Finite Time Identical Synchronization in Full Networks of n Linearly Coupled Dynamical Systems of the Hindmarsh-Rose 3D Type

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Abstract—**This paper proposes an adaptive nonlinear controller designed to achieve finite-time identical synchronization across a full network of n nodes. Each node is represented by a dynamical system of the Hindmarsh-Rose 3D type. The author also presents numerical results using R programming to verify the effectiveness of the theoretical findings.**

Index Terms—**controller, finite time identical synchronization, full network, Hindmarsh-Rose 3D model.**

I. Introduction

SYNCHRONIZATION is a captivating phenomenon that has been thoroughly explored across diverse fields and natural systems [3], [19], [10], [14], [11], [7], [13]. At its core, synchronization represents the remarkable ability of different systems to display identical behavior simultaneously [3]. In recent years, the allure of complex dynamical networks has surged, driven by their vast potential in critical domains such as information processing, the World Wide Web, biological systems, and neural networks [21], [22], [23], [8]. The implications of these studies are profound, promising to reshape our understanding of interconnected systems and enhance our capabilities in these essential areas.

In recent decades, complex networks have emerged as a focal point of research, captivating attention across various fields such as food webs, communication systems, and the Internet. The study of control and synchronization within these networks is not only essential for understanding numerous phenomena in nature and society, but it also holds immense potential for real-world applications [3], [5], [7], [8], [10], [11], [12], [13], [14], [19], [21], [22], [23]. Significant strides have already been made in unraveling the complexities of synchronization in these networks [1], [2], [3], [15], [16].

Despite the advances in this area, most existing research has centered around asymptotic synchronization wherein networks achieve synchronization over an infinite time [1], [2], [3], [6], [15], [16], [18]. However, in practical scenarios, the need for synchronization within a finite time frame is paramount. This pressing requirement has prompted a growing interest among scholars in exploring finite-time synchronization of complex dynamic networks.

This paper takes a bold step forward by focusing on finite-time identical synchronization in fully coupled networks of n linearly coupled dynamical systems modeled on the Hindmarsh-Rose 3D framework. By harnessing the principles of finite-time stability theory and devising a suitable controller along with an effective Lyapunov function, we demonstrate that these networks can indeed achieve finite-time identical synchronization. Furthermore, we lay out clear and sufficient conditions necessary for this synchronization to occur.

To bolster our theoretical claims, we present robust numerical results that showcase the effectiveness of our approach.

The structure of this paper is as follows: Section 2 provides a comprehensive overview of the model, foundational concepts, and key findings. In Section 3, we detail the numerical results that validate our theoretical framework, while Section 4 wraps up with insightful conclusions. Together, these contributions promise to advance our understanding of synchronization in complex networks, offering pathways for impactful applications in various domains.

II. Finite time identical synchronization in full networks of n dynamical systems of the Hindmarsh-Rose 3D type

In 1952, A. L. Hodgkin and A. F. Huxley published a groundbreaking paper that introduced a mathematical model consisting of four ordinary differential equations to approximate certain properties of neuronal membrane potential [6], [7], [10]. For their remarkable work, they were awarded the Nobel Prize. Building on their pioneering study, many scientists have sought to simplify Hodgkin-Huxleys model while retaining the significant energetic and biological properties of cells.

Among these researchers were J. L. Hindmarsh and R. M. Rose, who introduced a simpler system known as the Hindmarsh-Rose 3D model in 1984 [7], [9]. This model consists of three ordinary differential equations, which simplify Hodgkin-Huxleys system and provide insight into neuronal voltage dynamics. It includes two variables, u and v, along with a third variable w. The first variable, $u = u(t)$, represents the transmembrane voltage of the cell. The second variable, $v = v(t)$, and the third variable, $w = w(t)$, represent physical quantities such as the electrical conductivity of ion currents across the membrane. All those variables depend on time t. The ordinary differential equations of the Hindmarsh-Rose 3D model are given below [6], [7], [9]:

$$\begin{cases} \dfrac{du}{dt} = u_t = f(u) + v - w + I, \\ \\ \dfrac{dv}{dt} = v_t = 1 - bu^2 - v, \\ \\ \dfrac{dw}{dt} = w_t = r(s(u-c) - w), \end{cases} \tag{1}$$

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where $u = u(t), v = v(t), w = w(t)$; $f(u) = -u^3 + au^2$; a, b, c, r, s are constants $(a, b, r, s > 0)$; I presents the external current; t presents the time.

In this discussion, we redefine the dynamical system (1) as a neural model, paving the way for the development of a comprehensive network composed of n linearly coupled systems (1). This innovative approach promises to unlock new insights and deepen our understanding of complex interactions within neural structures. Specifically, a full network of n linearly coupled systems (1) is constructed as follows:

$$\begin{cases} u_{it} = f(u_i) + v_i - w_i + I - \sum\limits_{j=1, j\neq i}^{n} g_{syn}(u_i - u_j), \\ v_{it} = 1 - bu_i^2 - v_i, \\ w_{it} = r(s(u_i - c) - w_i), \\ i = 1, 2, ..., n, \end{cases} \tag{2}$$

where $(u_i, v_i, w_i), i = 1, 2, ..., n$, is defined as in (1); g_{syn} is a positive number presenting the coupling strength [6], [7].

In the following sections, we present a groundbreaking adaptive nonlinear controller designed to ensure identical synchronization of network (2) in a finite time frame. Before diving into the specifics, it's essential to revisit a crucial point that has been established in [18]. This foundation will reinforce our approach and underscore the significance of our findings.

Remark 1 (see [18])**.** The function f satisfies the following condition:

$$|f(u_i) - f(u_j)| \leq \alpha |u_i - u_j|, \tag{3}$$

where $u_i, u_j, i, j = 1, 2, ..., n$, present the transmembrane voltages, and α is a positive number.

Let the node errors of identical synchronization of the network (2) be $e_i^u = u_i - u_1, e_i^v = v_i - v_1, e_i^w = w_i - w_1$, for all $i = 2, ..., n$. The finite time identical synchronization problem of the network (2) can be defined as follows:

Definition 1. If there is a time $t^* > 0$ such that:

$$\lim_{t \to t^*} \sum_{i=2}^{n} (|e_i^u| + |e_i^v| + |e_i^w|) = 0,$$

and

$$\sum_{i=2}^{n} (|e_i^u| + |e_i^v| + |e_i^w|) \equiv 0, \text{ for all } t > t^*,$$

where t^* is called the setting time, then the full network (2) is identically synchrous in a finite time.

