Dynamic Analysis and Optimal Control of the SI1I2AR Investor Sentiment Propagation Model Under the Influence of Rumors

Mengjiao Ye, Yuhan Hu*

Abstract—**Under the influence of rumors, investor sentiment propagation often exacerbates stock market volatility, leading to frequent irrational decision-making. Therefore, this paper integrates the impact of rumors on the stock market with the heterogeneity of investor sentiment, constructing an investor sentiment propagation model based on an epidemic framework. First, we compute the basic reproduction number and analyze the local and global stability of the equilibrium points associated with sentiment dissipation and propagation. Then, using Pontryagin's maximum principle, we propose an optimal control strategy that effectively mitigates the negative effects of rumors on investor sentiment and reduces emotionally driven decisions. Our results demonstrate that reducing the interference of rumors on sentiment-optimistic investors decreases the number of sentiment-pessimistic investors. Additionally, optimizing the decision-making process of sentiment-hesitant investors helps stabilize market sentiment and suppresses excessive stock market volatility.**

Index Terms—**Investor sentiment propagation model; Stability analysis; Optimal control; Influence of rumors**

I. Introduction

INVESTOR sentiment plays a crucial role in financial markets, significantly influencing investment decisions and market volatility [1]. Investor sentiment propagation can not only trigger irrational market behavior but also lead to extreme phenomena such as panic selling or overly optimistic investment decisions [2]. Particularly in markets with information asymmetry, investors often rely on sentiment rather than rational analysis, further exacerbating market instability [3]. The spread of rumors can rapidly alter investor sentiment, resulting in dramatic market fluctuations [3]–[5]. Therefore, studying the mechanisms of investor sentiment propagation under the influence of rumors is essential for understanding market volatility and developing effective regulatory strategies.

The investor sentiment propagation models are strikingly similar to epidemic models [6], making the latter a valuable framework for studying sentiment dynamics. Among classical epidemiological models, the SIR model (originally proposed by Kermack and McKendrick) classifies populations into three states: susceptible (S), infected (I), and recovered (R) [7]. Later, Daley and Kendall extended the SIR framework to develop the DK rumor propagation model [8]. Maki and Thompson further refined the framework by proposing the MT model, which established critical theoretical foundations for social contagion modeling [9].

Theoretical understanding of sentiment propagation has been revolutionized through the application of epidemic modeling paradigms [10]–[15]. Current models encompass diverse propagation scenarios including social media diffusion, multilayer network dynamics, time-delay effects, and random processes. Yan et al.'s multilayer network model demonstrated that sentiment propagation dynamically enhances information diffusion, with interlayer coupling strength critically governing propagation dynamics [16]. Building on this, Yi et al.'s have developed an optimized SEIR model revealed that sentiment polarity enhances information velocity and coverage, where positive sentiments exhibit superior transmission efficiency [17], while Geng et al. have showed interlayer connectivity nonlinearly modulates opinion propagation in empirical networks [18]. Fundamental contributions include Hill et al.'s SISa framework quantifying large-scale contagion patterns [19], and Fan et al.'s competition model proving network topology and sentiment polarity co-determine propagation outcomes [20]. Chen et al. have further identified temporal competitive interactions induce nonlinear volatility [21], whereas Ma et al.'s SFPFNR model incorporated public sentiment pressure and forced silence mechanisms, revealing their complexity-enhancing effects [22]. Yin et al. proposed the MNE-SFI model, which formally quantified sentiment mutation dynamics. Their analysis revealed that nonlinear phase transitions during high-impact events induce superlinear acceleration of negative sentiment propagation [23]. To conclude, these works establish a comprehensive theoretical framework for analyzing sentiment propagation mechanisms across varied contexts.

Over the past decade, research on sentiment propagation research has expanded significantly from social media to financial market applications [24], with particular focus on the dynamic mechanisms of investor sentiment propagation and its market impacts. Epidemiological models have provided novel theoretical perspectives for understanding market volatility patterns, investor decision-making processes, and sentiment propagation effects [25]–[27]. Notably, Chen et al.'s SIRS model demonstrated that investor sentiment propagation substantially influences market volatility, particularly in short term scenarios [28], while Gao et al.'s ISRI model revealed how investor entry dynamics and network congestion jointly affect rumor propagation [29]. Kang et al.'s SPA2G2R model

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elucidated the competitive relationships between different investor sentiments, underscoring the dual importance of promoting positive sentiment propagation through social guidance while implementing regulatory measures to contain sentiment disorder [30]. Furthermore, Zhao et al.'s SEIR model incorporating time-delay effects showed these effects critically determine credit risk propagation velocity and coverage in P2P lending networks [31]. Liu et al.'s SISa model highlighted the combined influence of network structure and individual awareness states on sentiment propagation intensity, with awareness effectively mitigating negative sentiment spread [32]. Lei et al.'s comprehensive studies employing MTGNN model demonstrated the significant impacts of media reports, investor sentiment and attention on market volatility prediction, particularly revealing sentiment propagation's predominant short-term effects [33]. These studies collectively provide valuable theoretical perspectives and methodological innovations for understanding market volatility and sentiment propagation mechanisms.

Under the influence of rumors, the impact of investor sentiment propagation on stock market volatility has garnered significant attention. However, research on emotion-driven behavioral changes triggered by rumors and their dynamic propagation mechanisms remains relatively limited. To bridge this gap, we develop a novel dynamic sentiment propagation model that explicitly incorporates rumor effects, examining how rumor-induced emotional shifts alter investor decision making processes and their implications for market stability. Through rigorous analysis of sentiment propagation dynamics, this study provides both theoretical and practical contributions by offering evidence-based strategies for market participants and regulators to effectively mitigate rumor-driven market instability.

The remainder of this paper is structured as follows: Section 2 presents the investor sentiment propagation model formulation. Section 3 computes the basic reproduction number, the points of sentiment disappearance and sentiment propagation equilibrium, and further investigate the stability of these sentiment equilibrium points. Building on Pontryagin's maximum principle, Section 4 develops an optimal control strategy for sentiment regulation. Section 5 validates the theoretical findings and control efficacy through numerical simulations. Section 6 conducts sensitivity analysis to identify key model parameters. Finally, Section 7 concludes with research contributions and implications.

II. Model Building

In this study, we consider the population size in the stock market as time-varying, where $N(t)$ represents the total number of individuals in the stock market at time t. Based on the heterogeneity of investor sentiment states and trading behavior patterns, the population is further divided into five groups with distinct decision-making characteristics: $S(t)$ represents potential investors who have not yet been exposed to information but have investment intentions; $I_1(t)$ denotes sentiment-optimistic investors who are confident in the stock market and hold a positive outlook on investment opportunities; $I_2(t)$ indicates sentiment-pessimistic investors who are concerned about economic downturns or uncertainties in the stock market; $A(t)$ corresponds to sentiment-hesitant investors who struggle to make quick decisions when faced with stock market rumors; and $R(t)$ represents rational investors who remain unaffected by information and maintain a calm and rational mindset. The densities of these groups at time t are denoted as $S(t)$, $I_1(t)$, $I_2(t)$, $A(t)$, and $R(t)$, respectively.

Considering that the population of investors in the stock market is constantly in a state of flux, this paper constructs an open-system model capable of capturing such dynamic characteristics and proposes a series of research hypotheses:

In unit time, external investors enter the stock market at a constant inflow rate Λ, and this group is defined as potential investors S. It is assumed that investors migrate out of the system at a constant exit rate μ. When potential investors S interact with sentiment-optimistic investors I_1 or sentiment-pessimistic investors I_2, they transition to I_1 and I_2 with probabilities α and β, respectively. Sentiment optimistic investors I_1 and sentiment-pessimistic investors I_2 may transition to rational investors R at rates λ and γ, respectively, due to various factors, which involve market corrections during extreme conditions, knowledge upgrading through professional training, and adaptive risk preference calibration.

On this basis, a variable c is introduced to represent the influence intensity of rumors in the stock market. Competitors spread rumors to depress stock prices, thereby gaining market share or other competitive advantages. The propagation of rumors misleads investors into making irrational decisions, increasing the probability of stock market crashes for listed companies. Under the influence of stock market rumors, some sentiment-optimistic investors I_1 begin to worry about the prospects of the listed companies, leading to fear. When the influence of rumors is strong (i.e., c is large), they transition to sentiment-pessimistic investors I_2 with probability δc; when the influence of rumors is weak (i.e., c is small), they transition to hesitant investors A with probability $\theta(1-c)$. Hesitant investors A, after continuous reflection and repeated deliberation, make different decisions: some may transition to sentiment-pessimistic investors I_2 due to further influence of rumors, some may transition to rational investors through rational analysis, and others may regain confidence and transition back to sentiment-optimistic investors.

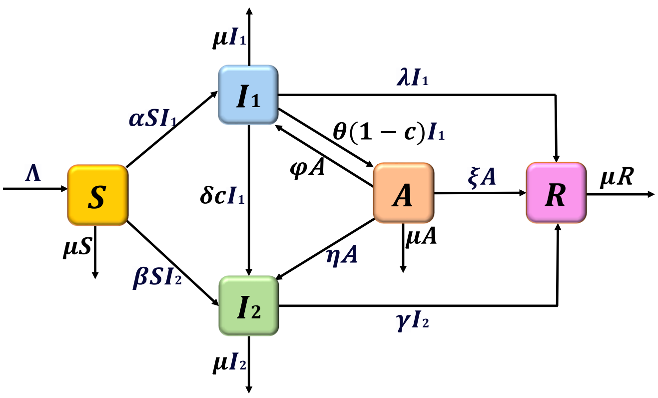


Fig. 1. The flowchart for the investor sentiment propagation model.

As shown in Fig. 1, the changes in the quantities of potential investors, sentiment-optimistic investors, sentiment pessimistic investors, sentiment-hesitant investors, and rational investors are illustrated as follows:

In unit time, the number of investors entering the system is Λ. When potential investors S interact with sentiment optimistic investors I_1, they may be influenced by the optimistic sentiment, believing that stock prices will continue to rise, and transition to sentiment-optimistic investors I_1 with probability α. Similarly, when potential investors S interact with sentiment-pessimistic investors I_2, they may be influenced by the pessimistic sentiment due to fear of stock market uncertainty and transition to sentiment-pessimistic investors I_2 with probability β. Additionally, due to certain factors (e.g., exiting the market or changes in sentiment), potential investors S migrate out of the system at a rate μ. Therefore, the change in the number of potential investors S per unit time is given by: $\Lambda - \alpha SI_1 - \beta SI_2 - \mu S$.

