Face Bimagic and 1-Antimagic Labelings of Rooted Product of Particular Graph Classes

Medini H R, Devadas Nayak C* and Sabitha D'Souza

***Abstract*—Graph labeling constitutes a significant area of research in graph theory, encompassing a wide variety of labeling methods and their multifaceted applications. This work specifically concentrates on the assignment of labels to a planar graph. It establishes the theoretical framework for face bimagic and 1-antimagic labelings of type $(1,0,0)$ and $(1,0,1)$ for the rooted product of P_n and $K_{2,m}$, outlining comprehensive criteria and methodologies. The paper also explores the behavior of the associated magic constant in various configurations. Through constructive examples and parameter variations, we demonstrate how label assignments affect the constancy or variability of sums across faces of a given graph, thus offering deeper insights into the labeling characteristics of planar graphs.**

***Index Terms*—Graph labeling, Planar graph, Face labeling, Magic labeling, Bimagic labeling, Antimagic labeling**

I. Introduction

Graph labeling is a fascinating area of graph theory, where labels are assigned to graph elements, such as vertices, edges, and/or faces, to satisfy specific properties. Magic labelings create uniform patterns by ensuring that the sums of labels around designated structures remain consistent. On the other hand, antimagic labelings focus on uniqueness, where the sums of labels around specified structures are distinct. In bimagic labeling, the sums of labels around designated structures give two magic constants (denoted as M_1 and M_2). Labeling types differ based on the graph elements involved. For instance, a labeling of type $(1,0,1)$ for a planar graph includes labeling of vertices and faces, whereas a type $(1,0,0)$ focuses solely on vertices. A magic labeling of type (a,b,c) for a planar graph G is called a super magic labeling of type (a,b,c) if $f(V(G)) = \{1,2,\dots,|V(G)|\}$.

Extensive research on graph labeling is documented in Gallian's survey [1], which provides a comprehensive overview of the field. Notably, in 1983, Lih introduced a magic-type labeling approach for the vertices, edges, and faces of planar graphs [2]. Baca later developed consecutive and magic labelings of type $(0,1,1)$ and a consecutive labeling of type $(1,1,1)$ for certain planar graphs with hexagonal faces [3]. Additionally, Baca and Miller defined the d-antimagic labeling of type $(1,1,1)$ for planar graphs [4]. Liu et al. explored the super $(a,0)$ edge-antimagic labeling of the rooted product of specific graphs [5]. Ahmed and Babujee explored face bimagic labeling of type $(1,1,0)$ for different graph structures, such as wheels, cylinders and the disjoint union of m copies of prism graphs [6]. Graph labeling encompasses a wide range of applications and plays a crucial role in the field of cryptography [7], [8].

This study is about the super face bimagic and 1-antimagic labelings of types, $(1,0,0)$ and $(1,0,1)$ for the rooted product of graphs, path P_n and $K_{2,m}$ ($P_n \circ K_{2,m}$). $K_{2,m}$ is a complete bipartite graph where the vertex set is divided into two disjoint subsets: One has 2 vertices and the other subset has m vertices. In a complete bipartite graph, every vertex in one subset is connected to every vertex in the other subset and there are no edges between vertices within the same subset. The total number of vertices and edges in $K_{2,m}$ are $2+m$ and $2m$, respectively. The rooted product of two graphs G and H is constructed by taking $|V(G)|$ copies of H and for every vertex v_i of G, identifying v_i with the root node of the i^{th} copy of H. This product is denoted by $G \circ H$.

II. Face labeling of $P_n \circ K_{2,m}$

Consider the planar graph $G \cong P_n \circ K_{2,m}$ with vertex set $V(G)$, edge set $E(G)$, and face set $F(G)$. Let M_1 and M_2 be the magic constants and W be the sum of the labels in the interior face of the graph (face weight). The total number of vertices, edges and faces in $P_n \circ K_{2,m}$ are $n(m+2)$, $2mn+n-1$, and $n(m-1)$, respectively.