Research indicates that while most theoretical studies predict that desired synchronization is achieved only as time t approaches infinity, numerical evidence reveals that synchronization can, in fact, occur in a finite time. This critical gap between theoretical predictions and numerical results underscores the need for further investigation, serving as a compelling motivation for our paper. Tackling this challenge is no simple task. To realize identical synchronization within a finite timeframe, it is essential to develop specific controllers for the network outlined in equation (2). This process involves carefully constructing and integrating controllers into neuron i (where $i \neq 1$), as detailed below:

$$\begin{cases} u_{1t} = f(u_1) + v_1 - w_1 + I - \sum\limits_{j=2}^{n} g_{syn}(u_1 - u_j), \\ v_{1t} = 1 - bu_1^2 - v_1, \\ w_{1t} = r(s(u_1 - c) - w_1), \\ u_{it} = f(u_i) + v_i - w_i + I - \sum\limits_{j=1, j\neq i}^{n} g_{syn}(u_i - u_j) + \Gamma_i^1, \\ v_{it} = 1 - bu_i^2 - v_i + \Gamma_i^2, \\ w_{it} = r(s(u_i - c) - w_i) + \Gamma_i^3, \\ i = 2, ..., n, \end{cases} \tag{4}$$

where the controllers $\Gamma_i^j = \Gamma_i^j(t), i = 2, 3, ..., n; j = 1, 2, 3$, will be designed as follows:

$$\begin{cases} \Gamma_i^1 = u_{1t} - f(u_1) - v_1 + w_1 - I \\ \qquad + \sum\limits_{j=1, j\neq i}^{n} g_{syn}(u_1 - u_j) - k_i e_i^u + G_i^1, \\ \Gamma_i^2 = v_{1t} - 1 + bu_1^2 + v_1 + G_i^2, \\ \Gamma_i^3 = w_{1t} - r(s(u_1 - c) - w_1) + G_i^3, \end{cases} \tag{5}$$

with the updated rules defined as follows:

$$k_{it} = r_i((e_i^u)^2 + \theta_i), \tag{6}$$

where $k_i = k_i(t)$; r_i is a arbitrary positive constant, for $i = 2, ..., n$, and $G_i^j = G_i^j(t), \theta_i = \theta_i(t), j = 1, 2, 3; i = 2, 3, ..., n$, are defined as follows:

$$\begin{cases} G_i^1 = -m.sign(e_i^u)|e_i^u|^\gamma, \\ G_i^2 = -m.sign(e_i^v)|e_i^v|^\gamma, \\ G_i^3 = -m.rs.sign(e_i^w)|e_i^w|^\gamma, \\ \theta_i = -m.sign(k_i - k).|k_i - k|^\gamma, \end{cases} \tag{7}$$

where $sign(.)$ represents a signum function; m is a given positive constant; $\gamma \in \mathbb{R}$ and satisfies $0 \leq \gamma < 1$; and k is a positive cosntant to be determined.

Under the action of the controllers designed as above, the error dynamic equations of the system (2) are described as:

$$\begin{aligned} e_{it}^u &= (u_{it} - u_{1t}) \\ &= f(u_i) + v_i - w_i + I - \sum_{j=1, j\neq i}^{n} g_{syn}(u_i - u_j) \\ &\quad - f(u_1) - v_1 + w_1 - I \\ &\quad + \sum_{j=1, j\neq i}^{n} g_{syn}(u_1 - u_j) - k_i e_i^u + G_i^1 \\ &= f(u_i) - f(u_1) + (v_i - v_1) - (w_i - w_1) \\ &\quad + (n-1) g_{syn}(u_i - u_1) - k_i e_i^u + G_i^1 \\ &= f(u_i) - f(u_1) + e_i^v - e_i^w \\ &\quad + (n-1) g_{syn} e_i^u - k_i e_i^u + G_i^1, \end{aligned} \tag{8}$$

$$\begin{aligned} e_{it}^v &= v_{it} - v_{1t} \\ &= 1 - bu_i^2 - v_i - 1 + bu_1^2 + v_1 + G_i^2 \\ &= -b(u_i + u_1) e_i^u - e_i^v + G_i^2, \end{aligned} \tag{9}$$

and

$$\begin{aligned} e^w_{it} &= w_{it} - w_{1t} \\ &= r(s(u_i - c) - w_i) - r(s(u_1 - c) - w_1) + G^3_i \\ &= rs(u_i - u_1) - r(w_i - w_1) + G^3_i \\ &= rse^u_i - re^w_i + G^3_i, \end{aligned} \tag{10}$$

for $i = 2, ..., n$.

Lemma 1 ([20]). *For every $a_i \in \mathbb{R}, i = 1, 2, ..., n$, if $p, q \in \mathbb{R}, 0 < p \leq 1, 0 < q < 2$, then we have:*

$$\sum_{i=1}^{n} |a_i|^q \geq \left(\sum_{i=1}^{n} |a_i|^2\right)^{\frac{q}{2}} \text{ and } \left(\sum_{i=1}^{n} |a_i|\right)^p \leq \sum_{i=1}^{n} |a_i|^p.$$

Lemma 2 ([4]). *Assume that a continuous, positive-definite function $V(t)$ satisfies the following differential inequality:*

$$\frac{dV(t)}{dt} \leq -\epsilon V^{\mu}(t), \text{ for all } t \geq 0, V(0) \geq 0,$$

where μ, ϵ are positive constants and $0 < \mu < 1$, then

$$\begin{cases} V^{1-\mu}(t) \leq V^{1-\mu}(0) - \epsilon(1-\mu)t, \quad 0 < t < t^*, \\ V(t) \equiv 0, \ t > t^* = \dfrac{V^{1-\mu}(0)}{\epsilon(1-\mu)}, \end{cases}$$

By employing Lyapunov functions alongside finite time stability theory, the author effectively tackles the critical challenge of achieving finite time identical synchronization in complex networks, as outlined in equation (2). The pivotal findings are encapsulated in the following theorem, which promises to advance our understanding of synchronization dynamics significantly.

Theorem 1. *The full network* (2) *can achieve identical synchronization in a finite time under the adaptive controller* (5) *and updated rule* (6). *The setting time is estimated as:*

$$t^* = \frac{V^{\frac{1-\gamma}{2}}(0)}{m\rho^{\frac{\gamma-1}{2}}(1-\gamma)},$$

where $\rho = \min\{2, 2rs, 2r_i\}, i = 2, 3, ..., n$.

Proof: We construct the Lyapunov function as follows:

$$V(t) = \frac{1}{2}\sum_{i=2}^{n}\left((e^u_i)^2 + (e^v_i)^2 + \frac{1}{rs}(e^w_i)^2 + \frac{1}{r_i}(k_i - k)^2\right). \tag{11}$$

Calculating the time derivative of $V(t)$ along the error systems (8) - (10), we get:

$$\begin{aligned} \frac{dV(t)}{dt} &= \sum_{i=2}^{n}\left[e^u_i e^u_{it} + e^v_i e^v_{it} + \frac{1}{rs}e^w_i e^w_{it} + \frac{1}{r_i}(k_i - k)k_{it}\right] \\ &= \sum_{i=2}^{n}\Big[e^u_i\left(f(u_i) - f(u_1) + e^v_i - e^w_i\right. \\ &\qquad \left. + (n-1)g_{syn}e^u_i - k_i e^u_i + G^1_i\right) \\ &\qquad - b(u_i + u_1)e^u_i e^v_i - (e^v_i)^2 \\ &\qquad + e^v_i G^2_i + k_i(e^u_i)^2 - k(e^u_i)^2 + (k_i - k)\theta_i \\ &\qquad + \frac{1}{rs}(rse^u_i e^w - r(e^w_i)^2 + e^w_i G^3_i)\Big] \\ &= \sum_{i=2}^{n}\Big[e^u_i\left(f(u_i) - f(u_1)\right) + e^u_i G^1_i \\ &\quad + (1 - b(u_i + u_1))e^u_i e^v_i - k(e^u_i)^2 + e^v_i G^2_i \\ &\qquad + (n-1)g_{syn}(e^u_i)^2 - (e^v_i)^2 \\ &\qquad + (k_i - k)\theta_i - \frac{1}{s}(e^w_i)^2 + \frac{1}{rs}e^w_i G^3_i\Big]. \end{aligned} \tag{12}$$