In unit time, when potential investors S interact with sentiment-optimistic investors I_1, they transition to sentiment optimistic investors I_1 with probability α. Simultaneously, under the influence of rumors, some optimistic investors I_1 develop fear and transition to sentiment-pessimistic investors I_2 with probability δ, where the number of such transitions is given by $\delta c I_1$. Additionally, some sentiment-optimistic investors I_1 transition to hesitant investors A with probability θ, where the number of such transitions is given by $\theta(1-c)I_1$. Hesitant investors A, after repeated deliberation, may revert to sentiment-optimistic investors I_1. Furthermore, due to external factors, sentiment-optimistic investors I_1 transition to rational investors R with probability λ and migrate out of the system at a rate μ. Therefore, the change in the number of sentiment-optimistic investors I_1 per unit time is given by: $\alpha SI_1 + \varphi A - [\theta(1-c) + \delta c + \lambda + \mu] I_1$.

In unit time, potential investors S transition to sentiment pessimistic investors I_2 at a rate of β after interacting with sentiment-pessimistic investors I_2. Due to the influence of rumors, some sentiment-optimistic investors I_1 change their investment decisions and transition to sentiment-pessimistic investors I_2, resulting in $\delta c I_1$ such transitions. Additionally, some hesitant investors A, after careful consideration, choose to believe the rumors and transition to sentiment-pessimistic investors I_2, resulting in ηA such transitions. On the other hand, some sentiment-pessimistic investors I_2 lose interest in sentiment propagation and transition to rational investors R, resulting in γI_2 such transitions. Furthermore, due to external factors, sentiment-pessimistic investors I_2 migrate out of the system at a rate μ. Therefore, the change in the number of sentiment-pessimistic investors I_2 per unit time is given by: $\beta SI_2 + \delta c I_1 + \eta A - (\gamma + \mu)I_2$.

In unit time, some sentiment-optimistic investors I_1 become skeptical of rumors and transition to hesitant investors A, resulting in $\theta(1-c)I_1$ such transitions. Hesitant investors A make different decisions after continuous reflection and deliberation: they transition back to sentiment-optimistic investors I_1 with probability φ, to sentiment-pessimistic investors I_2 with probability η, or to rational investors R with probability ξ. Additionally, hesitant investors A migrate out of the system at a rate μ. Therefore, the change in the number of hesitant investors A per unit time is given by: $\theta(1-c)I_1 - (\varphi + \eta + \mu + \xi)A$.

Over time, sentiment-optimistic investors I_1, sentiment pessimistic investors I_2, and hesitant investors A lose interest in sentiment propagation and transition to rational investors R, with the numbers of such transitions given by λI_1, γI_2, and ξA, respectively. Finally, rational investors R migrate out of the system at a rate μ. Therefore, the change in the number of rational investors R per unit time is given by: $\lambda I_1 + \gamma I_2 + \xi A - \mu R$.

Table I outlines the meanings of each parameter used in the *SI₁I₂AR* model.

TABLE I
PARAMETERS OF THE SI_1I_2AR MODEL.

Parameter	Description
$S(t)$	The number of potential investors at time t
$I_1(t)$	The number of sentiment optimistic investors at time t
$I_2(t)$	The number of sentiment pessimistic investors at time t
$A(t)$	The number of hesitant investors at time t
$R(t)$	The number of rational investors at time t
Λ	Individuals entering at time t
α	The probability that $S(t)$ converting to $I_1(t)$ at time t
β	The probability that $S(t)$ converting to $I_2(t)$ at time t
μ	The probability of group migration out at time t
φ	The transition rate from $A(t)$ to $I_1(t)$ at time t
θ	The transition rate from $I_1(t)$ to $A(t)$ at time t
γ	The transition rate from $I_2(t)$ to $R(t)$ at time t
ξ	The transition rate from $A(t)$ to $R(t)$ at time t
δ	The transition rate from $I_1(t)$ to $I_2(t)$ at time t
λ	The transition rate from $I_1(t)$ to $R(t)$ at time t
η	The transition rate from $A(t)$ to $I_2(t)$ at time t
c	The influence intensity of stock market rumors

Thereby, the SI_1I_2AR investor sentiment propagation model is established as follows:

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \alpha SI_1 - \beta SI_2 - \mu S, \\ \dfrac{dI_1(t)}{dt} = \alpha SI_1 + \varphi A - [\theta(1-c) + \delta c + \lambda + \mu] I_1, \\ \dfrac{dI_2(t)}{dt} = \beta SI_2 + \delta c I_1 + \eta A - (\gamma+\mu)I_2, \\ \dfrac{dA(t)}{dt} = \theta(1-c)I_1 - (\varphi + \eta + \mu + \xi)A, \\ \dfrac{dR(t)}{dt} = \lambda I_1 + \gamma I_2 + \xi A - \mu R. \end{cases} \tag{1}$$

where

$$\begin{aligned} &\Lambda, \mu, \varphi, \theta, \gamma, \xi, \delta, \lambda, \eta, c > 0, \alpha \in (0,1), \beta \in (0,1); \\ &S(0) = S_0 \geq 0, I_1(0) = I_{1_0} \geq 0, I_2(0) = I_{2_0} \geq 0, \\ &A(0) = A_0 \geq 0, R(0) = R_0 \geq 0, \end{aligned} \tag{2}$$

and

$$S(t) + I_1(t) + I_2(t) + A(t) + R(t) = N(t). \tag{3}$$

Where $N(t)$ is the total number of the population. We can know $\frac{dN(t)}{dt} = \Lambda - \mu N, N_0 = N(0)$, so $N(t) = \frac{\Lambda}{\mu} + e^{-\mu t}\left[N_0 - \frac{\Lambda}{\mu}\right]$, and then $\lim_{t\to\infty} N(t) = \frac{\Lambda}{\mu}$. Then the positively invariant set of the system (1) is:

$$\Omega = \left\{(S, I_1, I_2, A, R) \in \mathbb{R}_+^5;\ N \leq \frac{\Lambda}{\mu}\right\}. \tag{4}$$

III. MODEL ANALYSIS

A. The basic reproduction number R_0

The equilibrium point of the system's sentiment dissipation is $E_0 = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0, 0\right)^T$.

To better describe the stability of the equilibrium point of the system (1), we define the basic reproduction number R_0, which is calculated using the next-generation matrix approach. It reveals the average number of next-generation sentiment disseminators that can be produced by a single sentiment disseminator within a certain period, which plays an important role in formulating effective control strategies.

Let $X = (I_1, I_2, A, R, S)^T$, then the system can be written as

$$\frac{dX}{dt} = \mathcal{F}(X) - \mathcal{V}(X), \tag{5}$$

where

$$\begin{aligned} F(X) &= \begin{pmatrix} \alpha S I_1 \\ \beta S I_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\ V(X) &= \begin{pmatrix} -\varphi A + [\theta(1-c) + \delta c + \lambda + \mu] I_1 \\ -\delta c I_1 - \eta A + (\gamma+\mu) I_2 \\ -\theta(1-c) I_1 + (\varphi + \eta + \mu + \xi) A \\ -\lambda I_1 - \gamma I_2 - \xi A + \mu R \\ -\Lambda + \alpha S I_1 + \beta S I_2 + \mu S \end{pmatrix}. \end{aligned} \tag{6}$$

We can get,

$$\begin{aligned} F &= \begin{pmatrix} \alpha\dfrac{\Lambda}{\mu} & 0 \\ 0 & \beta\dfrac{\Lambda}{\mu} \end{pmatrix}, \\ V &= \begin{pmatrix} \theta(1-c) + \delta c + \lambda + \mu & 0 \\ -\delta c & \gamma + \mu \end{pmatrix}. \end{aligned} \tag{7}$$

The basic reproduction number R_0 of the system is the spectral radius of the matrix FV^{-1}, and the calculated basic reproduction number of the system is

$$R_0 = \max\{R_1, R_2\}, \tag{8}$$

where

$$\begin{aligned} R_1 &= \frac{\alpha\Lambda}{\mu\,[\theta(1-c) + \delta c + \lambda + \mu]}, \\ R_2 &= \frac{\beta\Lambda}{\mu(\gamma+\mu)}. \end{aligned} \tag{9}$$

B. Existence of the equilibrium point

It is necessary for us to demonstrate the existence of the equilibrium point of sentiment propagation E^* in the system (1).

Set the right-hand side of system (1) to zero to solve for the equilibrium points.

$$\begin{cases} \Lambda - \alpha S I_1 - \beta S I_2 - \mu S = 0, \\ \alpha S I_1 + \varphi A - [\theta(1-c) + \delta c + \lambda + \mu] I_1 = 0, \\ \beta S I_2 + \delta c I_1 + \eta A - (\gamma+\mu) I_2 = 0, \\ \theta(1-c) I_1 - (\varphi + \eta + \mu + \xi) A = 0, \\ \lambda I_1 + \gamma I_2 + \xi A - \mu R = 0. \end{cases} \tag{10}$$

Through calculation, we can obtain:

$$\begin{aligned} S^* &= \frac{[\delta c + \lambda + \theta(1-c) + \mu]\,(\xi + \eta + \mu) + \varphi(\delta c + \lambda + \mu)}{(\varphi + \xi + \eta + \mu)\alpha}, \\ I_1^* &= \frac{\Lambda - \mu S^*}{S^*(\alpha + \beta m)}, \\ I_2^* &= m I_1^*, \\ A^* &= \frac{\theta(1-c)}{\varphi + \xi + \eta + \mu} I_1^*, \\ R^* &= \frac{\lambda I_1^* + \gamma I_2^* + \xi A^*}{\mu}. \end{aligned} \tag{11}$$

Where $m = \frac{Y}{Z}$ with $Y = (\varphi+\xi+\eta+\mu)\delta c + \theta(1-c)\eta$ and $Z = (\varphi+\xi+\eta+\mu)\{\alpha(\gamma+\mu) - [\delta c + \lambda + \theta(1-c) + \mu]\beta\} + \varphi\theta(1-c)\beta$.

If $R_1 > R_2 > 1$, i.e., $\alpha(\gamma+\mu) > [\delta c + \lambda + \theta(1-c) + \mu]\beta$, $\alpha\Lambda > \mu[\delta c + \lambda + \theta(1-c) + \mu]$, then $m > 0$, and $\alpha\Lambda(\varphi + \xi + \eta + \mu) > \mu[\delta c + \lambda + \theta(1-c) + \mu](\varphi + \xi + \eta + \mu) - \varphi\theta(1-c)\mu$, so $S^* > 0$, $I_1^* > 0$, $I_2^* > 0$, $A^* > 0$, $R^* > 0$, and the equilibrium point $E^* = (S^*, I_1^*, I_2^*, A^*, R^*)$ exists.

C. Stability of the Equilibrium Point of Sentiment Extinction

Theorem 1. If $R_2 < 1$ and $\delta c + \lambda + \mu > \frac{\alpha\Lambda}{\mu}$, then the sentiment extinction equilibrium E_0 of the system is locally asymptotically stable.

