-2)}{8} = p.$$

Then the stress of l cycle vertices due to shortest path between cycle vertices to cycle vertices and $(n-l)$ path vertices is,
$p + 2(p + (n-l)) + 2(p + 2(n-l)) + ... + 2(p + (\frac{l}{2} - 1)(n-l)) + (p + l(n-l)).$
Hence, we get,

$$S_1 = \frac{l^2(l-2)}{8} + \frac{l(n-l)(l+2)}{4}.$$

The stress of the $n-l$ path vertices is given by,

$$\begin{aligned}S_2 &= (l+1)(n-(l+1)) + (l+2)(n-(l+2)) \\ &\quad + ... + (n-1)(n-(n-1)) \\ &= n((l+1)+(l+2)+\ldots+(n-1)) \\ &\quad -((l+1)^2+(l+2)^2+\ldots+(n-1)^2) \\ &= n(\sum_{i=1}^{n-1} i - \sum_{i=1}^{l} i) - (\sum_{i=1}^{n-1} i^2 - \sum_{i=1}^{l} i^2).\end{aligned}$$

Therefore, $S_1 + S_2$ gives,

$$st(G) = \frac{4n(n^2-1)+l(5l^2-6l-6nl+4)}{24}.$$

Case 2: When l is odd.

The stress of a cycle vertex associated with the shortest paths between two cycle vertices is

$$\frac{(l-3)(l-1)}{8} = q.$$

Then the stress of l cycle vertex due to shortest path between cycle vertices to cycle vertices and $(n-l)$ path vertices is,
$2q+2(q+(n-l))+2(q+2(n-l))+...+2(q+(\frac{l-1}{2}-1)(n-l))+(q+(l-1)(n-l))$.
Hence, we get,

$$S_3 = \frac{l(l-3)(l-1)}{8} + \frac{(n-1)(l^2-1)}{4}.$$

The stress of the $n-l$ path vertices is given by,

$$\begin{aligned}S_4 &= l(n-(l+1)) + (l+1)(n-(l+2)) + ... \\ &\quad + (n-2)(n-(n-1)) \\ &= n(l+(l+1)+\ldots+(n-2)) \\ &\quad -(l(l+1)+(l+1)(l+2)+\ldots+(n-2)(n-1)) \\ &= n(\sum_{i=1}^{n-2} i - \sum_{i=1}^{l-1} i) - (\sum_{i=1}^{n-2} i(i+1) - \sum_{i=1}^{l-1} i(i+1).\end{aligned}$$

Therefore, $S_3 + S_4$ gives,

$$st(G) = \frac{2n(2n^2-6n+1)+l(5l^2-12l-6nl+12n+7)}{24}.$$

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Definition 3.2: A cycle-star graph, denoted by $CS_{k,n-k}$, is a graph consisting of two parts: a cycle of length k and $n-k$ leaf vertices, each connected to a single vertex of the cycle.

Next, theorem discuss about stress of a cycle-star graph.

Theorem 3.2: Let $G = CS_{k,n-k}; 3 \leq k < n$, be a cycle star on n vertices. Then, $st(G) =$

$$\begin{cases} \frac{k^2(k-2)+2(n-k)(2n-2+k^2)}{8}; & \text{k is even} \\ \frac{k(k^2-4k+3)+2(n-k)(2n-2k-3+k^2)}{8}; & \text{k is odd} \end{cases}$$

Proof: Let $w \in V(G)$ be the vertex at which the $n-k$ leaf vertices are connected.

Case 1: When k is even.