By using Proposition 1, it is easy to obtain:

$$\begin{aligned} \frac{dV(t)}{dt} \leq \sum_{i=2}^{n}\Big[&\alpha(e^u_i)^2 + (n-1)g_{syn}(e^u_i)^2 - (e^v_i)^2 \\ &- \frac{1}{s}(e^w_i)^2 + e^u_i G^1_i + e^v_i G^2_i + \frac{1}{rs}e^w_i G^3_i - k(e^u_i)^2 \\ &+ (k_i - k)\theta_i + (1 + b(|u_i| + |u_1|))|e^u_i||e^v_i|\Big]. \end{aligned} \tag{13}$$

By using the Young's inequality for every $\delta > 0$, we can see:

$$\begin{aligned} &|e^u_i|\,|e^v_i|\,(1 + b(|u_i| + |u_1|)) \\ &\qquad \leq (1 + b(|u_i| + |u_1|))(\frac{1}{2\delta}(e^u_i)^2 + \frac{\delta}{2}(e^v_i)^2) \\ &\qquad \leq \frac{M}{2\delta}(e^u_i)^2 + \frac{M\delta}{2}(e^v_i)^2, \end{aligned} \tag{14}$$

where M is a positive constant, since $u_i, i = 1, 2, ..., n$ are bounded (see [17]).

Combining (13) and (14) yields:

$$\begin{aligned} \frac{dV(t)}{dt} \leq \sum_{i=2}^{n}\Big[&(\alpha - k + (n-1)g_{syn} + \frac{M}{2\delta})(e^u_i)^2 \\ &- (1 - \frac{M\delta}{2})\overline{e}_i{}^2 - \frac{1}{s}(e^w_i)^2 \\ &+ (k_i - k)\theta_i + e^u_i G^1_i + e^v_i G^2_i + \frac{1}{rs}e^w_i G^3_i\Big]. \end{aligned} \tag{15}$$

Chose $\delta > 0$ such that $1 - \dfrac{M\delta}{2} > 0$, and take

$$k > \alpha + (n-1)g_{syn} + \frac{M}{2\delta}, \tag{16}$$

then (15) can be estimated as:

$$\frac{dV(t)}{dt} \leq \sum_{i=2}^{n}\left[(k_i - k)\theta_i + e^u_i G^1_i + e^v_i G^2_i + \frac{1}{rs}e^w_i G^3_i\right]. \tag{17}$$

Besides that, we can see:

$$\sum_{i=2}^{n}\left((k_i-k)\theta_i+e_i^uG_i^1+e_i^vG_i^2+\frac{1}{rs}e_i^wG_i^3\right)=$$
$$\sum_{i=2}^{n}(-m.(k_i-k)sign(k_i-k).|k_i-k|^{\gamma}$$
$$-m.e_i^u sign(e_i^u)|e_i^u|^{\gamma}$$
$$-m.e_i^v sign(e_i^v)|e_i^v|^{\gamma}-m.e_i^w sign(e_i^w)|e_i^w|^{\gamma})$$
$$\leq\sum_{i=2}^{n}\left(-m(|k_i-k|^{\gamma+1}+|e_i^u|^{\gamma+1}+|e_i^v|^{\gamma+1}+|e_i^w|^{\gamma+1})\right). \tag{18}$$

Combining (17) - (18) yields:

$$\frac{dV(t)}{dt}\leq -m\sum_{i=2}^{n}\left(|k_i-k|^{\gamma+1}+|e_i^u|^{\gamma+1}+|e_i^v|^{\gamma+1}+|e_i^w|^{\gamma+1}\right). \tag{19}$$

By using Lemma 1, we have:

$$\left(\sum_{i=2}^{n}\left(|k_i-k|^{\gamma+1}+|e_i^u|^{\gamma+1}+|e_i^v|^{\gamma+1}+|e_i^w|^{\gamma+1}\right)\right)^{\frac{1}{\gamma+1}} \geq\left(\sum_{i=2}^{n}\left(|k_i-k|^{2}+|e_i^u|^{2}+|e_i^v|^{2}+|e_i^w|^{2}\right)\right)^{\frac{1}{2}}. \tag{20}$$

That yields:

$$\sum_{i=2}^{n}\left(|k_i-k|^{\gamma+1}+|e_i^u|^{\gamma+1}+|e_i^v|^{\gamma+1}+|e_i^w|^{\gamma+1}\right) \geq\left(\sum_{i=2}^{n}\left(|k_i-k|^{2}+|e_i^u|^{2}+|e_i^v|^{2}+|e_i^w|^{2}\right)\right)^{\frac{\gamma+1}{2}}. \tag{21}$$

Therefore, (19) becomes:

$$\begin{aligned}\frac{dV(t)}{dt}&\leq -m\left(\sum_{i=2}^{n}\left(|k_i-k|^{2}+|e_i^u|^{2}+|e_i^v|^{2}+|e_i^w|^{2}\right)\right)^{\frac{\gamma+1}{2}}\\ &\leq -m\rho^{\frac{\gamma+1}{2}}\left(\sum_{i=2}^{n}\frac{1}{2}\left(|e_i^u|^2+|e_i^v|^2+\frac{1}{rs}|e_i^w|^2+\frac{1}{r_i}|k_i-k|^2\right)\right)^{\frac{\gamma+1}{2}}\\ &\leq -m\rho^{\frac{\gamma+1}{2}}V^{\frac{\gamma+1}{2}}(t).\end{aligned} \tag{22}$$

where $\rho=\min\{2,2rs,2r_i\}, i=2,3,...,n$.

It is derived from Lemma 2 that $V(t)\equiv 0$ for

$$t>t^*=\frac{V^{\frac{1-\gamma}{2}}(0)}{m\rho^{\frac{\gamma-1}{2}}(1-\gamma)}.$$

Therefore,

$$\lim_{t\to t^*}\sum_{i=2}^{n}(|e_i^u|+|e_i^v|+|e_i^w|)=0,$$

and

$$\sum_{i=2}^{n}(|e_i|+|e_i|+|e_i^w|)\equiv 0 \text{ for } t>t^*.$$

This completes the proof. ■

III. Numerical results and discussion

In this section, we will rigorously evaluate the effectiveness of the controllers that have been developed. To achieve seamless integration of the system, we utilize R programming, a powerful tool for data analysis. The simulation results, which are derived from carefully selected parameter values, will provide insightful evidence of the system's performance.

$$f(u)=-u^3+au^2, a=3, b=5, s=4, r=0.008,$$
$$c=-\frac{1}{2}(1+\sqrt{5}), I=3.25.$$