Proof. The Jacobian matrix of the system at point E_0 is:

$$J(E_0) = \begin{pmatrix} -\mu & -\alpha\dfrac{\Lambda}{\mu} & -\beta\dfrac{\Lambda}{\mu} & 0 & 0 \\ 0 & \alpha\frac{\Lambda}{\mu} - [\delta c + \lambda + \theta(1-c) + \mu] & 0 & \varphi & 0 \\ 0 & \delta c & \beta\dfrac{\Lambda}{\mu} - (\gamma+\mu) & \eta & 0 \\ 0 & \theta(1-c) & 0 & -(\varphi + \eta + \mu + \xi) & 0 \\ 0 & \lambda & \gamma & \xi & -\mu \end{pmatrix}. \tag{12}$$

Let p be the eigenvalue, and the characteristic polynomial of $J(E_0)$ is:

$$\begin{aligned} |pE - J(E_0)| &= \begin{vmatrix} p+\mu & \alpha\dfrac{\Lambda}{\mu} & \beta\dfrac{\Lambda}{\mu} & 0 & 0 \\ 0 & p - \left\{\alpha\dfrac{\Lambda}{\mu} - [\delta c + \lambda + \theta(1-c) + \mu]\right\} & 0 & -\varphi & 0 \\ 0 & -\delta c & p - \left[\beta\dfrac{\Lambda}{\mu} - (\gamma+\mu)\right] & -\eta & 0 \\ 0 & -\theta(1-c) & 0 & p + (\varphi + \eta + \mu + \xi) & 0 \\ 0 & -\lambda & -\gamma & -\xi & p+\mu \end{vmatrix} \\ &= (p+\mu)^2\left(p - \beta\frac{\Lambda}{\mu} + \gamma - \mu\right)\left\{\left[p - \alpha\frac{\Lambda}{\mu} + \theta(1-c) + \delta c + \lambda + \mu\right](p + \varphi + \eta + \mu + \xi) - \varphi\theta(1-c)\right\} \\ &= 0. \end{aligned} \tag{13}$$

It can be obtained that: $p_1 = p_2 = -\mu, p_3 = \beta\frac{\Lambda}{\mu} - (\gamma+\mu)$, and $(p+m_1)(p+m_2) - \varphi\theta(1-c) = p^2 + (m_1+m_2)p + m_1 m_2 - \varphi\theta(1-c) = 0$.

Where $m_1 = -\alpha\frac{\Lambda}{\mu} + [\delta c + \lambda + \theta(1-c) + \mu]$, $m_2 = \varphi + \eta + \mu + \xi$.

Clearly, $p_1, p_2 < 0$, $m_2 > 0$. When $R_0 = \max\{R_1, R_2\} < 1$, that is, $\frac{\alpha\Lambda}{\mu} < \delta c + \lambda + \theta(1-c) + \mu$, $\frac{\beta\Lambda}{\mu} < \gamma + \mu$, then $-\alpha\frac{\Lambda}{\mu} + [\delta c + \lambda + \theta(1-c) + \mu] > 0$, $\frac{\beta\Lambda}{\mu} - (\gamma+\mu) < 0$. Thus, we obtain $p_3 < 0$ and $m_1 > 0$. The two roots of the characteristic polynomial satisfy: $p_4 + p_5 = -(m_1 + m_2) < 0$;

$$\begin{aligned} p_4 p_5 &= \left[-\alpha\frac{\Lambda}{\mu} + \delta c + \lambda + \theta(1-c) + \mu\right](\varphi + \eta + \mu + \xi) \\ &\quad - \varphi\theta(1-c) \\ &= \left[-\alpha\frac{\Lambda}{\mu} + \delta c + \lambda + \theta(1-c) + \mu\right](\eta + \mu + \xi) \\ &\quad + \varphi\left[-\alpha\frac{\Lambda}{\mu} + (\delta c + \lambda + \mu)\right]. \end{aligned} \tag{14}$$

When $-\alpha\frac{\Lambda}{\mu} + (\delta c + \lambda + \mu) > 0$, that is, $\delta c + \lambda + \mu > \alpha\frac{\Lambda}{\mu}$, we have $p_4 p_5 > 0$. Since $p_4 + p_5 < 0$, it follows that $p_4 < 0$ and $p_5 < 0$. The characteristic polynomial has no positive roots. According to the Routh-Hurwitz criterion, the sentiment extinction equilibrium E_0 of the system (1) is locally asymptotically stable when $R_2 < 1$ and $\delta c + \lambda + \mu > \alpha\frac{\Lambda}{\mu}$.

Theorem 2. If $\mu^2 > \max\{\alpha\Lambda, \beta\Lambda\}$, so the sentiment extinction equilibrium point E_0 of the system (1) is globally asymptotically stable.

Proof. Construct the Lyapunov function: $L(t) = I_1(t) + I_2(t) + A(t) + R(t)$.

$$\begin{aligned} L'(t) &= I_1'(t) + I_2'(t) + A'(t) + R'(t) \\ &= \alpha S I_1 + \varphi A - [\delta c + \lambda + \theta(1-c) + \mu] I_1 \\ &\quad + \beta S I_2 + \delta c I_1 + \eta A - (\gamma+\mu) I_2 \\ &\quad + \theta(1-c) I_1 - (\varphi + \eta + \mu + \xi) A \\ &\quad + \lambda I_1 + \gamma I_2 + \xi A - \mu R \\ &= \alpha S I_1 + \beta S I_2 - \mu(I_1 + I_2 + A + R) \\ &= (\alpha S - \mu) I_1 + (\beta S - \mu) I_2 - \mu(A + R). \end{aligned} \tag{15}$$

Since $S \le \frac{\Lambda}{\mu}$, we obtain:

$$\begin{aligned} L'(t) &= (\alpha S - \mu) I_1 + (\beta S - \mu) I_2 - \mu(A+R) \\ &\le \left(\alpha\frac{\Lambda}{\mu} - \mu\right) I_1 + \left(\beta\frac{\Lambda}{\mu} - \mu\right) I_2 - \mu(A+R). \end{aligned} \tag{16}$$

When $\mu^2 > \max\{\alpha\Lambda, \beta\Lambda\}$, the condition $L'(t) \le 0$ holds, and the sentiment extinction equilibrium point E_0 is globally asymptotically stable.

D. Stability of the equilibrium point of sentiment propagation

Theorem 3. When $A_3 A_4 > \theta(1-c)\varphi$, $a_3(A_1 + A_4) > a_4 \delta c$, and $B_3(B_1 B_2 - B_3) > B_1^2 B_4$, the sentiment propagation equilibrium point E^* is locally asymptotically stable.

Proof. The Jacobian matrix of the system at the sentiment propagation equilibrium point E^* is:

$$J(E^*) = \begin{pmatrix} -A_1 & -a_3 & -a_4 & 0 & 0 \\ a_1 & -A_4 & 0 & \varphi & 0 \\ a_2 & \delta c & -A_2 & \eta & 0 \\ 0 & \theta(1-c) & 0 & -A_3 & 0 \\ 0 & \lambda & \gamma & \xi & -\mu \end{pmatrix}. \tag{17}$$

Let $a_1 = \alpha I_1^*$, $a_2 = \beta I_2^*$, $a_3 = \alpha S^*$, $a_4 = \beta S^*$, and q be the eigenvalue.

Where

$$\begin{aligned} &A_1 = a_1 + a_2 + \mu, A_2 = -a_4 + \gamma + \mu, \\ &A_3 = \varphi + \eta + \mu + \xi, A_4 = -a_3 + \theta(1-c) + \delta c + \lambda + \mu. \end{aligned} \tag{18}$$

According to the relationship of the equilibrium point, it can be inferred that $A_2, A_4 > 0$, and thus $A_1, A_2, A_3, A_4 > 0$.

The characteristic equation of $J(E^*)$ is:

$$\begin{aligned} |qE - J(E^*)| &= \begin{vmatrix} q + A_1 & a_3 & a_4 & 0 & 0 \\ -a_1 & q + A_4 & 0 & -\varphi & 0 \\ -a_2 & -\delta c & q + A_2 & -\eta & 0 \\ 0 & -\theta(1-c) & 0 & q + A_3 & 0 \\ 0 & -\lambda & -\gamma & -\xi & q + \mu \end{vmatrix} \\ &= (q+\mu)(q^4 + B_1 q^3 + B_2 q^2 + B_3 q + B_4) \\ &= 0. \end{aligned} \tag{19}$$

It is easy to obtain $q_1 = -\mu$, and the other eigenvalues satisfy the following conditions:

$$q^4 + B_1 q^3 + B_2 q^2 + B_3 q + B_4 = 0, \tag{20}$$

where

$$\begin{aligned} B_1 &= A_1 + A_2 + A_3 + A_4, \\ B_2 &= A_1 A_2 + a_1 a_3 + a_2 a_4 + (A_1 + A_2)(A_3 + A_4) \\ &\quad + A_3 A_4 - \theta(1-c)\varphi, \\ B_3 &= (A_1 + A_2)[A_3 A_4 - \theta(1-c)\varphi] + a_1 a_4 \delta c \\ &\quad + (A_1 A_2 + a_2 a_4)(A_3 + A_4) + a_1 a_3 (A_2 + A_3), \\ B_4 &= (A_1 A_2 + a_2 a_4)[A_3 A_4 - \theta(1-c)\varphi] \\ &\quad + [\delta c A_3 + \eta\theta(1-c)] a_1 a_4 + a_1 a_3 A_2 A_3. \end{aligned} \tag{21}$$

Based on the Routh-Hurwitz theorem, the necessary and sufficient condition for the local asymptotic stability of E^* is that its coefficients satisfy the following relationship:

$$\begin{aligned} &B_1 > 0, B_2 > 0, B_3 > 0, B_4 > 0, \\ &B_1 B_2 - B_3 > 0, \\ &B_3(B_1 B_2 - B_3) > B_1^2 B_4. \end{aligned} \tag{22}$$

When $A_3 A_4 > \theta(1-c)\varphi$, we have $B_1 > 0$, $B_2 > 0$, $B_3 > 0$, and $B_4 > 0$.

Let $T = A_3 A_4 - \theta(1-c)\varphi$, and consider $B_1 B_2 - B_3 > 0$.

$$\begin{aligned} B_1 B_2 - B_3 &= (A_1 + A_2 + A_3 + A_4)[A_1 A_2 + a_1 a_3 + a_2 a_4 \\ &\quad + (A_1 + A_2)(A_3 + A_4) + T] \\ &\quad - [(A_1 + A_2)T + a_1 a_4 \delta c + a_1 a_3 (A_2 + A_3) \\ &\quad + (A_1 A_2 + a_2 a_4)(A_3 + A_4)] \\ &= -a_1 a_4 \delta c + a_1 a_3 (A_1 + A_4) \\ &\quad + (A_3 + A_4)[(A_1 + A_2)^2 + T] \\ &\quad + (A_1 + A_2)[(A_3 + A_4)^2 + A_1 A_2 + a_2 a_4]. \end{aligned} \tag{23}$$

Then, when $a_3(A_1 + A_4) > a_4 \delta c$, the inequality $B_1 B_2 - B_3 > 0$ holds.