Theorem 1: The graph $P_n \circ K_{2,m}$, where $m \geq 3$ and $n \geq 2$, admits a super bimagic labeling of type $(1,0,0)$ with the following magic constants:

i) $M_1 = 2mn + 4n + 2$

ii) $M_2 = 2mn + 5n + 2$.

Proof: Let G be the rooted product of graphs P_n and $K_{2,m}$ with vertex set $V(G)$ and $|V(G)| = n(m+2)$. The vertices of G are labeled as described below.

The vertices of $P_n \circ K_{2,m}$ are labeled in general for $m \geq 3$ and $n \geq 2$ as shown in Fig. 1 and Fig. 2. The graph in Fig. 1 contains the vertex labels when m is odd and the graph in Fig. 2 contains the vertex labels when m is even.

A vertex labeling function $g : V(G) \longrightarrow \{1,2,\dots,n(m+2)\}$, which is a bijection, is defined and the vertices v_i, v_p^i, u_q^i and w_i, where $1 \leq i \leq n$ are labeled as follows:

$$v_i = i,$$
$$v_p^i = pn + i, \quad 1 \leq p \leq \left\lceil \frac{m}{2} \right\rceil,$$
$$u_q^i = (q+1)n + 1 - i, \quad \left\lceil \frac{m+2}{2} \right\rceil \leq q \leq m,$$
$$w_i = (m+2)n + 1 - i.$$

Let all the faces be divided into two categories, F_1 and F_2. Each category will contain the alternate faces of $P_n \circ K_{2,m}$.

Manuscript received February 17, 2025; revised July 18, 2025.

Medini H R is a Research Scholar in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India, 576104 (e-mail: medinihr@gmail.com).

*Devadas Nayak C is an Associate Professor in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India, 576104 (corresponding author; phone: 9900507548; e-mail: devadas.nc@manipal.edu).

Sabitha D'Souza is an Associate Professor in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India, 576104 (e-mail: sabitha.dsouza@manipal.edu).