A. Example 1.

In this compelling example, we explore a full network of two nodes and strategically design a controller, as outlined in the theoretical section, to achieve identical synchronization within a finite time frame. Our objective is to create effective controllers for this network based on the groundbreaking theoretical results illustrated in systems (5) to (7). Moreover, we will rigorously test the performance of these controllers through numerical simulations to ensure their efficacy. This full network of two nodes, equipped with the designed controllers, paves the way for achieving seamless synchronization, as depicted in the following system:

$$\begin{cases}u_{1t}=f(u_1)+v_1-w_1+I-g_{syn}(u_1-u_2),\\ v_{1t}=1-bu_1^2-v_1,\\ w_{1t}=r(s(u_1-c)-w_1),\\ u_{2t}=f(u_2)+v_2-w_2+I-g_{syn}(u_2-u_1)+\Gamma_2^1,\\ v_{2t}=1-bu_2^2-v_2+\Gamma_2^2,\\ w_{2t}=r(s(u_2-c)-w_2)+\Gamma_2^3.\end{cases} \tag{23}$$

where

$$\begin{cases}\Gamma_2^1=u_{1t}-f(u_1)-v_1+w_1-I-k_2e_2^u+G_2^1,\\ \Gamma_2^2=v_{1t}-1+bu_1^2+v_1+G_2^2,\\ \Gamma_2^3=w_{1t}-r(s(u_1-c)-w_1)+G_2^3,\end{cases} \tag{24}$$

with the updated rules defined as follows:

$$k_{2t}=r_2((e_2^u)^2+\theta_2), \tag{25}$$

where $k_2=k_2(t)$; r_2 is a arbitrary positive constant; $e_2^u=u_2-u_1, e_2^v=v_2-v_1, e_2^w=w_2-w_1$; and $G_2^j=G_2^j(t), \theta_2=\theta_2(t), j=1,2,3$, are defined as follows:

$$\begin{cases}G_2^1=-m.sign(e_2^u)|e_2^u|^{\gamma},\\ G_2^2=-m.sign(e_2^v)|e_2^v|^{\gamma},\\ G_2^3=-m.rs.sign(e_2^w)|e_2^w|^{\gamma},\\ \theta_2=-m.sign(k_2-k).|k_2-k|^{\gamma},\end{cases} \tag{26}$$

where $sign(.)$ represents a signum function, m is a given positive constant, $\gamma\in\mathbb{R}$ and satisfies $0\leq\gamma<1$.

In this example, we take:

$$m=0.55;\ \gamma=0.55; r_2=0.001, k=3.$$

Let $|e_2^u|+|e_2^v|+|e_2^w|$ denote the identical synchronization error that is a crucial measure of the system's performance. We assert that the network defined by system (23) achieves identical synchronization within a finite time frame if this synchronization error converges to zero as t approaches a specific finite value. To effectively demonstrate this, we will establish the initial conditions for the system in (23) as follows:

$$(u_1(0), v_1(0), u_2(0), v_2(0)) = (-0.5, -0.5, -0.5, 0.5, 0.5, 0.5).$$

Fig. 1 powerfully illustrates the challenge of achieving identical synchronization error within the network defined by system (23). In particular, Fig. 1(a) showcases a simulation of this network devoid of the controllers specified in systems (24)-(26). With a coupling strength set at $g_{syn} = 0.0001$ and over the time interval $t \in [0, 200000]$, the results are striking: the identical synchronization error fails to approach zero, highlighting that the phenomenon of identical synchronization remains elusive, even when time is extended to significant lengths.

Moreover, Fig. 2 compellingly presents the time series data for all variables in the system governed by (23), also without the aforementioned controllers. In Fig. 2(a), u_1 is depicted by a solid line, while u_2 is represented by a dotted line. This clear distinction carries through to Fig. 2(b) for v_1 and v_2, and likewise in Fig. 2(c) for w_1 and w_2. The stark contrast between the solid and dotted lines serves as undeniable evidence that the behaviors of these variables do not align, reinforcing the conclusion that the desired phenomenon of identical synchronization is not realized in this case.

In Fig. 1(b), we present a compelling simulation of the network defined by system (23), utilizing the controllers specified in systems (24) through (26). With a carefully chosen coupling strength of $g_{syn} = 0.0001$ and over the time frame $t \in [0, 10000]$, the results are striking. They demonstrate that the identical synchronization error converges to zero in a remarkably short time, even with the same coupling strength as in the earlier case, emphasizing the effectiveness of our approach.

As t approaches a finite value, we can confidently assert:

$$u_1(t) \approx u_2(t); \quad v_1(t) \approx v_2(t); \quad w_1(t) \approx w_2(t).$$

Moreover, Fig. 3 powerfully depicts the time series behavior of all variables in the system governed by (23) with the applied controllers. In Fig. 3(a), the captivating solid line representing u_1 synchronously tracks the dotted line of u_2, mirroring this phenomenon in Fig. 3(b) for v_1 and v_2, and in Fig. 3(c) for w_1 and w_2. The clear alignment of these lines underscores the significant identical synchronization achieved in this scenario, occurring decisively within a finite time frame (notably, $t < 4000$). This compelling evidence not only illustrates the success of our methods but also reinforces the importance of harnessing synchronization in complex networks.

B. Example 2.

In this compelling example, we delve into a full network made up of three nodes, where we strategically design a controller based on the theoretical foundations provided earlier. Our goal is to achieve identical synchronization in a finite amount of time. To that end, we must meticulously construct controllers for this network, leveraging the robust theoretical results captured in systems (5) to (7). This investigation will reveal whether these controllers can deliver the promised results in practice. Presented below is the system that outlines a full network of three nodes, equipped with controllers that drive us toward achieving flawless identical synchronization:

$$\begin{cases} u_{1t} = f(u_1) + v_1 - w_1 + I \\ \qquad -g_{syn}(u_1 - u_2) - g_{syn}(u_1 - u_3), \\ v_{1t} = 1 - bu_1^2 - v_1, \\ w_{1t} = r(s(u_1 - c) - w_1), \\ u_{2t} = f(u_2) + v_2 - w_2 + I \\ \qquad -g_{syn}(u_2 - u_1) - g_{syn}(u_2 - u_3) + \Gamma_2^1, \\ v_{2t} = 1 - bu_2^2 - v_2 + \Gamma_2^2, \\ w_{2t} = r(s(u_2 - c) - w_2) + \Gamma_2^3, \\ u_{3t} = f(u_3) + v_3 - w_3 + I \\ \qquad -g_{syn}(u_3 - u_1) - g_{syn}(u_3 - u_2) + \Gamma_3^1, \\ v_{3t} = 1 - bu_3^2 - v_3 + \Gamma_3^2, \\ w_{3t} = r(s(u_3 - c) - w_3) + \Gamma_3^3. \end{cases} \tag{27}$$

where

$$\begin{cases} \Gamma_2^1 = u_{1t} - f(u_1) - v_1 + w_1 - I \\ \qquad +g_{syn}(u_1 - u_3) - k_2 e_2^u + G_2^1, \\ \Gamma_2^2 = v_{1t} - 1 + bu_1^2 + v_1 + G_2^2, \\ \Gamma_2^3 = w_{1t} - r(s(u_1 - c) - w_1) + G_2^3, \\ \Gamma_3^1 = u_{1t} - f(u_1) - v_1 + w_1 - I \\ \qquad +g_{syn}(u_1 - u_2) - k_3 e_3^u + G_3^1, \\ \Gamma_3^2 = v_{1t} - 1 + bu_1^2 + v_1 + G_3^2, \\ \Gamma_3^3 = w_{1t} - r(s(u_1 - c) - w_1) + G_3^3, \end{cases} \tag{28}$$

with the updated rules defined as follows:

$$\begin{cases} k_{2t} = r_2((e_2^u)^2 + \theta_2), \\ k_{3t} = r_3((e_3^u)^2 + \theta_3), \end{cases} \tag{29}$$

where $k_2 = k_2(t), k_3 = k_3(t)$; r_2, r_3 are arbitrary positive constants; $e_i^u = u_i - u_1, e_i^v = v_i - v_1, e_i^w = w_i - w_1, i = 2, 3$; and $G_i^j = G_i^j(t), \theta_i = \theta_i(t), i = 2, 3, j = 1, 2, 3$, are defined as follows:

$$\begin{cases} G_2^1 = -m.sign(e_2^u)|e_2^u|^\gamma, \\ G_2^2 = -m.sign(e_2^v)|e_2^v|^\gamma, \\ G_2^3 = -m.rs.sign(e_2^w)|e_2^w|^\gamma, \\ G_3^1 = -m.sign(e_3^u)|e_3^u|^\gamma, \\ G_3^2 = -m.sign(e_3^v)|e_3^v|^\gamma, \\ G_3^3 = -m.rs.sign(e_3^w)|e_3^w|^\gamma, \\ \theta_2 = -m.sign(k_2 - k).|k_2 - k|^\gamma, \\ \theta_3 = -m.sign(k_3 - k).|k_3 - k|^\gamma, \end{cases} \tag{30}$$

where $sign(.)$ represents a signum function, m is a given positive constant, $\gamma \in \mathbb{R}$ and satisfies $0 \leq \gamma < 1$.