The stress of a cycle vertex associated with shortest paths between two cycle vertices is

$$\frac{k(k-2)}{8} = a$$

The stress of the k cycle vertices associated with shortest paths between cycle vertices to other cycle vertices and leaf vertices is
$a+2(a+(n-k))+2(a+2(n-k))+...+2(a+(\frac{k}{2}-1)(n-k))+(a+k(n-k))$.
Hence, we get,

$$\frac{k^2(k-2)}{8} + \frac{k(n-k)(k+2)}{4}.$$

The vertex w has an additional stress as it lies in the shortest path between every two leaf vertices and this stress amounts to

$$\frac{(n-k)(n-k-1)}{2}.$$

Therefore,

$$st(G) = \frac{k^2(k-2)+2(n-k)(2n-2+k^2)}{8}.$$

Case 2: When k is odd.

The stress of a cycle vertex associated with the shortest paths between two cycle vertices is

$$\frac{(k-3)(k-1)}{8} = b.$$

The stress of the k cycle vertices associated with shortest paths between cycle vertices to other cycle vertices and leaf vertices is
$2b+2(b+(n-k))+2(b+2(n-k))+...+2(b+(\frac{k-1}{2}-1)(n-k))+(b+(k-1)(n-k))$.
Hence, we get,

$$\frac{k(k-3)(k-1)}{8} + \frac{(n-k)(k-1)(k+1)}{4}.$$

The vertex w has an additional stress as it lies in the shortest path between every two leaf vertices and this stress amounts to

$$\frac{(n-k)(n-k-1)}{2}.$$

Therefore,

$$st(G) = \frac{k(k^2-4k+3)+2(n-k)(2n-2k-3+k^2)}{8}.$$

■

Definition 3.3: The rooted product graph denoted by $C_n[P_m]$ is obtained by taking n copies of P_m and joining each vertex of C_n with a pendant vertex of a P_m.

In the following when n is even and odd we gave stress of the rooted product graph separately.

Theorem 3.3: Let $G = C_n[P_m], n \geq 3, m \geq 1$ be a rooted product graph on $n(m+1)$ vertices. Then,

$$st(G) = \begin{cases} \frac{n^2(m+1)(nm+6m+n-2)}{8} + \\ \frac{nm(m-1)(3nm+3n+m+1)}{6}; & \text{n is even} \\ \frac{n(n-1)(m+1)(nm+5m+n-3)}{8} + \\ \frac{nm(m-1)(3nm+3n-2m-2)}{6}; & \text{n is odd.} \end{cases}$$

Proof: Let $\{v_1, v_2, v_3, ..., v_n\}$ be the cycle vertices.

Case 1: When n is even

The total stress of v_i, $1 \leq i \leq n$ is due to the following:

1) The stress of v_i due to the shortest paths between two vertices of the cycle is given by
$$\frac{n(n-2)}{8}.$$
2) Stress of v_i due to shortest paths from $m+1$ vertices of P_m+v_j,$i\neq j$ to other path vertices is given by,
$$\frac{m(m+1)(n^2+2n+8)}{8}.$$
3) Stress of v_i due to shortest paths from m vertices of the P_m connected to v_i to $m+1$ vertices of P_m+v_j,$i\neq j$, not considered above is given by,
$$\frac{m(m+1)(n-2)}{2}.$$
4) Stress of v_i because of shortest paths from m vertices of the P_m connected to v_i to cycle vertices not considered above is given by,
$$\frac{mn(n-2)}{8}.$$

Therefore, stress of n such cycle vertices in $C_n[P_m]$ is,
$$S_1=\frac{n^2(m+1)(nm+6m+n-2)}{8}.$$
Let $\{w_1,w_2,w_3,\ldots,w_m\}$ be the vertices in a P_m with $w_1\sim v_i$. Note that $st(w_m)=0$ as it is a pendant vertex. So, $st(w_i)=(m-i)(n(m+1)+i)$. Then, stress of m vertices in one P_m is given by $\sum_{i=1}^{m-1}(m-i)((m+1)n+i)=\frac{nm(m^2-1)}{2}+\frac{m^2(m-1)}{2}-\frac{m(m-1)(2m-1)}{6}$.
Hence stress of n such P_m's is given by,
$$S_2=\frac{nm(m-1)(3mn+m+3n+1)}{6}$$
Therefore, adding S_1 and S_2 we get $st(C_n[P_m])=\frac{n^2(m+1)(nm+6m+n-2)}{8}+\frac{nm(m-1)(3mn+m+3n+1)}{6}$.