Due to the analytical complexity involved in evaluating the expression $B_3(B_1 B_2 - B_3) - B_1^2 B_4$, establishing a rigorous theoretical proof of local asymptotic stability at the sentiment propagation equilibrium point proves particularly challenging. Consequently, our study employs numerical methods to prove the model's local stability properties, with detailed results to be presented in the numerical simulation section. It should be noted that our analysis focuses specifically on identifying sufficient conditions for model stability, while recognizing the potential existence of additional numerical solutions beyond those explicitly considered in this work.

Combining the above analysis, we conclude that when $A_3A_4 > \theta(1-c)\varphi, a_3(A_1+A_4) > a_4\delta c$, and $B_3(B_1B_2 - B_3) > B_1^2B_4$, the sentiment propagation equilibrium E^* of the system is locally asymptotically stable.

IV. Optimal control

In the stock market, investors' decision-making behaviors are often influenced by both emotions and information. When positive news emerges or stock prices rise rapidly in the market, investors are easily influenced, leading to a herd mentality and a decision to buy stocks. At the same time, rumors and misinformation in the market may exacerbate such emotional fluctuations, causing some investors to make decisions without thorough rational analysis. To effectively guide investors to maintain a calm decision-making attitude and reduce the occurrence of emotional trading, this paper proposes corresponding control strategies based on the *SI₁I₂AR* sentiment propagation model. Specifically, the conversion rate α and the control variable c have significant impacts on the investor sentiment propagation model. Therefore, this paper transforms them into time-dependent control variables $\alpha(t)$ and $c(t)$. The control variable $c(t)$ aims to reduce the number of investors converting into sentiment pessimistic investors due to rumor propagation, thereby effectively mitigating the spread of negative sentiment. Meanwhile, the control variable $\alpha(t)$ is designed to promote the conversion of potential investors, enhancing the positive sentiment atmosphere in the market. By precisely controlling these two variables, this paper aims to reduce the negative impact of emotional decision on the stock market, thereby helping investors make more rational and scientific investment choices.

Therefore, the following objective function is given:

$$J(\alpha, c)=\int_0^{t_f}\left[I_1(t)+I_2(t)+A(t)+\frac{c_1}{2}\alpha^2(t)+\frac{c_2}{2}c^2(t)\right]dt. \tag{24}$$

The variables in the functional expression satisfy the state equation system in the subsequent analysis:

$$\begin{cases}\dfrac{dS(t)}{dt}=\Lambda-\alpha(t)SI_1-\beta SI_2-\mu S,\\ \dfrac{dI_1(t)}{dt}=\alpha(t)SI_1+\varphi A-\left[\theta(1-c(t))+\delta c(t)+\lambda+\mu\right]I_1,\\ \dfrac{dI_2(t)}{dt}=\beta SI_2+\delta c(t)I_1+\eta A-(\gamma+\mu)I_2,\\ \dfrac{dA(t)}{dt}=\theta(1-c(t))I_1-(\varphi+\eta+\mu+\xi)A,\\ \dfrac{dR(t)}{dt}=\lambda I_1+\gamma I_2+\xi A-\mu R.\end{cases} \tag{25}$$

The initial conditions for the system are:

$$\begin{aligned}&S(0)=S_0, I_1(0)=I_{1_0},\\ &I_2(0)=I_{2_0}, A(0)=A_0, R(0)=R_0,\end{aligned} \tag{26}$$

where

$$\alpha(t), c(t)\in U \triangleq \left\{(\alpha, c)\,\middle|\, \begin{aligned}&(\alpha(t), c(t)) \text{ measurable},\\ &0\le\alpha(t), c(t)\le 1,\ \forall t\in[0,t_f]\end{aligned}\right\}. \tag{27}$$

U is the admissible set of control, and the time interval is controlled within 0 and t_f. c_1, c_2 are positive weight coefficients, denoting the control strength and significance of the two control measures.

A. Existence of optimal control

Theorem 4. Regarding the optimal control problem of the system, there exists an optimal control $u^*=(\alpha^*, c^*)\in U$ as follows:

$$J(\alpha^*, c^*)=\min\left\{J(\alpha, c):(\alpha, c)\in U\right\}. \tag{28}$$

The existence of the optimal control needs to satisfy the following conditions:

1) The set of control variables and state variables is not empty;

2) The control set U is both convex and closed;

3) The right side of the state system is a linear and bounded function of both control and state variables;

4) The integrand of the objective functional is convex over U;

5) There exist constants $d_1, d_2 > 0$ and $\rho > 1$ such that the integrand of the objective functional holds true.

$$L(t;\alpha;c)\triangleq I_1(t)+I_2(t)+A(t)+\frac{c_1}{2}\alpha^2(t)+\frac{c_2}{2}c^2(t), \tag{29}$$

$$L(t;\alpha;c)\ge d_1\left(|\alpha|^2+|c|^2\right)^{\frac{\rho}{2}}-d_2. \tag{30}$$

Proof. If the existence of optimal control is to be proven, all five of the above conditions must be satisfied. Conditions 1-3 are clearly met, and only conditions 4 and 5 need to be verified.

Let $N(t)=S(t)+I_1(t)+I_2(t)+A(t)+R(t)$, and the following inequality holds:

$$\begin{cases}S'\le\Lambda,\\ I_1'\le\alpha SI_1+\varphi A,\\ I_2'\le\beta SI_2+\delta cI_1+\eta A,\\ A'\le\theta(1-c)I_1,\\ R'\le\lambda I_1+\gamma I_2+\xi A.\end{cases} \tag{31}$$

It can be concluded that condition 4 holds true.

Next, the verification of the last condition is carried out:

$$\begin{aligned}-L(t;\alpha;c)&=\frac{c_1\alpha^2(t)+c_2c^2(t)}{2}-I_1(t)-I_2(t)-A(t)\\ &\ge d_1\left(|\alpha|^2+|c|^2\right)^{\frac{\rho}{2}}-2M.\end{aligned} \tag{32}$$

Let $d_1=\min\left\{\frac{c_1}{2},\frac{c_2}{2}\right\}$, $d_2=2M$, and $\rho=2$. It is known that condition 5 is satisfied.

Therefore, the optimal control is proved.

B. Optimal control strategy

After the proof of the optimal control, the calculation of the optimal control strategy is carried out. The Hamiltonian function with a penalty term is defined as follows:

$$\begin{aligned}H=&I_1(t)+I_2(t)+A(t)+\frac{c_1}{2}\alpha^2(t)+\frac{c_2}{2}c^2(t)\\&+\lambda_1\left[\Lambda-\alpha(t)SI_1-\beta SI_2-\mu S\right]\\&+\lambda_2\left\{\alpha(t)SI_1+\varphi A-\left[\theta(1-c(t))+\delta c(t)+\lambda+\mu\right]I_1\right\}\\&+\lambda_3\left[\beta SI_2+\delta c(t)I_1+\eta A-(\gamma+\mu)I_2\right]\\&+\lambda_4\left[\theta(1-c(t))I_1-(\varphi+\eta+\mu+\xi)A\right]\\&+\lambda_5\left[\lambda I_1+\gamma I_2+\xi A-\mu R\right]\\&-\omega_{11}\alpha(t)-\omega_{12}(1-\alpha(t))-\omega_{21}c(t)-\omega_{22}(1-c(t)).\end{aligned}\tag{33}$$

Among them, $\omega_{ij}(t)\geq 0$ is the penalty operator and satisfies:

$$\begin{cases}\omega_{11}(t)\alpha^*=\omega_{12}(t)(1-\alpha^*)=0,\\\omega_{21}(t)c^*=\omega_{22}(t)(1-c^*)=0.\end{cases}\tag{34}$$

There exist adjoint variables λ_i ($i=1,2,3,4,5$), which satisfy:

$$\begin{cases}\lambda_1'=&\lambda_1(\alpha(t)I_1+\beta I_2+\mu)-\lambda_2\alpha(t)I_1-\lambda_3\beta I_2,\\\lambda_2'=&-1+\lambda_1\alpha(t)S\\&-\lambda_2\left\{\alpha(t)S-\left[\theta(1-c(t))+\delta c(t)+\lambda+\mu\right]\right\}\\&-\lambda_3\delta c(t)-\lambda_4\theta(1-c(t))-\lambda_5\lambda,\\\lambda_3'=&-1+\lambda_1\beta S-\lambda_3(\beta S-\gamma-\mu)-\lambda_5\gamma,\\\lambda_4'=&-1-\lambda_2\varphi-\lambda_3\eta+\lambda_4(\varphi+\eta+\mu+\xi)-\lambda_5\xi,\\\lambda_5'=&\lambda_5\mu.\end{cases}\tag{35}$$

Thus, the expression of the optimal control is (α^*,c^*):

$$\begin{cases}\alpha^*=\min\left\{1,\max\left\{0,\frac{1}{c_1}(\lambda_1-\lambda_2)S(t)I_1(t)\right\}\right\},\\c^*=\min\left\{1,\max\left\{0,\frac{1}{c_2}\left[(\lambda_2-\lambda_3)\delta+(\lambda_4-\lambda_2)\theta\right]I_1(t)\right\}\right\}.\end{cases}\tag{36}$$

Proof. Based on Pontryagin's maximum principle, the reciprocals of the Hamiltonian operators for each state variable are computed, yielding the synergetic system.

$$\begin{gathered}\lambda_1'=-\frac{\partial H}{\partial S},\quad\lambda_2'=-\frac{\partial H}{\partial I_1},\quad\lambda_3'=-\frac{\partial H}{\partial I_2},\\\lambda_4'=-\frac{\partial H}{\partial A},\quad\lambda_5'=-\frac{\partial H}{\partial R}.\\\lambda_i(t_f)=0,\quad i=1,2,3,4,5.\end{gathered}\tag{37}$$

Next, we discuss how to obtain the optimal conditions. Taking the partial derivative of the Hamiltonian operator with respect to the control variable $U=(\alpha,c)$, and setting the derivative to zero, we get:

$$\begin{cases}\frac{\partial H}{\partial\alpha}=&c_1\alpha(t)-\lambda_1S(t)I_1(t)+\lambda_2S(t)I_1(t)\\&-\omega_{11}+\omega_{12}=0,\\\frac{\partial H}{\partial c}=&c_2c(t)+\lambda_2(\theta-\delta)I_1(t)+\lambda_3\delta I_1(t)-\lambda_4\theta I_1(t)\\&-\omega_{21}+\omega_{22}=0.\end{cases}\tag{38}$$

Solve for the optimal control expression therefrom:

$$\begin{cases}\alpha^*(t)=\frac{1}{c_1}\left[(\lambda_1-\lambda_2)S(t)I_1(t)+\omega_{11}-\omega_{12}\right],\\c^*(t)=\frac{1}{c_2}\left[(\lambda_2-\lambda_3)\delta I_1(t)+(\lambda_4-\lambda_2)\theta I_1(t)+\omega_{21}-\omega_{22}\right].\end{cases}\tag{39}$$

First, consider α^*.