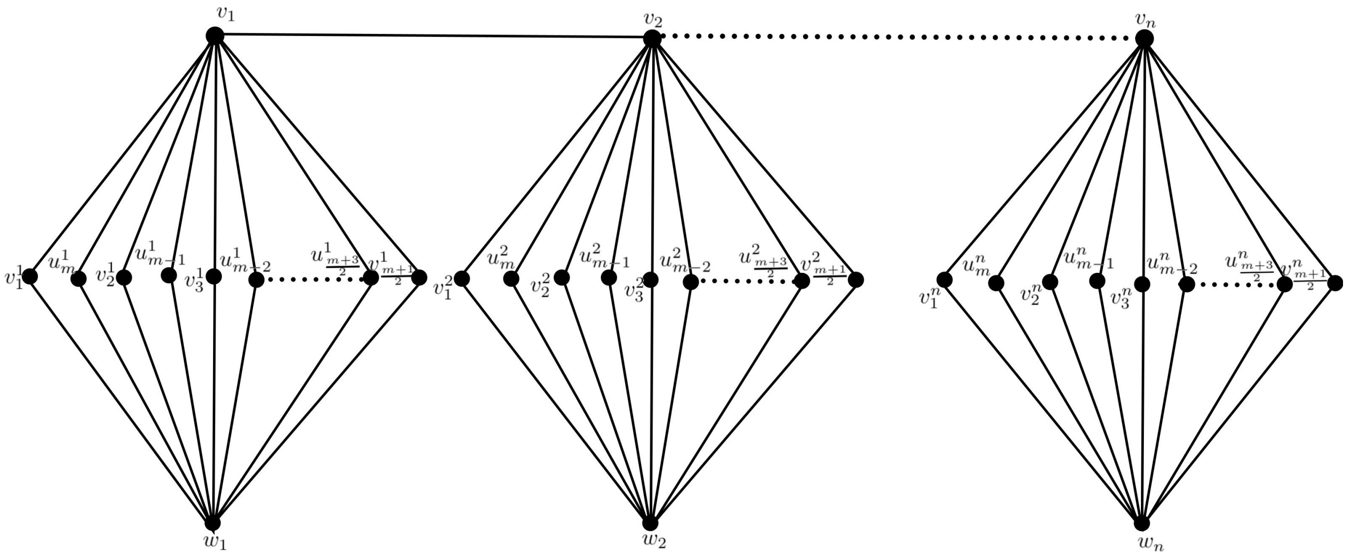

Fig 1. $P_n \circ K_{2,m}$, where m is odd

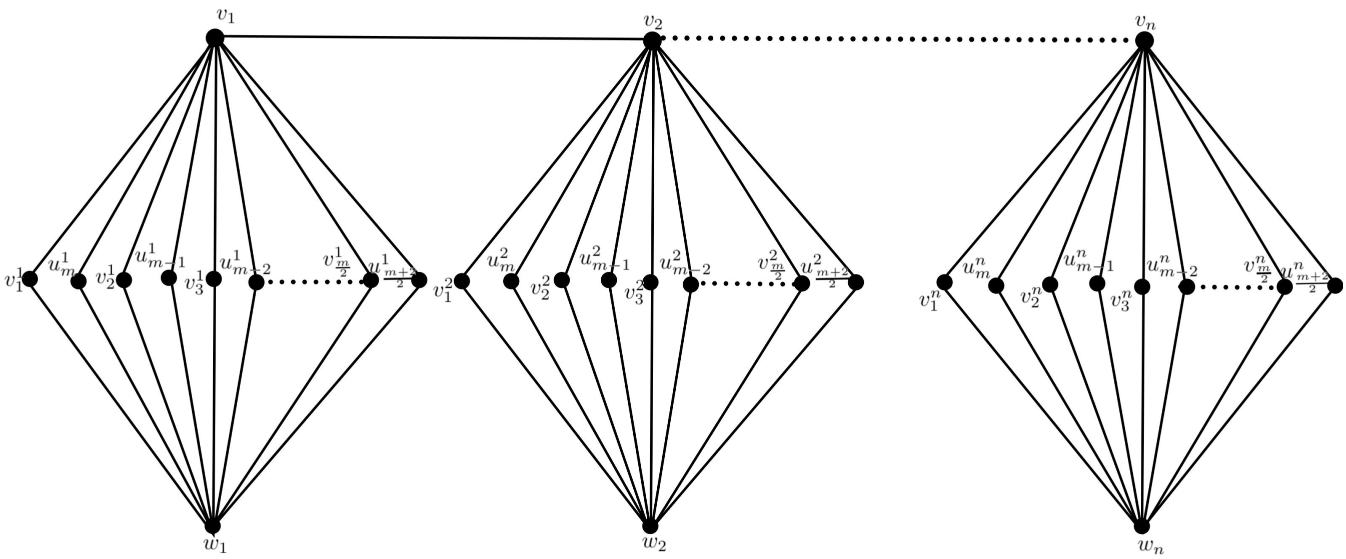

Fig 2. $P_n \circ K_{2,m}$, where m is even

■

i) $M_1 = 2mn + 4n + 2$

The expression is proved by using the induction method three times. To prove the expression $M_1 = 2mn+4n+2$ for $m \geq 3$ and $n \geq 2$, first we will use induction on all faces in F_1, induction on n (for a fixed m) and then induction on m. The proof uses the assumption for $k-1$ to prove for k.

Step 1: The proof consists of two cases: when m is even and when m is odd, to prove the expression holds for all faces in F_1.

Case 1: m is odd.
Consider $F_1 = \{f_1, f_3, \ldots, f_{m-2}\}$. First, we will prove $M_1 = 2mn + 4n + 2$ for all $f_1, f_3, \ldots, f_{m-2}$ by induction.