Case 2: When n is odd
The total stress of v_i,$1\leq i\leq n$ is due to the following:

1) Stress of v_i due to shortest paths between two cycle vertices is given by,
$$\frac{(n-1)(n-3)}{8}.$$
2) Stress of v_i due to shortest paths from $m+1$ vertices of P_m+v_j,$i\neq j$ to other path vertices is given by,
$$\frac{m(m+1)(n^2-1)}{8}.$$
3) Stress of v_i because of shortest paths from m vertices of the P_m connected to v_i to $m+1$ vertices of P_m+v_j,$i\neq j$, not considered above is given by,
$$\frac{m(m+1)(n-1)}{2}.$$
4) Stress of v_i because of shortest paths from m vertices of the P_m connected to v_i to cycle vertices not considered above is given by,
$$\frac{m(n-1)(n-3)}{8}.$$

Therefore, stress of n such cycle vertices in $C_n[P_m]$ is,
$$S_3=\frac{n(n-1)(m+1)(nm+5m+n-3)}{8}.$$
Let $\{w_1,w_2,w_3,...,w_m\}$ be the vertices in a P_m with $w_1\sim v_i$. Note that $st(w_m)=0$ as it is a pendant vertex.Then, stress of m vertices in one P_m is given by
$$\sum_{i=1}^{m-1}(m-i)((m+1)(n-1)+i).$$
Hence stress of n such P_m's is given by,
$$S_4=\frac{nm(m-1)(3mn-2m+3n-2)}{6}$$
Therefore, adding S_3 and S_4 we get $st(C_n[P_m])$ which is equal to $\frac{n(n-1)(m+1)(nm+5m+n-3)}{8}+\frac{nm(m-1)(3mn-2m+3n-2)}{6}$. ■

Definition 3.4: A chain graph is a bipartite graph $G(V_1\cup V_2,E)$ with the property that the neighborhoods of vertices of each partite set form a chain with respect to the partial ordering of set inclusion.
Given a chain graph $G(V_1\cup V_2,E)$, each of the color classes V_i $(i=1,2)$ can be further partitioned into h nonempty cells $V_1=V_{1,1}\cup V_{1,2}\cup\ldots\cup V_{1,h}$ and $V_2=V_{2,1}\cup V_{2,2}\cup\ldots\cup V_{2,h}$ satisfying $N_G(u)=V_{2,1}\cup\cdots\cup V_{2,h-i+1}$, $\forall\ u\in V_{1,i}$, $1\leq i\leq h$. In light of this nesting property, chain graphs are called Double Nested Graphs (DNG in short). If $m_i=|V_{1,i}|$ and $n_i=|V_{2,i}|$, then we write $G=DNG(m_1,m_2,\ldots,m_h;n_1,n_2,\ldots,n_h)$.
If $m_i=n_i=1$ for all $1\leq i\leq h$, then the graph is called a half graph.

Theorem 3.4: Let G be a half graph on $2n$ vertices. Then,
$$st(G)=\frac{n(n^2-1)(n+6)}{12}.$$