In this example, we take:

$$m = 0.0055; \ \gamma = 0.06; r_2 = 0.0001; r_3 = 0.001; k = 4.$$

Let us define the identical synchronization error as $|e^{u_2}| + |e^{v_2}| + |e^{w_2}| + |e^{u_3}| + |e^{v_3}| + |e^{w_3}|$. We assert that the network described by system (27) achieves identical synchronization in a remarkably short time if this synchronization error approaches zero as t tends to a finite value. We will establish the initial conditions for the system in (27) as follows:

$$(u_1(0), v_1(0), w_1(0)) = (-1, -1, -1),$$
$$(u_2(0), v_2(0), w_2(0)) = (0, 0, 0),$$
$$(u_3(0), v_3(0), w_3(0)) = (1, 1, 1).$$

Figure 4 compellingly demonstrates the identical synchronization error within the network outlined in system (27). In panel (a) of Fig. 4, we conducted a simulation of the network without the controllers specified in systems (28)-(30), employing a coupling strength of $g_{syn} = 0.00001$ over the time interval $t \in [0, 500000]$. The findings are striking: the identical synchronization error fails to converge to zero, highlighting that the phenomenon of identical synchronization does not materialize, regardless of how extensively we extend the time t.

Moreover, Fig. 5 powerfully illustrates the time series behavior of all variables in the system defined by (27) without the influence of controllers. In panel (a), we see u_1 represented by a solid line, u_2 by a dotted line, and u_3 by a dashed line. This pattern continues in panel (b) for v_1, v_2, and v_3; and in panel (c) for w_1, w_2, and w_3. It is abundantly clear that the dotted and dashed lines do not mirror the behavior of the solid line, firmly reinforcing the conclusion that identical synchronization is unattainable in this instance.

In stark contrast, panel (b) of Fig. 4 reveals the simulation of the network incorporating controllers from equations (28)-(30). Here, we again use the coupling strength $g_{syn} = 0.00001$, but over a different time interval $t \in [0, 200000]$. The results are nothing short of transformative: the identical synchronization error remarkably reaches zero within a finite time. This achievement occurs even with an identical coupling strength as before and with a significantly shorter time t. As a result, as t approaches this decisive finite point, we observe a striking alignment:

$$u_1(t) \approx u_2(t) \approx u_3(t);$$
$$v_1(t) \approx v_2(t) \approx v_3(t);$$
$$w_1(t) \approx w_2(t) \approx w_3(t.$$

This data unequivocally underscores the vital impact of implementing controllers within network dynamics, effectively facilitating identical synchronization where it was otherwise unattainable.

Figure 6 powerfully depicts the time series of all variables within the system outlined by systems (27) and governed by (28)-(30). In panel (a), the variable u_1 is illustrated with a solid line, while u_2 and u_3 are represented by dotted and dashed lines, respectively. This clear distinction continues in panel (b) with v_1, v_2, and v_3, and again in panel (c) with w_1, w_2, and w_3. What stands out is the striking similarity in behavior between the dotted and dashed lines and their solid counterparts. This compelling evidence of synchronization underscores a remarkable phenomenon that occurs within a finite time frame ($t < 200000$). Such synchronization not only emphasizes the effectiveness of the control strategy but also highlights the interconnected nature of the system's dynamics.

Remark 2. The two examples presented clearly demonstrate that achieving identical synchronization is impossible without the innovative controllers proposed in this study, even when the time t is exceedingly large. Additionally, networks investigated without controllers could still numerically achieve the desired synchronization in a finite time, provided that the coupling strength is sufficiently large [15], [16]. However, this does not negate the theoretical results. It is important to note that we have not proven that synchronization can be achieved in a finite time without controllers. Remarkably, when we incorporate the controllers into the investigated networks, identical synchronization can be realized in a finite time, even with relatively small coupling strength and a short duration. This compelling evidence highlights the effectiveness of the controllers introduced in this paper and underscores their critical role in achieving synchronization.

IV. Conclusion

In this paper, we tackle the critical challenge of finite-time identical synchronization in a full network of n linearly coupled dynamical systems of the Hindmarsh-Rose 3D type. By strategically designing a robust controller and employing the powerful tools of Lyapunov functions alongside finite-time stability theory, we derive compelling sufficient conditions as a set of inequalities that guarantee successful synchronization among all node systems. Remarkably, this allows them to achieve the desired synchronous solution within a finite time frame. Furthermore, the effectiveness and practicality of our innovative approach are convincingly demonstrated through numerical simulations, highlighting its potential to advance the field of dynamical system synchronization.

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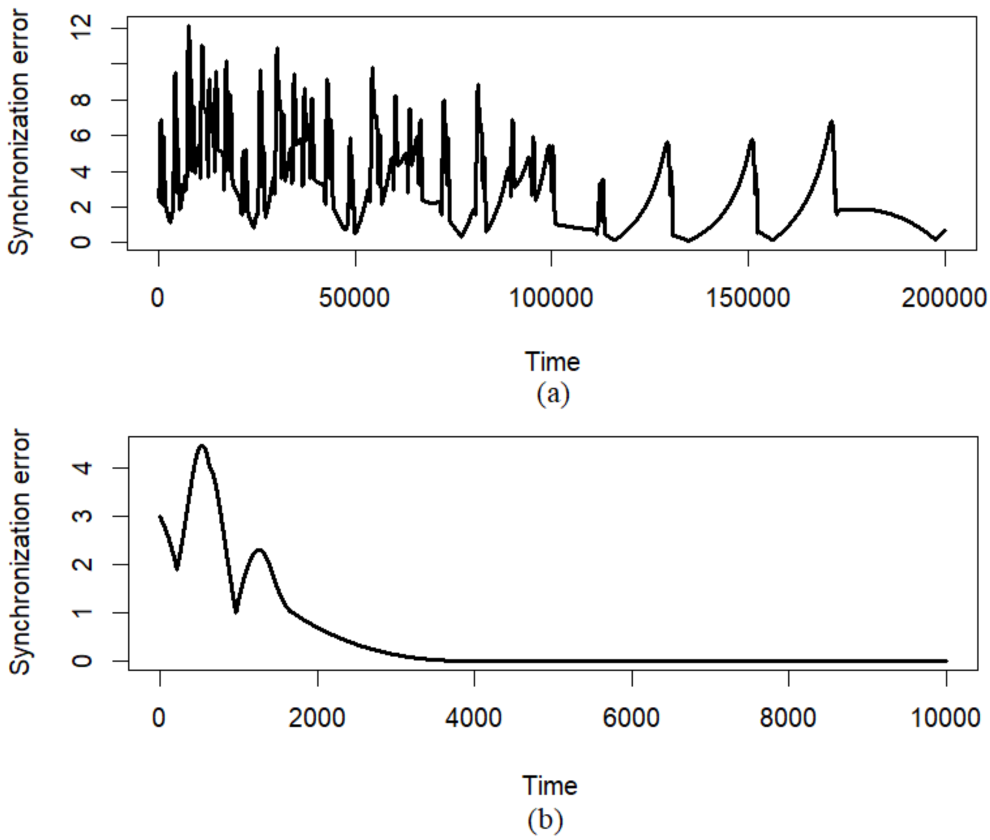


Fig. 1. Identical synchronization errors of the network (23): (a) without controllers (24)-(26); (b) with controllers (24)-(26) according to the coupling strength $g_{syn} = 0.0001$, and $t \in [0; 10000]$.