1) On the set $0<\alpha^*(t)<1$, let $\omega_{11}(t)=\omega_{12}(t)=0$, so $\alpha^*(t)=\frac{1}{c_1}\left[(\lambda_1-\lambda_2)S(t)I_1(t)\right]$.

2) On the set $\alpha^*(t)=1$, let $\omega_{11}(t)=0$, so $1=\alpha^*(t)=\frac{1}{c_1}\left[(\lambda_1-\lambda_2)S(t)I_1(t)-\omega_{12}\right]$.

3) On the set $\alpha^*(t)=0$, let $\omega_{12}(t)=0$, so $0=\alpha^*(t)=\frac{1}{c_1}\left[(\lambda_1-\lambda_2)S(t)I_1(t)+\omega_{11}\right]$.

Therefore,

$$\alpha^*=\min\left\{1,\max\left\{0,\frac{1}{c_1}\left[(\lambda_1-\lambda_2)S(t)I_1(t)\right]\right\}\right\}.\tag{40}$$

Similarly, the expression of another control variable is:

$$c^*=\min\left\{1,\max\left\{0,\frac{1}{c_2}\left[(\lambda_2-\lambda_3)\delta+(\lambda_4-\lambda_2)\theta\right]I_1(t)\right\}\right\}.\tag{41}$$

The optimal system is as follows:

$$\begin{cases}S'=\Lambda-\alpha(t)SI_1-\beta SI_2-\mu S,\\I_1'=\alpha(t)SI_1+\varphi A-\left[\theta(1-c(t))+\delta c(t)+\lambda+\mu\right]I_1,\\I_2'=\beta SI_2+\delta c(t)I_1+\eta A-(\gamma+\mu)I_2,\\A'=\theta(1-c(t))I_1-(\varphi+\eta+\mu+\xi)A,\\R'=\lambda I_1+\gamma I_2+\xi A-\mu R,\\\lambda_1'=\lambda_1(\alpha(t)I_1+\beta I_2+\mu)-\lambda_2\alpha(t)I_1-\lambda_3\beta I_2,\\\lambda_2'=-1+\lambda_1\alpha(t)S\\\qquad-\lambda_2\left\{\alpha(t)S-\left[\theta(1-c(t))+\delta c(t)+\lambda+\mu\right]\right\}\\\qquad-\lambda_3\delta c(t)-\lambda_4\theta(1-c(t))-\lambda_5\lambda,\\\lambda_3'=-1+\lambda_1\beta S-\lambda_3(\beta S-\gamma-\mu)-\lambda_5\gamma,\\\lambda_4'=-1-\lambda_2\varphi-\lambda_3\eta+\lambda_4(\varphi+\eta+\mu+\xi)-\lambda_5\xi,\\\lambda_5'=\lambda_5\mu,\\S(0)=S_0,I_1(0)=I_{1_0},I_2(0)=I_{2_0},A(0)=A_0,R(0)=R_0,\\\lambda_i(t_f)=0,i=1,2,3,4,5.\end{cases}\tag{42}$$

V. NUMERICAL SIMULATION

A. Numerical simulation of system stability

In this section, numerical simulations will be conducted to validate the stability conditions of the equilibrium points derived previously and to analyze the impact of different parameters on the dynamic changes in the investor population. Additionally, the optimal control problem will be thoroughly verified to evaluate the feasibility of the proposed control strategies.

In the process of numerical simulation, the selection of parameter values is not fixed. By reviewing the literature related to investor sentiment propagation, it is found that although these parameters do not have a definitive range of values, they are typically positive and must satisfy the stability conditions [34]. Therefore, this paper refers to the parameter settings in existing studies and, in combination with the requirements of the stability conditions, assigns specific values to the parameters in the model.

Let $\Lambda=1$, $\alpha=0.7$, $\beta=0.3$, $\theta=0.2$, $\delta=0.05$, $\xi=0.8$, $\eta=0.3$, $\lambda=0.6$, $\varphi=0.2$, $\gamma=0.6$, $\mu=0.6$, and $c=0.5$. Substituting these into the formula, we obtain $R_2=\frac{\beta\Lambda}{\mu(\gamma+\mu)}=0.6<1$, and $\delta c+\lambda+\mu-\frac{\alpha\Lambda}{\mu}=0.0583>0$. As shown in Fig. 2, over time, $S(t)$ gradually increases and stabilizes after reaching 1. $R(t)$ rises briefly at first, then declines rapidly, eventually approaching 0. $I_1(t)$, $I_2(t)$, and $A(t)$ decrease quickly and eventually become 0. In the end, only the potential investors $S(t)$ remain in the system, while $I_1(t)$, $I_2(t)$, $A(t)$, and $R(t)$ all become 0. This indicates that the sentiment extinction equilibrium is stable.

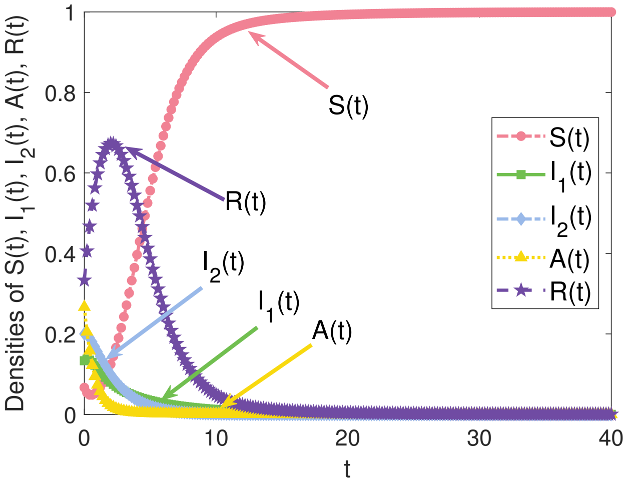


Fig. 2. Stability of equilibrium point E_0.

Let $\Lambda = 1$, $\alpha = 0.6$, $\beta = 0.5$, $\theta = 0.33$, $\delta = 0.05$, $\xi = 0.8$, $\eta = 0.7$, $\lambda = 0.1$, $\varphi = 0.3$, $\gamma = 0.6$, $\mu = 0.4$, and $c = 0.3$. Calculations yield $R_1 = 2.0107 > R_2 = 1.25 > 1$, $A_3A_4 - \theta(1-c)\varphi = 4.4 \times 10^{-10} > 0$, $a_3(A_1 + A_4) - a_4\delta c = 0.6136 > 0$, and $B_3(B_1B_2 - B_3) - B_1^2B_4 = 13.75697 > 0$. From Fig. 3(a), it can be observed that the number of rational investors initially increases and then decreases, while the densities of potential investors and optimistic spreaders gradually increase from the initial time. At the same time, the densities of pessimistic investors and hesitant investors gradually decline. Nevertheless, sentiment propagation does not completely vanish, and the densities of various types of sentiment investors eventually stabilize within a certain range, indicating that the system tends to the sentiment propagation equilibrium. Fig. 3(b) shows that under the same parameter conditions, even if the initial densities of investors in different states vary, the system ultimately converges to a unique sentiment propagation equilibrium E^*, determined by the system parameters.

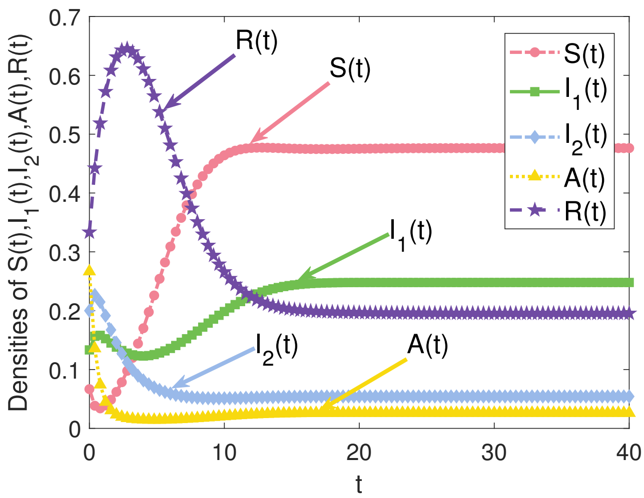
TABLE II
THE COMPARTMENTS AND VALUES OF THE PARAMETERS.

Compartments and parameters	Values
$S(t)$	50
$I_1(t)$	50
$I_2(t)$	50
$A(t)$	50
$R(t)$	50
Λ	100
α	0.2
β	0.1
μ	0.4
φ	0.3
θ	0.4
γ	0.6
ξ	0.8
δ	0.7
λ	0.1
η	0.7
c	0.3

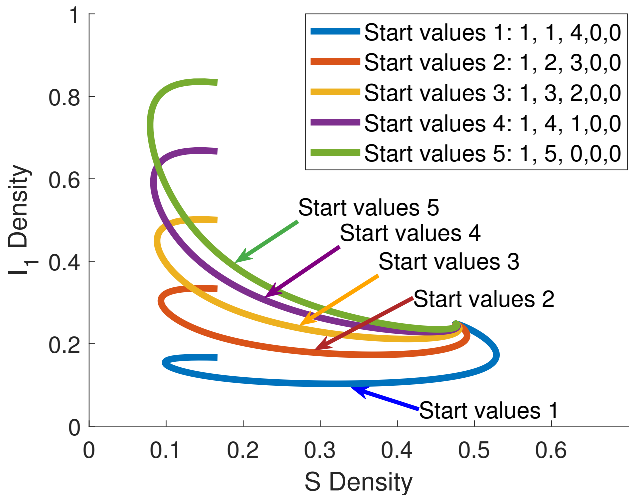
To explore the impact of different parameters in the system on the investor sentiment propagation model, this paper constructs Table II based on the assumption of uniform distribution. Using the data in Table II, dynamic change curves of $I_1(t)$, $I_2(t)$, and $A(t)$ under varying parameters are plotted, and the patterns reflected by these changes are further analyzed.

Fig. 4 illustrates the trend of the number of sentiment optimistic investors I_1 with respect to the parameters α, δ, c, and θ. As can be seen from the figure, the parameters α is positively correlated with $I_1(t)$, while the parameters δ, c, and θ are negatively correlated with $I_1(t)$. Therefore, to increase the number of sentiment-optimistic investors, one can increase the parameters α while decreasing the parameters δ, c, and θ. This approach provides a clear and effective strategy for influencing investor sentiment in a positive direction.

Fig. 5 depicts the variation of the number of sentiment pessimistic investors I_2 with respect to the parameters β, δ, c, and η. It can be observed from the figure that the parameters β, δ, c, and η are positively correlated with $I_2(t)$, indicating that an increase in these parameters leads to an increase in the number of sentiment-pessimistic investors. To effectively reduce the number of sentiment-pessimistic investors, it is recommended to decrease the values of key parameters including β, δ, c, and η. This strategy can help reduce the negative impact of pessimistic sentiment and foster a more balanced market environment.



(a) Stability of equilibrium point E_1.



(b) Convergence to E^* under the different initial densities.