Base case: In f_1, p and q are 1 and m, respectively.
The first face f_1 of $K_{2,m}$ attached to i consists of the labels v_i, v_1^i, w_i and u_m^i. The first magic constant is obtained by summing the vertex labels around the face f_1 as follows.

$$M_1 = i + pn + i + (q+1)n + 1 - i + (m+2)n + 1 - i$$

Substitute $p = 1$ and $q = m$.

$$M_1 = n + (m+1)n + 1 + (m+2)n + 1$$

$$= 2mn + 4n + 2.$$

The base case holds.

Assume the expression holds for f_{m-4}.
Substitute $p = \frac{m-3}{2}$ and $q = \frac{m+5}{2}$ in the expression.

$$M_1 = \left(\frac{m-3}{2}\right)n + \left(\frac{m+5}{2} + 1\right)n + 1 + (m+2)n + 1$$

$$= 2mn + 4n + 2.$$

We need to prove the expression holds for f_{m-2}.
Substitute $p = \frac{m-1}{2}$ and $q = \frac{m+3}{2}$ in the expression.

$$M_1 = \left(\frac{m-1}{2}\right)n + \left(\frac{m+3}{2}+1\right)n + 1 + (m+2)n + 1$$
$$= 2mn + 4n + 2.$$

Thus, the expression holds for all faces in F_1 when m is odd.

Case 2: m is even.
Consider $F_1 = \{f_1, f_3, \ldots, f_{m-1}\}$. Now, we will prove $M_1 = 2mn + 4n + 2$ for all $f_1, f_3, \ldots, f_{m-1}$ by induction.

Base case: In f_1, p and q are 1 and m, respectively.
This is similar to the base step in Case 1. The base case holds.

Assume the expression holds for f_{m-3}.
Substitute $p = \frac{m-2}{2}$ and $q = \frac{m+4}{2}$ in the expression.

$$M_1 = \left(\frac{m-2}{2}\right)n + \left(\frac{m+4}{2}+1\right)n + 1 + (m+2)n + 1$$
$$= 2mn + 4n + 2.$$

We need to prove the expression holds for f_{m-1}.
Substitute $p = \frac{m}{2}$ and $q = \frac{m+2}{2}$ in the expression.

$$M_1 = \left(\frac{m}{2}\right)n + \left(\frac{m+2}{2}+1\right)n + 1 + (m+2)n + 1$$
$$= 2mn + 4n + 2.$$

Thus, the expression holds for all faces in F_1 when m is even.

To prove the expression $M_1 = 2mn + 4n + 2$ for $m \geq 3$ and $n \geq 2$, we will use induction on n first (for a fixed m) and then induction on m. The proof uses the assumption for $k-1$ to prove for k.

Step 2: Induction on n (for fixed $m = 3$).

Base case: $n = 2$. Substitute $m = 3$ and $n = 2$ into the expression:
$M_1 = 2mn + 4n + 2 = 22$. The base case holds.

Assume the expression holds for $n = k-1$. That is,

$$M_1 = 2m(k-1) + 4(k-1) + 2.$$

We need to prove the expression holds for $n = k$. Use the expression for $n = k-1$ in the inductive hypothesis.

$$M_1 = 2m(k-1) + 4(k-1) + 2.$$

Write $k = (k-1)+1$ in the above expression. For $n = k$:

$$M_1 = 2m[(k-1)+1] + 4[(k-1)+1] + 2$$
$$= 2m(k-1) + 2m + 4(k-1) + 4 + 2$$
$$= 2mk + 4k + 2.$$

Thus, the expression holds for $n = k$.
By induction, the expression is proven for all $n \geq 2$.

Step 3: Induction on m (for fixed $n \geq 2$).

Base case: $m = 3$. From Step 2, we proved that the expression holds for $m = 3$ and all $n \geq 2$.