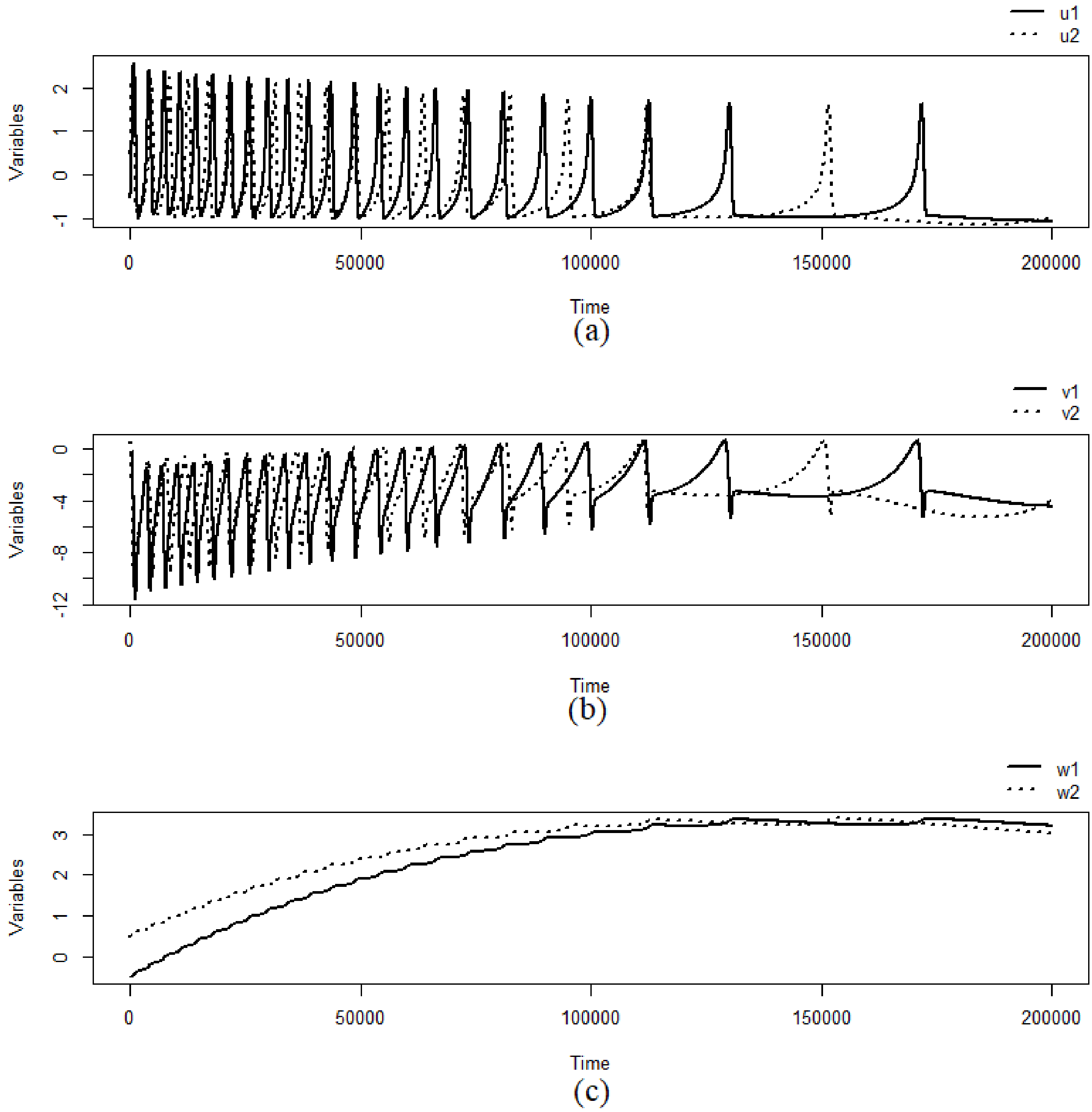
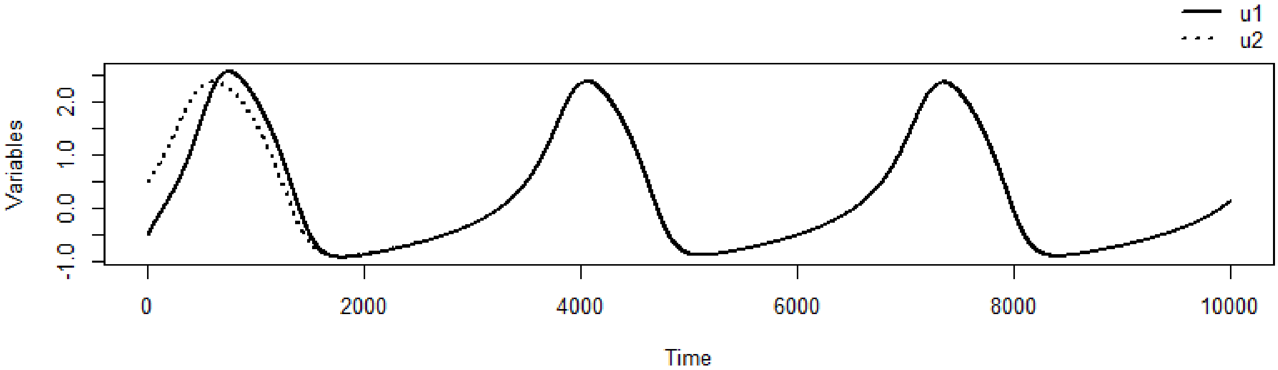
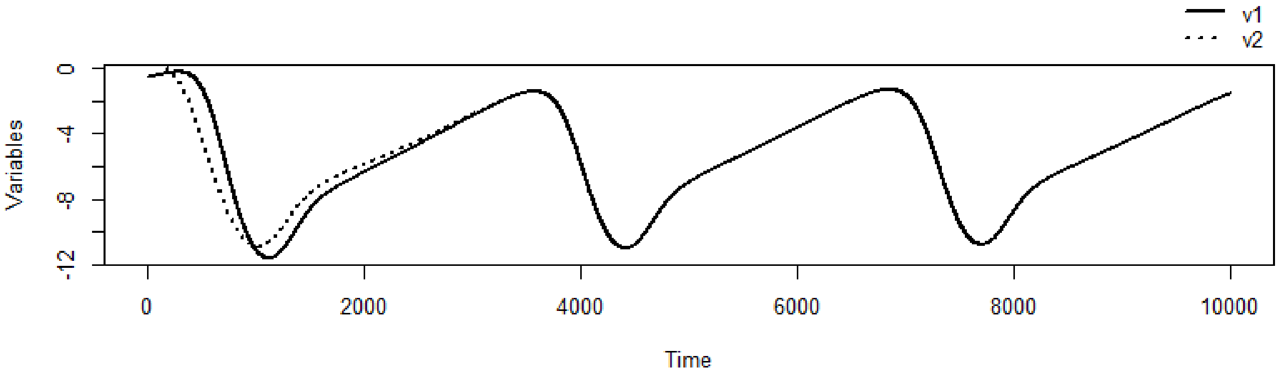


Fig. 2. Time series of all variables of the system (23) without controllers (24)-(26) accroding to the coupling strength $g_{syn} = 0.0001$, and $t \in [0; 200000]$.