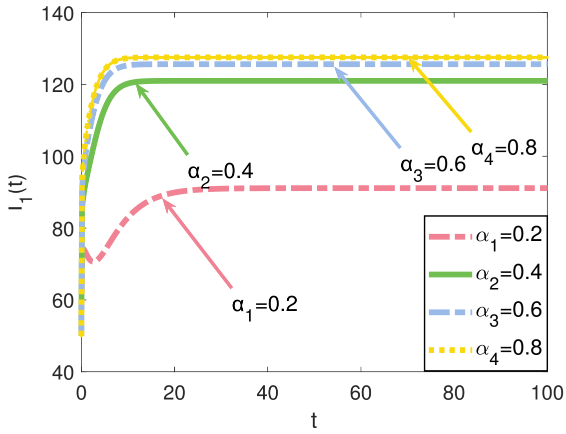
Fig. 3. The stability analysis of equilibrium point E^*.

Fig. 6 illustrates the variation of the number of hesitant investors A with respect to the parameters φ, c, η, and ξ. As can be observed from the figure, the parameters φ, c, η, and ξ exhibit a negative correlation with $A(t)$. Specifically, an increase in these parameters leads to a decrease in the number of hesitant investors A. Therefore, to effectively reduce the number of hesitant investors A, it is recommended to increase the values of φ, c, η, and ξ.

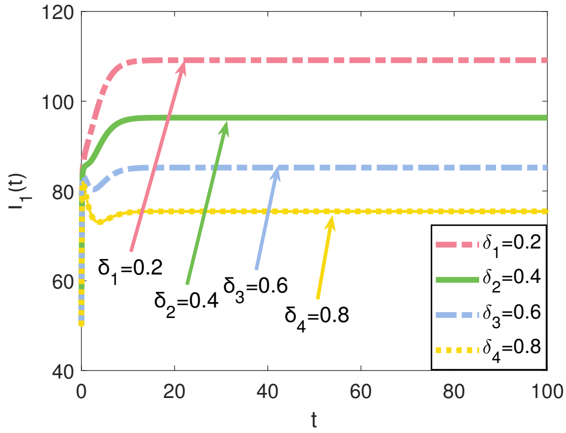
After thoroughly exploring the influence of individual parameters on the dynamic changes of sentiment-optimistic spreaders, sentiment-pessimistic spreaders, and hesitant investors, this paper will further investigate the evolution of the peak values of sentiment spreaders under the interaction of two parameters. Fig. 7 and 8 illustrate the peak values of the density change curves for sentiment-optimistic investors and sentiment-pessimistic investors in the system.

When $\Lambda = 1$, $\beta = 0.5$, $\theta = 0.1$, $\delta = 0.6$, $\xi = 0.2$, $\eta = 0.2$, $\lambda = 0.1$, $\varphi = 0.3$, $\gamma = 0.6$, and $\mu = 0.4$. In the Fig. 7, It is observed that the combined effect of increasing α and decreasing c significantly accelerates the rate at which sentiment-optimistic spreaders attain higher density peaks. This is because an increase in α enhances the conversion rate of potential investors into sentiment-optimistic spreaders, while a decrease in c reduces the likelihood of optimistic spreaders being influenced by rumors and converting into pessimistic spreaders, thereby further strengthening the growth trend of sentiment-optimistic spreaders. Conversely, when α decreases or c increases, the density peak of sentiment optimistic spreaders significantly decreases. Specifically, when α is fixed at 0.8, an increase in c leads to a decline in the peak of sentiment-optimistic spreaders, as more optimistic spreaders are influenced by rumors and then convert into pessimistic spreaders. On the other hand, when c is fixed at 0.2, an increase in α causes the peak of sentiment-optimistic spreaders to rise, as more potential investors convert into sentiment-optimistic investors, driving the growth of their density.

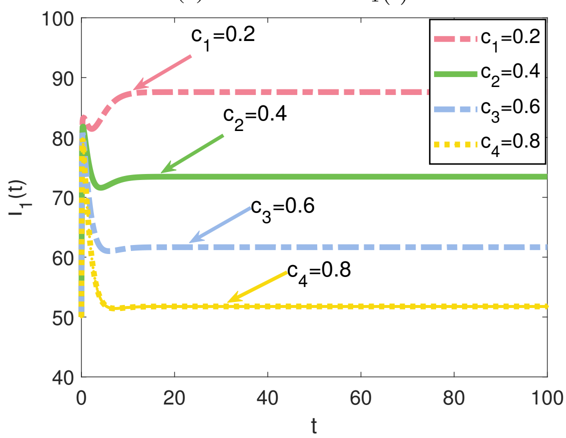
Let $\Lambda = 1$, $\alpha = 0.6$, $\theta = 0.1$, $\delta = 0.6$, $\xi = 0.2$, $\eta = 0.2$, $\lambda = 0.1$, $\varphi = 0.3$, $\gamma = 0.6$, and $\mu = 0.4$. In Fig. 8, the trend of the peak density of sentiment-pessimistic investors can be clearly observed. When the values of parameters β and c increase, the effect of sentiment-pessimistic propagation is significantly enhanced, leading to a continuous rise in the density peak and reaching a higher level. Specifically, an increase in β enhances the conversion rate of potential investors into sentiment-pessimistic spreaders, while an increase in c raises the probability of sentiment-optimistic spreaders being influenced by rumors and converting into sentiment-pessimistic spreaders. The combined effect of these two factors further amplifies the pessimistic propagation. In particular, when β is fixed at 0.8, an increase in c leads to a rise in the peak of sentiment-pessimistic investors, as more sentiment-optimistic spreaders are influenced by rumors and convert into sentiment-pessimistic spreaders. On the other hand, when c is fixed at 0.8, an increase in β also causes the peak to rise significantly, as more potential investors directly convert into sentiment-pessimistic spreaders, thereby exacerbating the number of sentiment-pessimistic spreaders. This phenomenon indicates that the interaction between β and c has a significant synergistic enhancement effect on the peak density of sentiment-pessimistic spreaders.



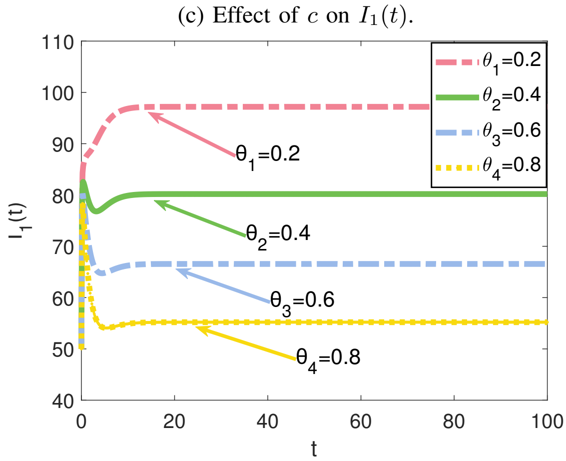
(a) Effect of α on $I_1(t)$.



(b) Effect of δ on $I_1(t)$.

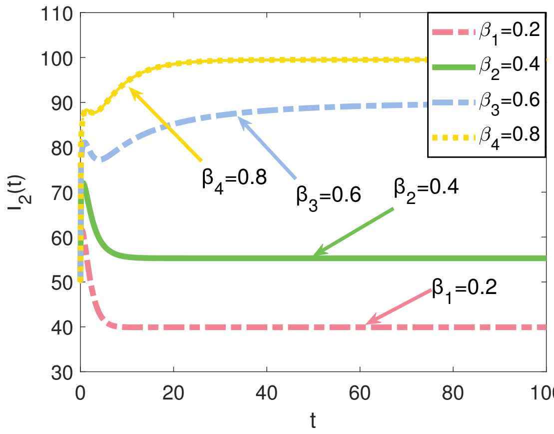


(c) Effect of c on $I_1(t)$.

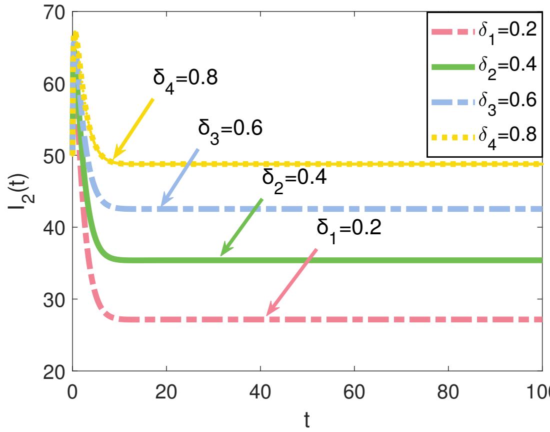


(d) Effect of θ on $I_1(t)$.

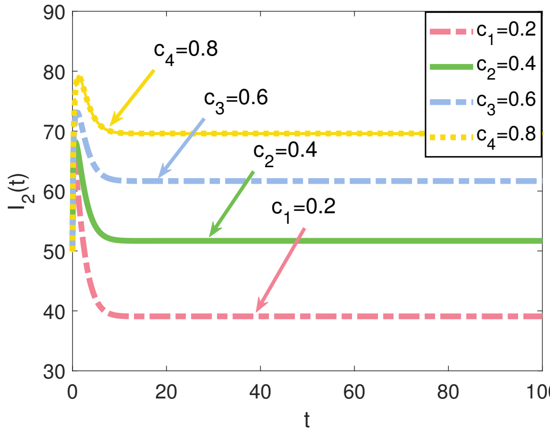
Fig. 4. The effect of $\alpha, \delta, c, \theta$ on the number of $I_1(t)$.



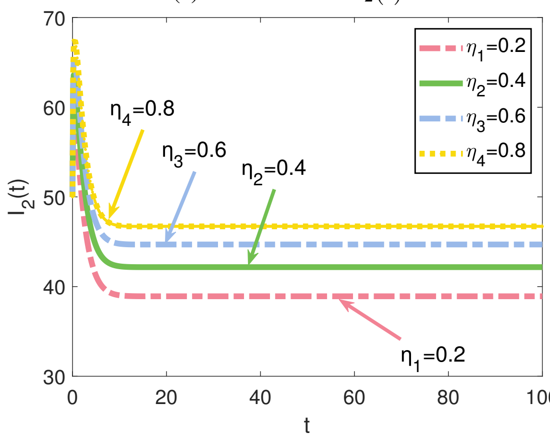
(a) Effect of β on $I_2(t)$.



(b) Effect of δ on $I_2(t)$.

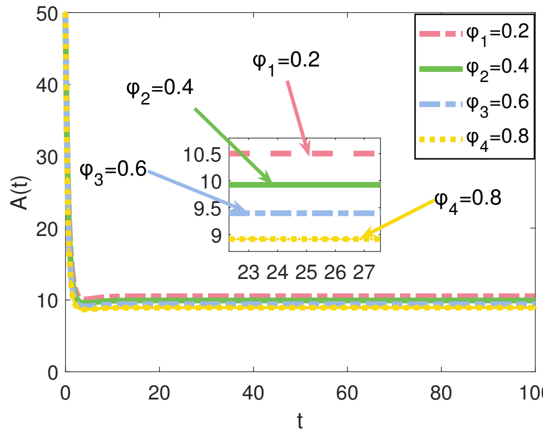


(c) Effect of c on $I_2(t)$.