Proof: Let G be a half graph with vertex set partitioned into $V_1=\{v_1,v_2,...,v_n\}$ and $V_2=\{w_1,w_2,...,w_n\}$ with $|V_1|=|V_2|=n$. Let the vertex v_n be adjacent to the n vertices of V_2.
Stress of v_n due to shortest paths between every two vertices in V_2 is,
$$\frac{n(n-1)}{2}.$$
Stress of v_n due to shortest paths between vertices in V_2 to vertices in V_1 is given by,
$$(1)+(1+2)+(1+2+3)+...+(1+2+3+...+(n-1))=\sum_{k=1}^{n-1}a_k$$

where
$$a_k = \frac{k(k+1)}{2}.$$
Therefore,
$$st(v_n) = \frac{n(n-1)}{2} + \sum_{k=1}^{n-1} a_k.$$
This is true for all $v_x \in V_1$ and $w_x \in V_2$ with corresponding values of $x, 1 \le x \le n$
Hence,
$$st(G) = 2\sum_{x=1}^{n}(\frac{x(x-1)}{2} + \sum_{k=1}^{x-1}\frac{k(k+1)}{2}).$$
Therefore,
$$st(G) = \frac{n(n^2-1)(n+6)}{12}.$$
■

Theorem 3.5: Let $G = DNG(m_1, m_2; n_1, n_2)$ be a chain graph on n vertices. Then,
$st(G) = \frac{m_1n_1(m_1+2m_2+n_1+2n_2+4m_2n_2-2)}{2} + \frac{m_1n_2(m_1+n_2-2)+m_2n_1(m_2+n_1-2)}{2}$.

Proof: $G = DNG(m_1, m_2; n_1, n_2)$
Then $|V_{1,1}| = m_1, |V_{1,2}| = m_2, |V_{2,1}| = n_1, |V_{2,2}| = n_2$.

1) The stress of a vertex present in $V_{1,1}$ due to shortest paths between every two vertex in $V_{2,1} \cup V_{2,2}$ is given by,
$$\sum_{i=1}^{n_1+n_2-1} i.$$
The vertex in $V_{1,1}$ also lies in the shortest paths between vertices in $V_{2,2}$ to vertices in $V_{1,2}$ and this gives, $m_2n_1n_2$.
Therefore, the stress on m_1 such vertices of $V_{1,1}$ is,
$$m_1 \sum_{i=1}^{n_1+n_2-1} i + m_1m_2n_1n_2.$$
2) The stress of m_2 vertices in $V_{1,2}$ due to shortest paths between every two vertices of $V_{2,1}$ is given by,
$$\frac{m_2n_1(n_1-1)}{2}.$$
3) The stress of a vertex present in $V_{2,1}$ due to shortest paths between every two vertex in $V_{1,1} \cup V_{1,2}$ is given by,
$$\sum_{i=1}^{m_1+m_2-1} i.$$
The vertex in $V_{2,1}$ also lies in the shortest paths between vertices in $V_{1,2}$ to vertices in $V_{2,2}$ and this gives, $m_1m_2n_2$.
Therefore, the stress on n_1 such vertices of $V_{2,1}$ is,
$$n_1 \sum_{i=1}^{m_1+m_2-1} i + m_1m_2n_1n_2.$$
4) The stress of n_2 vertices in $V_{2,2}$ due to shortest paths between every two vertices of $V_{1,1}$ is given by,
$$\frac{n_2m_1(m_1-1)}{2}.$$

Therefore,
$st(G) = \frac{m_1n_1(m_1+2m_2+n_1+2n_2+4m_2n_2-2)}{2} + \frac{m_1n_2(m_1+n_2-2)+m_2n_1(m_2+n_1-2)}{2}$. ■

Definition 3.5: A windmill graph $K_n^{(l)}$ is obtained by taking l copies of K_n and joining them at one vertex. So, it has $(n-1)l+1$ vertices.