Assume the expression holds for $m = k-1$. That is,

$$M_1 = 2(k-1)n + 4n + 2.$$

We need to prove the expression holds for $m = k$.
Use the expression for $m = k-1$ in the inductive hypothesis.

$$M_1 = 2(k-1)n + 4n + 2.$$

Write $k = (k-1)+1$ in the above expression. For $m = k$:

$$M_1 = 2[(k-1)+1]n + 4n + 2$$
$$= 2(k-1)n + 2n + 4n + 2$$
$$= 2kn + 4n + 2.$$

Thus, the expression holds for $m = k$.
By induction on m and n, the expression $M_1 = 2mn+4n+2$ is proven for all $m \geq 3$ and $n \geq 2$, using the assumption for $k-1$ in both steps.

ii) $M_2 = 2mn + 5n + 2$
The steps involved in this proof are similar to the steps described in the proof for M_1. To prove the expression $M_2 = 2mn + 5n + 2$ for $m \geq 3$ and $n \geq 2$, first we will use induction on all faces in F_2, induction on n (for a fixed m) and then induction on m.

Step 1: The proof consists of two cases: when m is even and when m is odd, to prove the expression holds for all faces in F_2.

Case 1: m is odd.
Consider $F_2 = \{f_2, f_4, \ldots, f_{m-1}\}$. First, we will prove $M_2 = 2mn + 5n + 2$ for all $f_2, f_4, \ldots, f_{m-1}$ by induction.

Base case: In f_2, p and q are 2 and m, respectively.
f_2 of $K_{2,m}$ attached to i consists of the labels v_i, v_2^i, w_i and u_m^i. The second magic constant is obtained by summing the vertex labels around the face f_2 as follows.

$$M_2 = i + pn + i + (q+1)n + 1 - i + (m+2)n + 1 - i$$

Substitute $p = 2$ and $q = m$.

$$M_2 = 2n + (m+1)n + 1 + (m+2)n + 1$$
$$= 2mn + 5n + 2.$$

The base case holds.

Assume the expression holds for f_{m-3}.
Substitute $p = \frac{m-1}{2}$ and $q = \frac{m+5}{2}$ in the expression.

$$M_2 = \left(\frac{m-1}{2}\right)n + \left(\frac{m+5}{2}+1\right)n + 1 + (m+2)n + 1$$
$$= 2mn + 5n + 2.$$

We need to prove the expression holds for f_{m-1}.
Substitute $p = \frac{m+1}{2}$ and $q = \frac{m+3}{2}$ in the expression.

$$M_2 = \left(\frac{m+1}{2}\right)n + \left(\frac{m+3}{2}+1\right)n + 1 + (m+2)n + 1$$
$$= 2mn + 5n + 2.$$

Thus, the expression holds for all faces in F_2 when m is odd.

Case 2: m is even.
Consider $F_2 = \{f_2, f_4, \ldots, f_{m-2}\}$. Now, we will prove $M_2 = 2mn + 5n + 2$ for all $f_2, f_4, \ldots, f_{m-2}$ by induction.

Base case: In f_2, p and q are 2 and m, respectively.
This is similar to the base step in Case 1. The base case holds.

Assume the expression holds for f_{m-4}.
Substitute $p = \frac{m-2}{2}$ and $q = \frac{m+6}{2}$ in the expression.

$$M_1 = \left(\frac{m-2}{2}\right)n + \left(\frac{m+6}{2} + 1\right)n + 1 + (m+2)n + 1$$
$$= 2mn + 5n + 2.$$

We need to prove the expression holds for f_{m-2}.
Substitute $p = \frac{m}{2}$ and $q = \frac{m+4}{2}$ in the expression.

$$M_1 = \left(\frac{m}{2}\right)n + \left(\frac{m+4}{2} + 1\right)n + 1 + (m+2)n + 1$$
$$= 2mn + 5n + 2.$$

Thus, the expression holds for all faces in F_2 when m is even.

To prove the expression $M_2 = 2mn + 5n + 2$ for $m \geq 3$ and $n \geq 2$, we will use induction on n first (for a fixed m) and then induction on m. The proof uses the assumption for $k-1$ to prove for k.