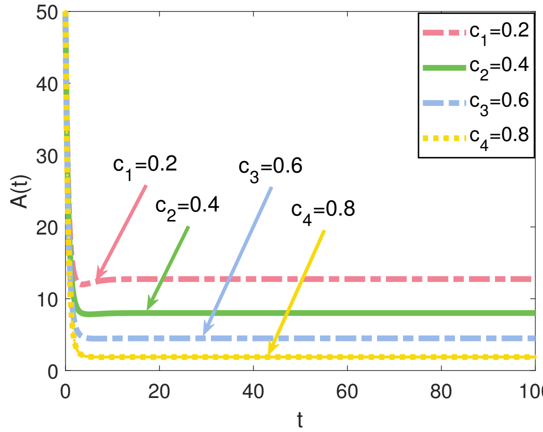


(d) Effect of η on $I_2(t)$.

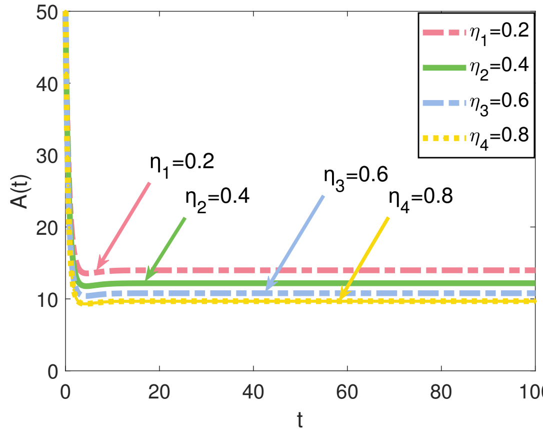
Fig. 5. The effect of β, δ, c, η on the number of $I_2(t)$.



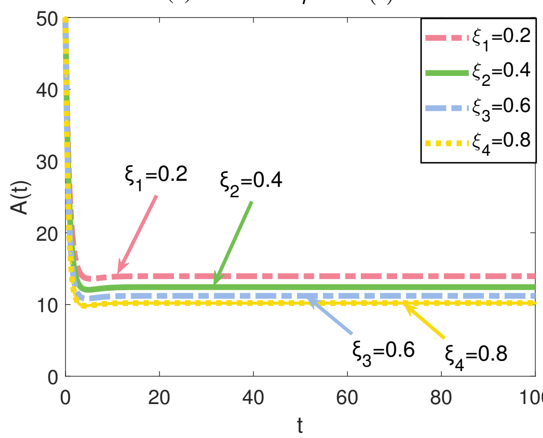
(a) Effect of φ on $A(t)$.



(b) Effect of c on $A(t)$.



(c) Effect of η on $A(t)$.



(d) Effect of ξ on $A(t)$.

Fig. 6. The effect of φ, c, η, ξ on the number of $A(t)$.

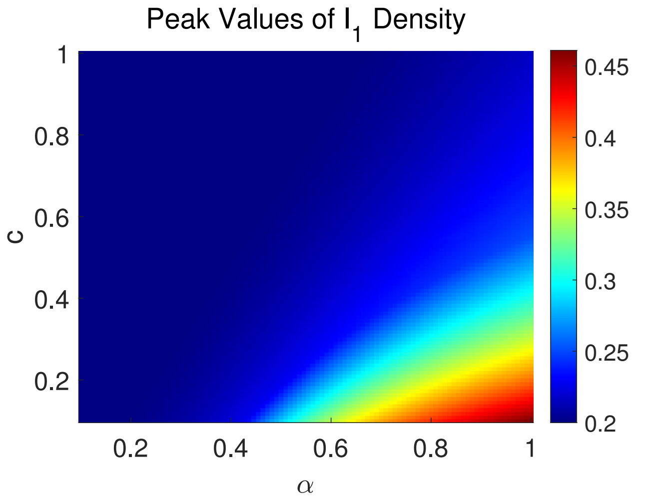


Fig. 7. The peak values of the density variation curve of $I_1(t)$ when α and c take different values.

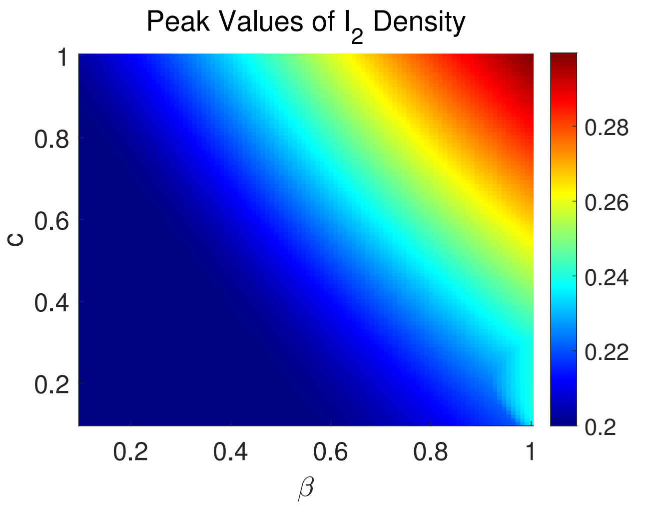


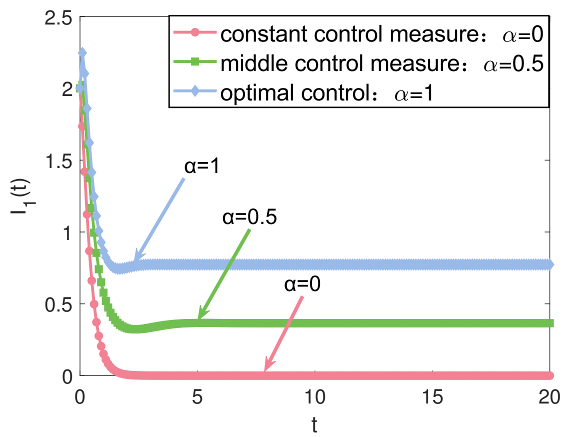
Fig. 8. The peak values of the density variation curve of $I_2(t)$ when β and c take different values.

B. Optimal control analysis

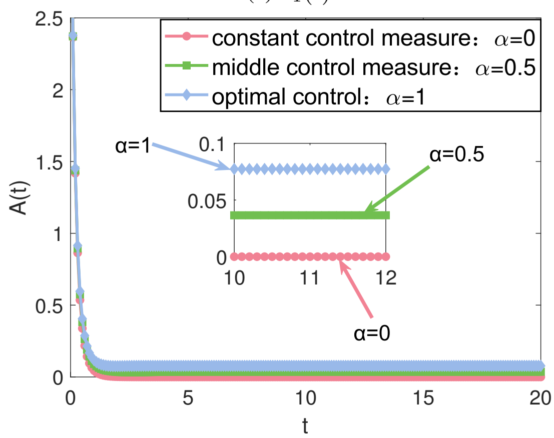
Based on the above analysis, we now proceed to perform optimal control on the parameters α and c. Because the optimal control primarily targets the three groups of sentiment optimistic investors, sentiment-pessimistic investors, and hesitant investors affected by rumors, only $I_1(t)$, $I_2(t)$, and $A(t)$ are displayed in the plotting process.

First, let $\Lambda = 1$, $\beta = 0.5$, $\theta = 0.4$, $\delta = 0.05$, $\xi = 0.8$, $\eta = 0.7$, $\lambda = 0.1$, $\varphi = 0.3$, $\gamma = 0.6$, $\mu = 0.4$, and $c = 0.3$, and control α. Fig. 9 show the trends of the densities of sentiment optimistic investors $I_1(t)$ and hesitant investors $A(t)$ over time under different control strategies. The results indicate that the optimal control strategy performs significantly better than the intermediate control strategy, which in turn outperforms the no-control strategy. Under the optimal control strategy, the number of sentiment-optimistic investors reaches its maximum, demonstrating that this strategy effectively promotes the propagation of positive sentiment. At the same time, as the number of sentiment-optimistic investors I_1 increases, the number of hesitant investors A also rises accordingly. This is because more potential investors are converted into sentiment optimistic investors under the influence of the optimal control α, while some sentiment-optimistic investors may transition into a hesitant state during the decision-making process. Therefore, the optimal control strategy not only significantly increases the number of sentiment-optimistic investors but also indirectly leads to an increase in the hesitant group.

Next, let $\Lambda = 1$, $\alpha = 0.9$, $\beta = 0.5$, $\theta = 0.5$, $\delta = 0.7$, $\xi = 0.3$, $\eta = 0.2$, $\lambda = 0.2$, $\varphi = 0.3$, $\gamma = 0.4$, $\mu = 0.4$, and control c. Fig. 10 show the trends of the densities of $I_1(t)$, $I_2(t)$, and $A(t)$ over time under different control strategies. As can be seen from the figures, under the optimal control strategy for rumor propagation, the numbers of sentiment-optimistic investors and hesitant investors both reach their maximum values, while the number of sentiment-pessimistic investors drops to its minimum. This result indicates that the optimal control strategy can effectively suppress the negative impact of rumors, promote the propagation of positive sentiment, and simultaneously reduce the number of sentiment-pessimistic spreaders, thereby validating the effectiveness of the control strategy.



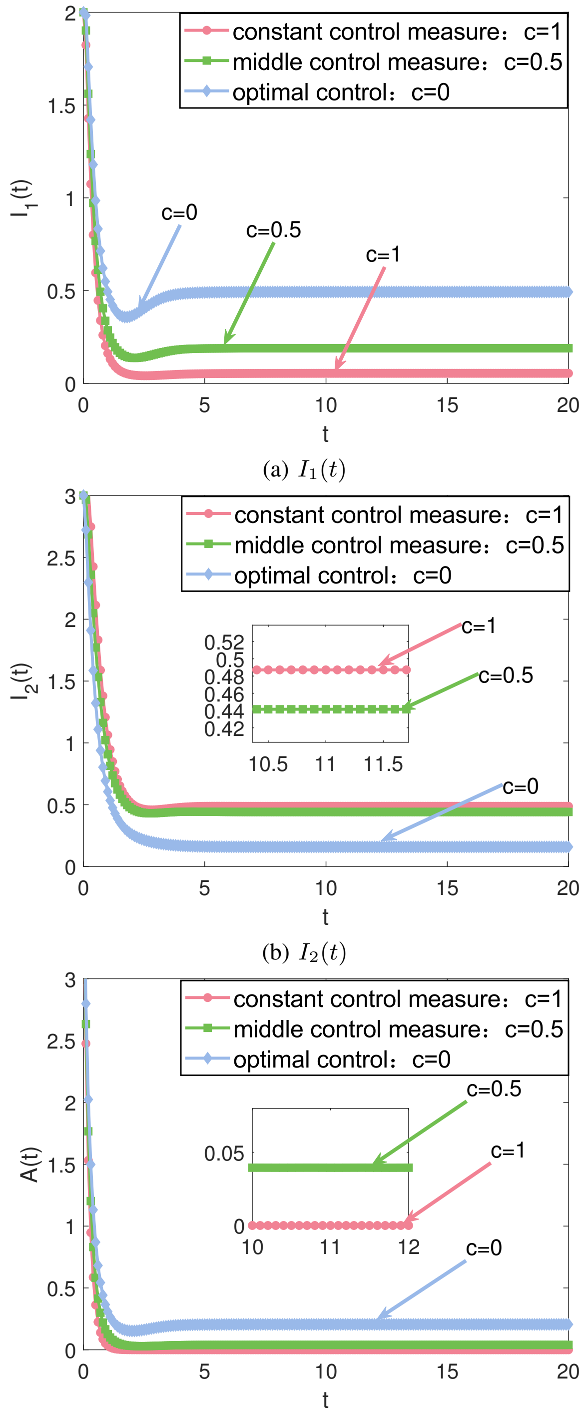
(a) $I_1(t)$



(b) $A(t)$

Fig. 9. The density variations of $I_1(t), A(t)$ with different values of α.