Theorem 3.6: Let $G = K_n^{(l)}$ be a windmill graph on $(n-1)l+1$ vertices. Then,
$$st(G) = \frac{l(l-1)(n-1)^2}{2}.$$

Proof: Let $w \in V(G)$ be the vertex at which the l copies of K_n are joined. Then, $st(v_i) = 0$ for all $v_i \in V(G) - \{w\}$ as $G[N(v_i)]$ is complete. The vertex w lies in the shortest paths between the $n-1$ vertices of any one copy of K_n to the $n-1$ vertices of all the other $l-1$ copies of K_n's. As there are $\binom{l}{2}$ such combinations, we get,
$$st(w) = st(G) = \frac{l(l-1)(n-1)^2}{2}.$$
■

A subdivision of a graph G, denoted as $S(G)$, is a new graph created by subdividing each edge of G. For each edge (u, v) in G, a new vertex w is added, and the edge (u, v) is replaced by two new edges (u, w) and (w, v). Observe that $S(C_n)$ is C_{2n} and $S(P_n)$ is P_{2n-1}. We obtain stress of $S(K_n)$ in the following. Observe that $S(K_n)$ is a graph with $n + \binom{n}{2}$ vertices and $n(n-1)$ edges.

Theorem 3.7: Let $G = K_n$. Then,
$$st(S(G)) = \frac{n(n-1)(3n^2-10n+9)}{2}.$$

Proof: Let $V(G) = \{u_1, u_2, \ldots, u_n\}$. Then $V(S(G)) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_{\frac{n(n-1)}{2}}\}$. Observe that $st(u_i) = \frac{(n-1)(n-2)(2n-3)}{2}, 1 \le i \le n$ and $st(v_j) = n^2 - 3n + 3, 1 \le j \le \binom{n}{2}$. Hence, $st(S(G)) = \frac{n(n-1)(n-2)(2n-3)}{2} + \binom{n}{2}(n^2-3n+3)$. ■

Theorem 3.8: Let $G = K_{1,n-1}$. Then,
$$st(S(G)) = (n-1)(4n-7).$$

Proof: Let $V(G) = \{u_1, u_2, \ldots, u_{n-1}, v\}$. Then $V(S(G)) = \{v, u_1, u_2, \ldots, u_{n-1}, v_1, v_2, \ldots, v_{n-1}\}$. Observe that $st(u_i) = 0, st(v_i) = 2n-3, 1 \le i \le n-1$ and $st(v) = 2(n-1)(n-2)$. Hence, $st(S(G)) = 2(n-1)(n-2) + (n-1)(2n-3)$. ■

IV. Graphs with Maximum and Minimum Stress in a Given Family of Graphs

It is necessary to understand which graphs in a given family of graphs are with minimum or maximum stress. Given a family $\mathcal{F}$ of graphs with same number of vertices, we denote $G_{\max}(\mathcal{F})$ (similarly, $G_{\min}(\mathcal{F})$) to represent the set all graphs in the family $\mathcal{F}$ with maximum (minimum) stress. In other words,
$$G_{\max}(\mathcal{F}) = \{G \in \mathcal{F} : \mathrm{st}(G) \ge \mathrm{st}(F),\ \forall F \in \mathcal{F}\},$$
and
$$G_{\min}(\mathcal{F}) = \{G \in \mathcal{F} : \mathrm{st}(G) \le \mathrm{st}(F),\ \forall F \in \mathcal{F}\}.$$

In the present paper, we consider the two family of unicyclic graphs of order n which are P_n^l and $CS_{s,n-k}$.

Theorem 4.1: Let $\mathcal{F}$ be a family of graphs P_n^l. Then, $G_{max}(\mathcal{F}) = P_n^4$.

Proof: The stress of $G = P_n^l$ is given by,

$$st(G) = \begin{cases} \dfrac{4n(n^2-1)+l(5l^2-6l+4-6nl)}{24}; & \text{l is even} \\ \dfrac{2n(2n^2-6n+1)+l(5l^2-12l-6nl+12n)}{24}; & \text{l is odd.} \end{cases}$$