Step 2: Induction on n (for fixed $m = 3$).
Base case: $n = 2$. Substitute $m = 3$ and $n = 2$ into the expression:
$M_2 = 2mn + 5n + 2 = 24$. The base case holds.

Assume the expression holds for $n = k - 1$. That is,

$$M_2 = 2m(k-1) + 5(k-1) + 2.$$

We need to prove the expression holds for $n = k$.
Use the expression for $n = k-1$ in the inductive hypothesis.

$$M_2 = 2m(k-1) + 5(k-1) + 2.$$

Write $k = (k-1) + 1$ in the above expression. For $n = k$:

$$M_2 = 2m[(k-1)+1] + 5[(k-1)+1] + 2$$
$$= 2m(k-1) + 2m + 5(k-1) + 5 + 2$$
$$= 2mk + 5k + 2.$$

Thus, the expression holds for $n = k$.
By induction, the expression is proven for all $n \geq 2$.

Step 3: Induction on m (for fixed $n \geq 2$).
Base case: $m = 3$. In Step 2, we proved that the expression holds for $m = 3$ and all $n \geq 2$.

Assume the expression holds for $m = k - 1$. That is,

$$M_2 = 2(k-1)n + 5n + 2.$$

We need to prove the expression holds for $m = k$. Use the expression for $m = k-1$ in the inductive hypothesis.

$$M_2 = 2(k-1)n + 5n + 2.$$

Write $k = (k-1) + 1$ in the above expression. For $m = k$:

$$M_2 = 2[(k-1)+1]n + 5n + 2$$
$$= 2(k-1)n + 2n + 5n + 2$$
$$= 2kn + 5n + 2.$$

Thus, the expression holds for $m = k$.
By induction on m and n, the expression $M_2 = 2mn+5n+2$ is proven for all $m \geq 3$ and $n \geq 2$, using the assumption for $k-1$ in both steps.

Thus, all the faces of $P_n \circ K_{2,m}$ can have magic constant either $M_1 = 2mn + 4n + 2$ or $M_2 = 2mn + 5n + 2$. Hence, $P_n \circ K_{2,m}$ admits a super bimagic labeling of type $(1, 0, 0)$.

Corollary 1.1: The difference between the two magic constants M_1 and M_2 in $P_n \circ K_{2,m}$ is n.

Corollary 1.2: For the magic constant $M_1 = 2mn+4n+2$, where $m \geq 3$ and $n \geq 2$, the difference between the magic constants when n increases by 1 is $2m + 4$.

Proof: Substituting $n = k$ into the given expression.

$$M_1 = 2mk + 4k + 2$$

Substituting $n = k - 1$ into the given expression.

$$M_1 = 2m(k-1) + 4(k-1) + 2$$

$$\triangle M_1 = [2mk+4k+2] - [2m(k-1)+4(k-1)+2] = 2m+4.$$

■

Corollary 1.3: For the magic constant $M_2 = 2mn+5n+2$, where $m \geq 3$ and $n \geq 2$, the difference between the magic constants when n increases by 1 is $2m + 5$.

Example: The graph $P_3 \circ K_{2,4}$ satisfies super bimagic labeling of type $(1, 0, 0)$. The labels are assigned to the vertices of the rooted product of P_3 and $K_{2,4}$ as shown in Fig. 3. The magic constant of each face is computed using the expression given in Theorem 1. The magic constants obtained using the expressions for $P_3 \circ K_{2,4}$ are 38 and 41, which are the same as the values obtained by summing the vertex labels around each face of the graph.