(a)



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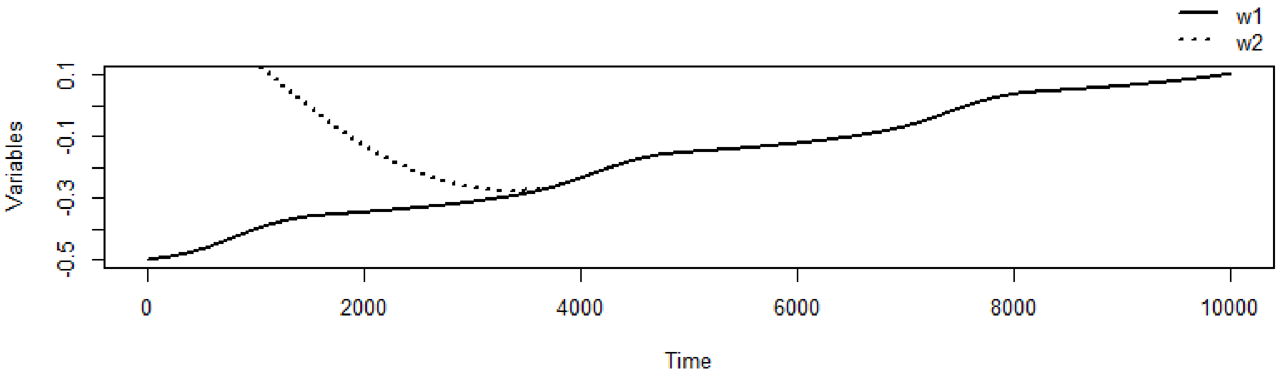


(c)

Fig. 3. Time series of all variables of the system (23) with controllers (24)-(26) according to the coupling strength $g_{syn} = 0.0001$, and $t \in [0; 10000]$.

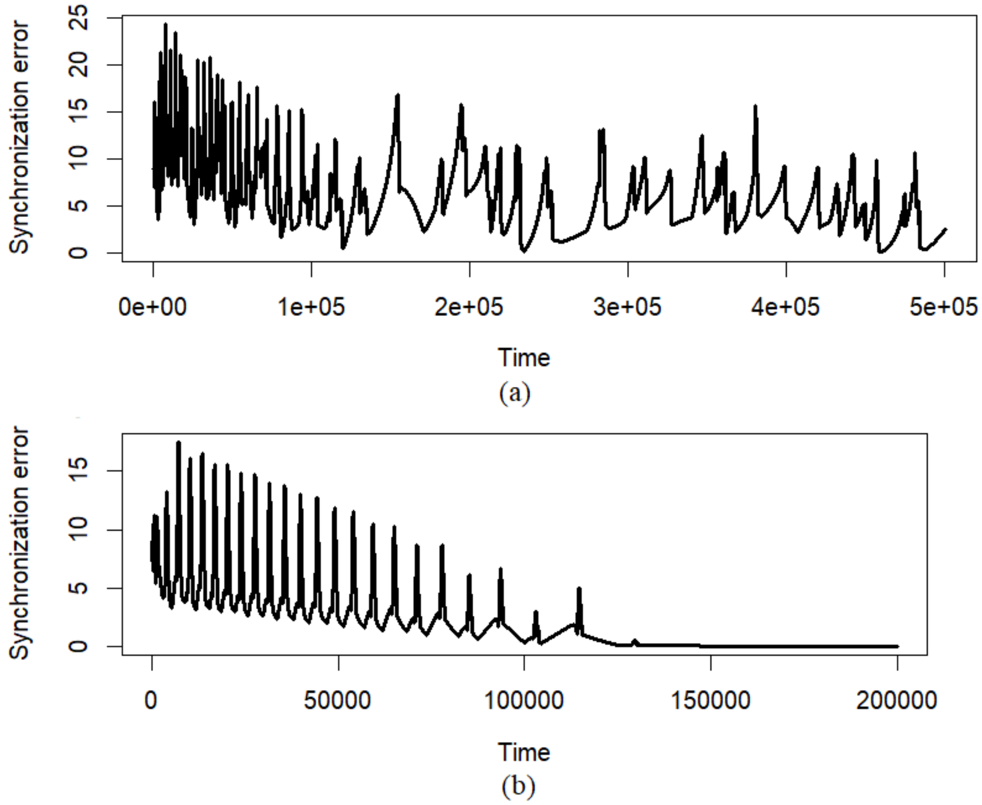
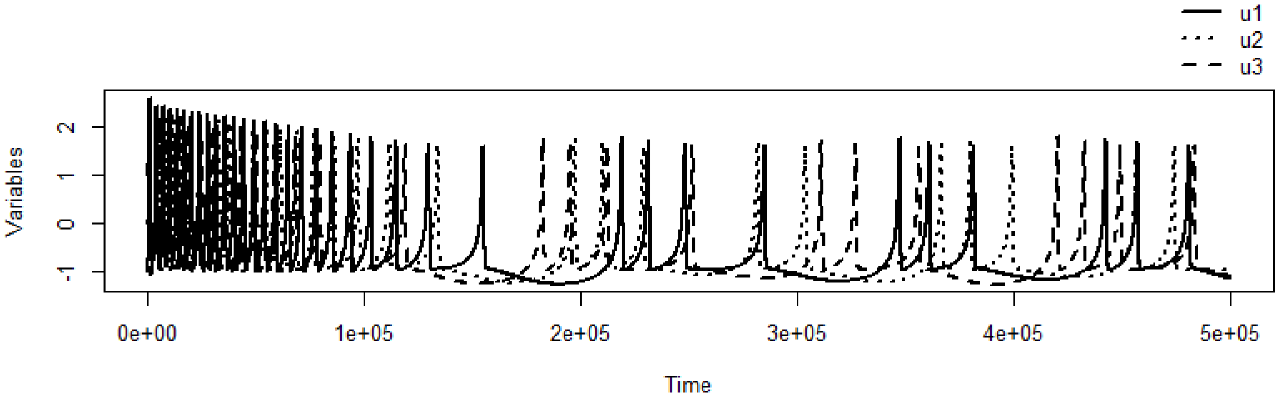
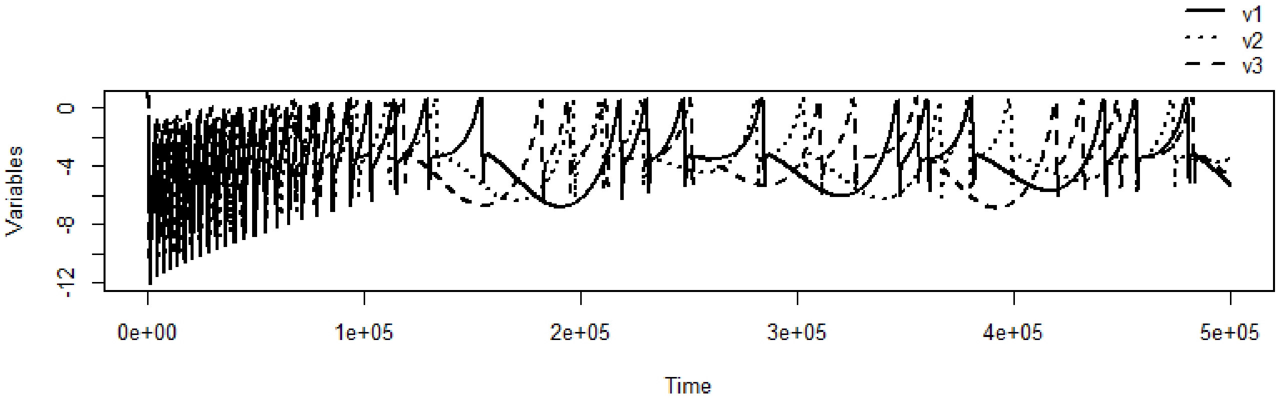