When $l = 4$ we get,

$$\frac{4n^3-100n+240}{24} = R_1.$$

When $l = 2m$; $m \geq 3$, we get,

$$\frac{4n^3-4n+8m-4m^2-24nm}{24} = R_2.$$

Now, $R_1 - R_2$ for all $m \geq 0$ is,

$$\frac{(m-2)(4m+24n)+240}{24} \geq 0.$$

Therefore, highest stress when l is even is for $l = 4$.
Similarly, highest stress when l is odd is for $l = 3$ and is given by,

$$\frac{4n^3-12n^2-16n+27}{24} = R_3.$$

Now, $R_1 - R_3$ for all n is,

$$\frac{12n^2-84n+213}{24} \geq 0.$$

Hence $G_{max}(\mathcal{F}) = P_n^4$. ■

Theorem 4.2: Let $\mathcal{F}$ be a family of cycle star graphs $CS_{k,n-k}$. Then,

$$G_{min}(\mathcal{F}) = CS_{3,n-3}$$

and

$$G_{max}(\mathcal{F}) = \begin{cases} CS_{n-2,2}; & n \text{ is even} \\ CS_{n-1,1}; & n \text{ is odd.} \end{cases}$$

Proof: The stress of $G = CS_{k,n-k}$ is given by,

$$\begin{cases} \dfrac{k^2(k-2)+2(n-k)(2n-2+k^2)}{8}; & k \text{ is even} \\ \dfrac{k(k^2-4k+3)+2(n-k)(2n-2k-3+k^2)}{8}; & k \text{ is odd} \end{cases}$$

When $k = 3$, we get,

$$\frac{4n^2-12n}{8} = C_1.$$

When $k = 2m+1; m \geq 2$, we get,

$$\frac{8m^2n-8mn-8m^3+4m^2+16m+4n^2-10n}{8} = C_2.$$

Now,$C_1 - C_2$ is,

$$\frac{4m(2m^2-m-4)-2n(4m^2-4m+1)}{8} < 0, \forall m \geq 2.$$

Therefore, least stress when k is odd is for $k = 3$.
Similarly, least stress when k is even is for $k = 4$ and is given by,

$$\frac{4n^2+12n-80}{8} = C_3.$$

Now, $C_1 - C_3$, for all $n \geq 4$ is,

$$\frac{-24n+8}{8} < 0.$$

Hence, $G_{min}(\mathcal{F}) = CS_{3,n-3}$.

When $k = n-2$, we get,

$$\frac{n^3-4n^2+12n-8}{8} = D_1.$$

When $k = 2m$; $2 \leq m \leq \dfrac{n}{2} - 1$, we get,

$$\frac{4n^2-4n-8m^3-8m^2+8nm^2-8nm+8m}{8} = D_2.$$

Now, $D_1 - D_2$ is given by,

$$\frac{(n-2)^3+8m(m^2+m-nm+n-1)}{8} > 0;\ 2 \leq m \leq \frac{n}{2} - 1.$$

Hence, $G_{max}(\mathcal{F}) = CS_{n-2,2}$ when n is even.
When $k = n-1$,we get,

$$\frac{n^3-5n^2+10n-8}{8} = D_3.$$

When $k = 2m+1$;$2 \leq m \leq \dfrac{n}{2} - 1$, we get,

$$\frac{4n^2-12n-8m^3+8nm^2-12m^2+12m+8}{8} = D_4.$$

Now, $D_3 - D_4$ is,

$$\frac{(n-4)^2(n-1)+4m(2m-2n+3)(m-1)}{8} > 0;$$

$2 \leq m \leq \dfrac{n}{2} - 1$. Hence, $G_{max}(\mathcal{F}) = CS_{n-1,1}$ when n is odd. ■

V. Conclusion

In this article, we have obtained stress centrality for a few standard classes of graphs. Also, for two classes of unicyclic graph on a given number of vertices, we have obtained the maximum and minimum values of stress. Recently, centrality measures are gaining a lot of researchers' interest. As a future scope, many other centrality measures can also be studied and characterized. Topological indices are gaining a lot of interest nowadays [10]–[12]. One can also try to get stress based topological indices of these graphs.

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