Finally, let $\Lambda = 2$, $\beta = 0.5$, $\theta = 0.5$, $\delta = 0.05$, $\xi = 0.3$, $\eta = 0.4$, $\lambda = 0.2$, $\varphi = 0.1$, $\gamma = 0.5$, $\mu = 0.25$, and control both α and c. The results reveal that the effectiveness of the optimal control strategy is significantly enhanced when the parameters α and c are controlled simultaneously, as illustrated in Fig. 11. An increase in α improves the conversion rate of potential investors into sentiment-optimistic investors, while a decrease in c suppresses the negative impact of rumors on sentiment optimistic investors. This dual control mechanism maximizes the number of $I_1(t)$, while the number of $A(t)$ also rises due to the decision-making hesitation of some investors. Additionally, a decrease in c effectively reduces the number of sentiment-pessimistic investors. Therefore, under the optimal strategy of simultaneously controlling α and c, the numbers of sentiment-optimistic and hesitant investors both reach their peaks, while the number of sentiment-pessimistic investors drops to its minimum, fully demonstrating the substantial effectiveness of strategy in optimizing sentiment propagation.



(a) $I_1(t)$

(b) $I_2(t)$

(c) $A(t)$

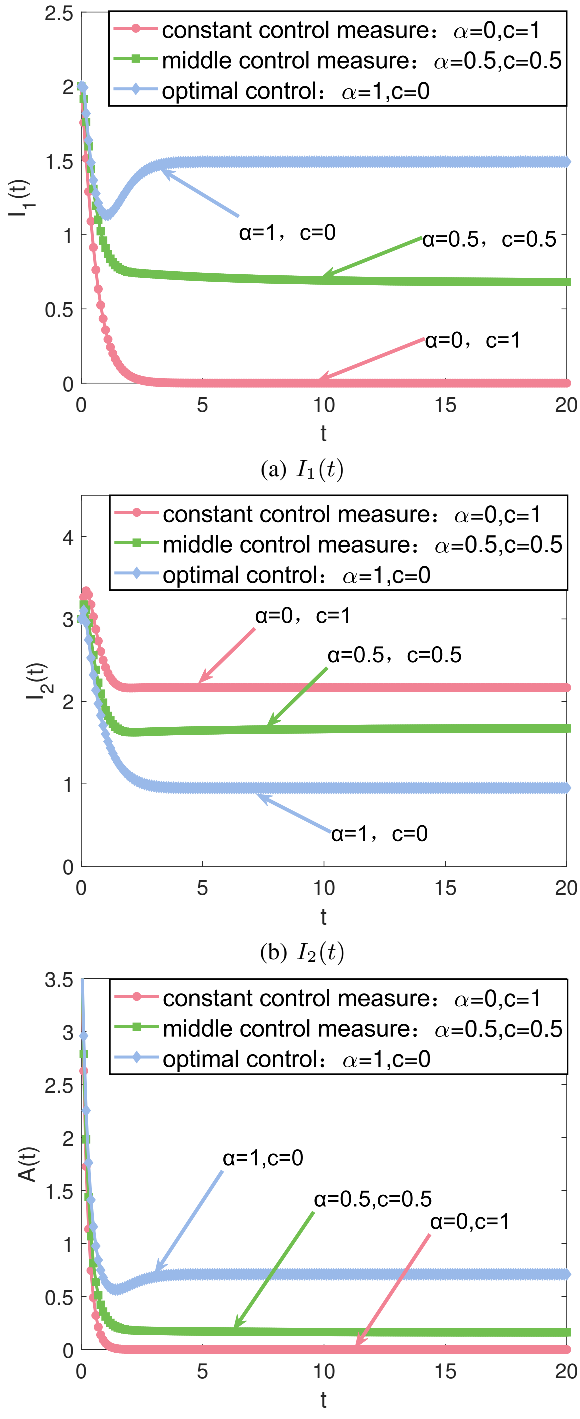
Fig. 10. The density variations of $I_1(t), I_2(t), A(t)$ with different values of c.

VI. Sensitivity analysis

To discuss the influence of parameters α and λ on R_1 as well as β and γ on R_2, we analyzed the basic reproduction numbers. Additionally, the corresponding three-dimensional plots are shown in Fig. 12 and 13, respectively.

It can be calculated,

$$\begin{aligned} \frac{\partial R_1}{\partial \alpha} &= \frac{\Lambda}{\mu\left[\theta(1-c)+\delta c+\lambda+\mu\right]} > 0, \\ \frac{\partial R_1}{\partial \lambda} &= -\frac{\alpha\Lambda}{\mu\left[\theta(1-c)+\delta c+\lambda+\mu\right]^2} < 0. \end{aligned} \tag{43}$$



(a) $I_1(t)$

(b) $I_2(t)$

(c) $A(t)$

Fig. 11. The density variations of $I_1(t), I_2(t), A(t)$ with different values of α, c.

Fig. 12 identifies $\alpha = 0.9$ and $\lambda = 0.1$ as the parameter combination yielding maximum R_1 values, indicating peak system sensitivity. Notably, when the parameter α increases, the basic reproduction number R_1 also increases. This indicates that the higher the contact rate between potential investors and positive spreaders, the greater the number of positive spreaders. On the other hand, when the parameter λ increases, the basic reproduction number R_1 decreases. This indicates that as the rate of transformation of positive spreaders into rational investors increases, the number of positive investors gradually decreases. Thus, by increasing the contact rate of positive spreaders and optimizing the sentiment propagation environment, the dynamic balance of sentiment propagation can be effectively regulated.

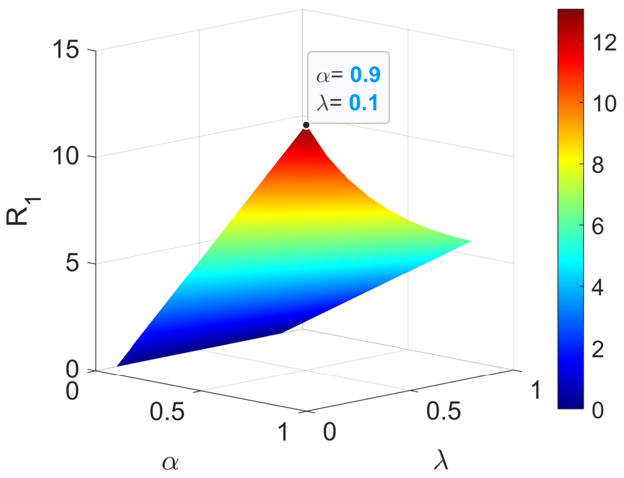


Fig. 12. The sensitivity analysis of R_1.

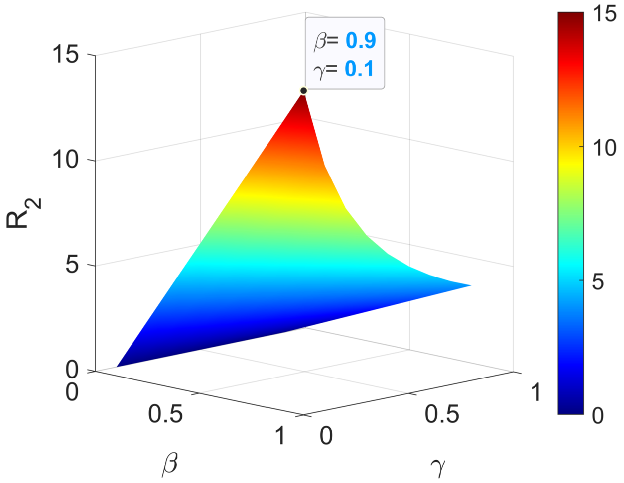


Fig. 13. The sensitivity analysis of R_2.

$$\frac{\partial R_2}{\partial \beta} = \frac{\Lambda}{\mu(\gamma+\mu)} > 0, \frac{\partial R_2}{\partial \gamma} = -\frac{\beta\Lambda}{\mu(\gamma+\mu)^2} < 0. \quad (44)$$

From the Fig. 13, it is clear that R_2 maximizes at $\beta = 0.9$ and $\gamma = 0.1$, indicating peak sensitivity. Specifically, when the parameter β increases, the basic reproduction number R_2 also increases. This indicates that the higher the contact rate between potential investors and negative spreaders, the greater the number of negative spreaders. On the other hand, when the parameter γ increases, the basic reproduction number R_2 decreases. This shows that as the rate at which negative spreaders transform into rational investors increases, the number of negative spreaders gradually decreases. Therefore, by reducing the contact rate between potential investors and negative spreaders and increasing the conversion rate of negative spreaders, the spread of negative sentiment can be effectively suppressed.

VII. Conclusion

Investor sentiment serves as a pivotal factor in financial markets, where the spread of rumors significantly amplifies emotional volatility, leading to a loss of market stability and distorting investor decision-making processes. To address this phenomenon, this study develops a novel investor sentiment propagation model that explicitly incorporates the influence of rumors. Through rigorous analysis of the existence and stability conditions for both sentiment extinction equilibrium and sentiment propagation equilibrium, we propose an optimal control strategy. The validity of our model is systematically verified via comprehensive numerical simulations. The key conclusions are summarized below.

Our analysis reveals that heightened rumor propagation intensity leads to a marked increase in sentiment-pessimistic investors while significantly suppressing sentiment-optimistic investors. Notably, during uncertain market conditions, rumor dissemination amplifies investor panic, triggering frequent irrational decision-making behaviors. To counteract these adverse effects, policymakers and regulators should prioritize two key measures: tightening control over rumor spread to curb false information, and enhancing investors' financial literacy and information-discernment capabilities. What's more, our findings suggest that dynamically adjusting critical sentiment propagation parameters—such as the sentiment propagation rate α and the rumor influence coefficient c—can effectively mitigate rumor-driven sentiment volatility and promote market stability, while future research should explore time-delay effects and external shocks to better reflect real world dynamics.

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