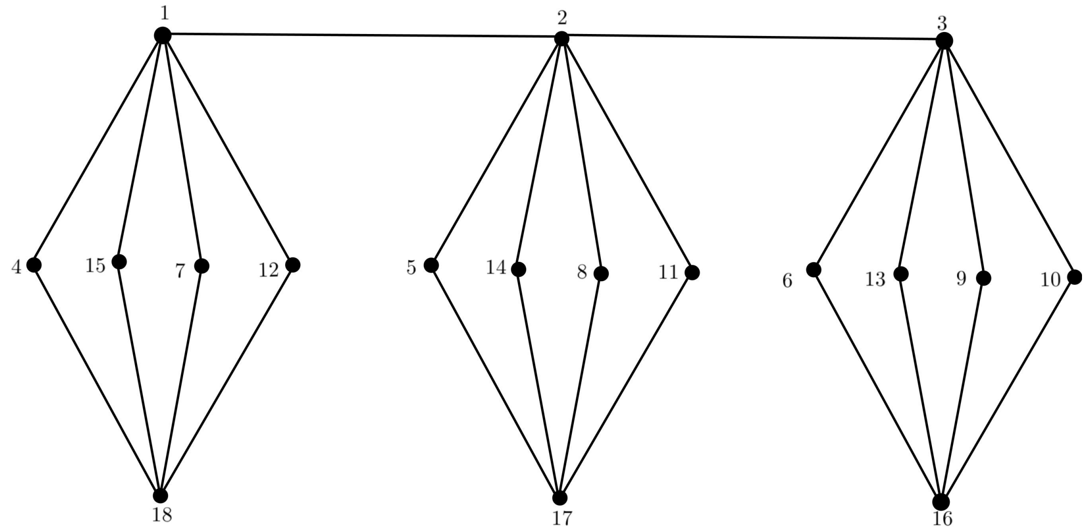

Fig 3. A super bimagic labeling of type (1,0,0) of $P_3 \circ K_{2,4}$

Table 1 presents the values of the magic constant $M_1 = 2mn + 4n + 2$ for the graph $P_n \circ K_{2,m}$, obtained by varying n from 2 to 10 and m from 3 to 10.

This is verified by constructing the graphs, assigning appropriate values to the vertices of $P_n \circ K_{2,m}$, and computing the magic constants by summing all the vertex labels around each face for various values of m and n up to 10.

Table 1: The values of magic constant M_1 for $P_n \circ K_{2,m}$ by varying n and m

$n \backslash m$	3	4	5	6	7	8	9	10
2	22	26	30	34	38	42	46	50
3	32	38	44	50	56	62	68	74
4	42	50	58	66	74	82	90	98
5	52	62	72	82	92	102	112	122
6	62	74	86	98	110	122	134	146
7	72	86	100	114	128	142	156	170
8	82	98	114	130	146	162	178	194
9	92	110	128	146	164	182	200	218
10	102	122	142	162	182	202	222	242

Table 2 presents the values of the magic constant $M_2 = 2mn + 5n + 2$ for the graph $P_n \circ K_{2,m}$, obtained by varying n from 2 to 10 and m from 3 to 10.

Table 2: The values of magic constant M_2 for $P_n \circ K_{2,m}$ by varying n and m

$n \backslash m$	3	4	5	6	7	8	9	10
2	24	28	32	36	40	44	48	52
3	35	41	47	53	59	65	71	77
4	46	54	62	70	78	86	94	102
5	57	67	77	87	97	107	117	127
6	68	80	92	104	116	128	140	152
7	79	93	107	121	135	149	163	177
8	90	106	122	138	154	170	186	202
9	111	119	137	155	173	191	209	227
10	112	132	152	172	192	212	232	252

Note: The graph $P_n \circ K_{2,2}$ admits a super magic labeling of type (1,0,0) with magic constant $8n + 2$.

Example: The graph $P_5 \circ K_{2,2}$ satisfies super magic labeling of type (1,0,0) and the magic constant is 42, which is obtained by summing the vertex labels around each face of the graph shown in Fig. 4.