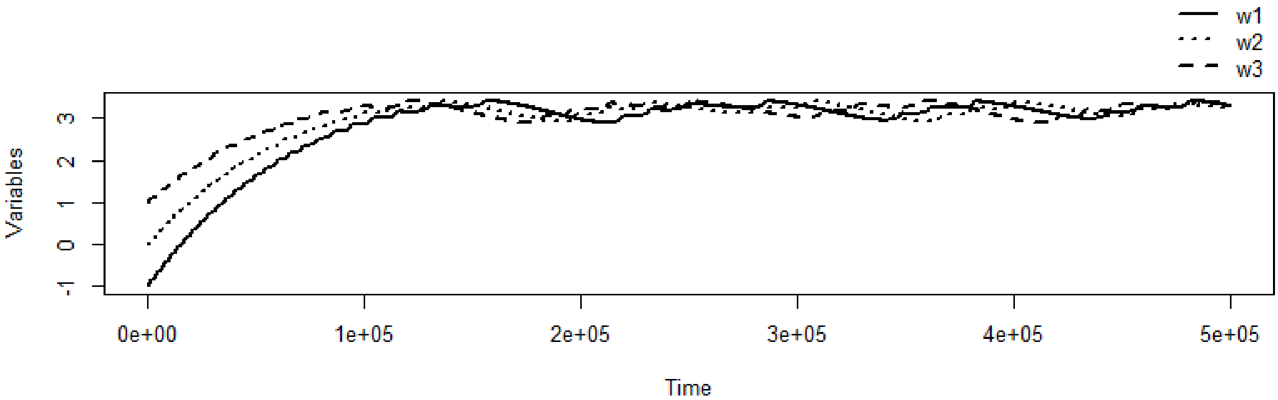
Fig. 4. Identical synchronization errors of the network (27): (a) without controllers (28)-(30); (b) with controllers (28)-(30) according to the coupling strength $g_{syn} = 0.00001$, and $t \in [0; 200000]$.



(a)



(b)



(c)

Fig. 5. Time series of all variables of the system (27) without controllers (28)-(30) accroding to the coupling strength $g_{syn} = 0.00001$, and $t \in [0; 500000]$.

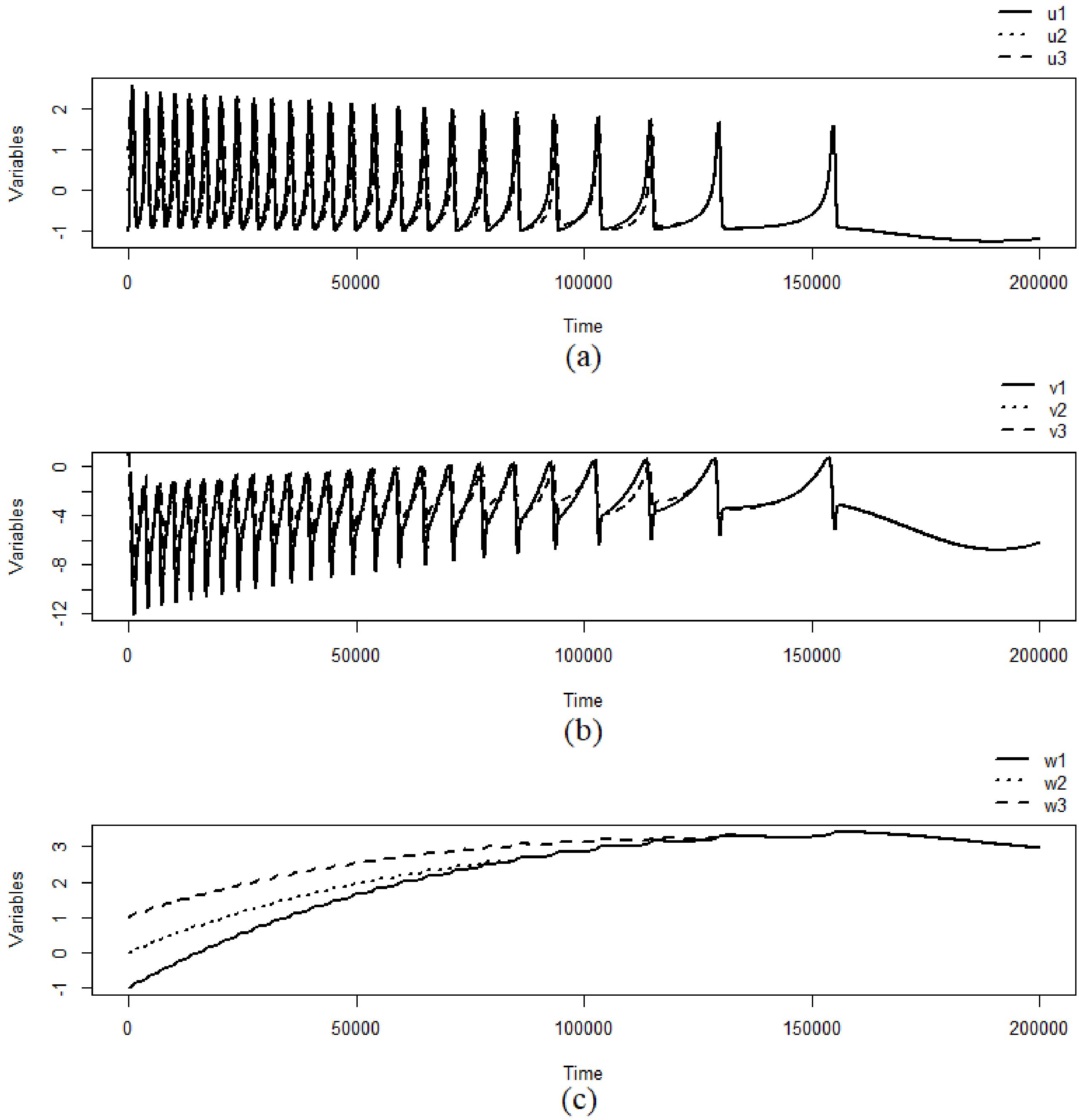


Fig. 6. Time series of all variables of the system (27) with controllers (28)-(30) according to the coupling strength $g_{syn} = 0.00001$, and $t \in [0; 200000]$.