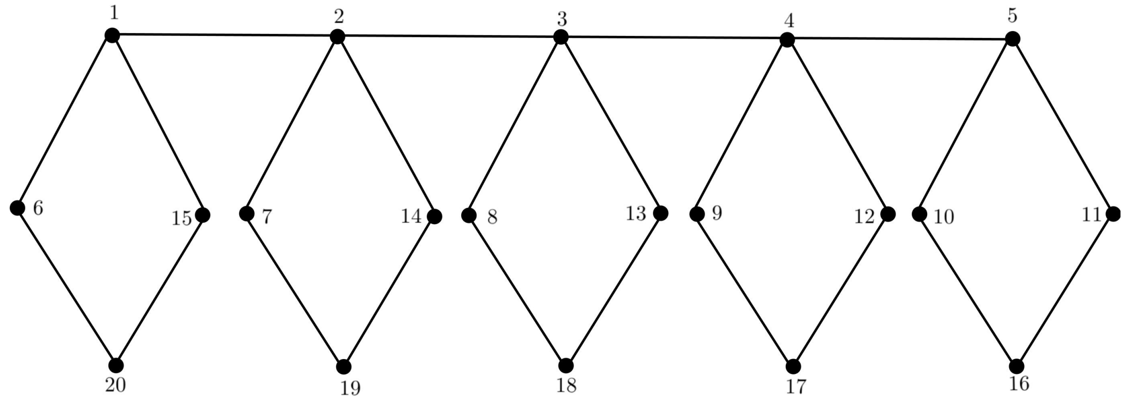

Fig 4. A super magic labeling of type (1,0,0) of $P_5 \circ K_{2,2}$

III. 1-ANTIMAGIC LABELING OF TYPE (1,0,1) OF $P_n \circ K_{2,m}$

Theorem 2: For odd m, the graph $P_2 \circ K_{2,m}$ admits a super 1-antimagic labeling of type $(1, 0, 1)$.

Proof: Let G be the rooted product of graphs P_2 and $K_{2,m}$ with vertex set $V(G)$, where $|V(G)| = 2(m+2)$ and face set $F(G)$, where $|F(G)| = 2(m-1)$. Fig. 5 shows the vertex and face labels of $P_2 \circ K_{2,m}$ in general.

The vertex labels are similar to the labels in $(1, 0, 0)$ as described in Theorem 1. So, the magic constants obtained in Theorem 1 are added to the face label to find the face weight W of each face in $P_2 \circ K_{2,m}$. The faces are labeled as described below.
A face labeling function $h : F(G) \longrightarrow \{mn+2n+1, mn+2n+2, \ldots, 2mn+n\}$, a bijection which is defined as follows:

$$h(f_r) = 2mn + n + 1 - r.$$

For $n = 2$,

$$h(f_r) = 4m + 3 - r.$$

We know that the sum of the vertex labels around all the faces in F_1 is $M_1 = 2mn + 4n + 2$.
The sum of the vertex labels and the face label around all the faces in F_1 with $n = 2$ is calculated as follows:

$$W = 2mn + 4n + 2 + 2mn + n + 1 - r$$

$$= 8m + 13 - r.$$

This will give the alternate values since only alternate faces are considered and all values will be even.
We know that the sum of the vertex labels around all the faces in F_2 is $M_2 = 2mn + 5n + 2$.

The sum of the vertex labels and the face labels around all the faces in F_2 with $n = 2$ is calculated as follows:

$$W = 2mn + 5n + 2 + 2mn + n + 1 - r$$

$$= 8m + 15 - r.$$

This will also give the alternate values since only alternate faces are considered and all values will be odd.

Since $8m + 13 - r$ generates all even numbers and $8m + 15 - r$ generates all odd numbers and they appear in an alternating pattern with the difference between successive terms from these two expressions being 1, they together form a continuous sequence. This ensures that all integers appear consecutively, satisfying the conditions for 1-antimagic labeling. Thus, $P_2 \circ K_{2,m}$ admits a super 1-antimagic labeling of type $(1, 0, 1)$, when m is odd. ∎

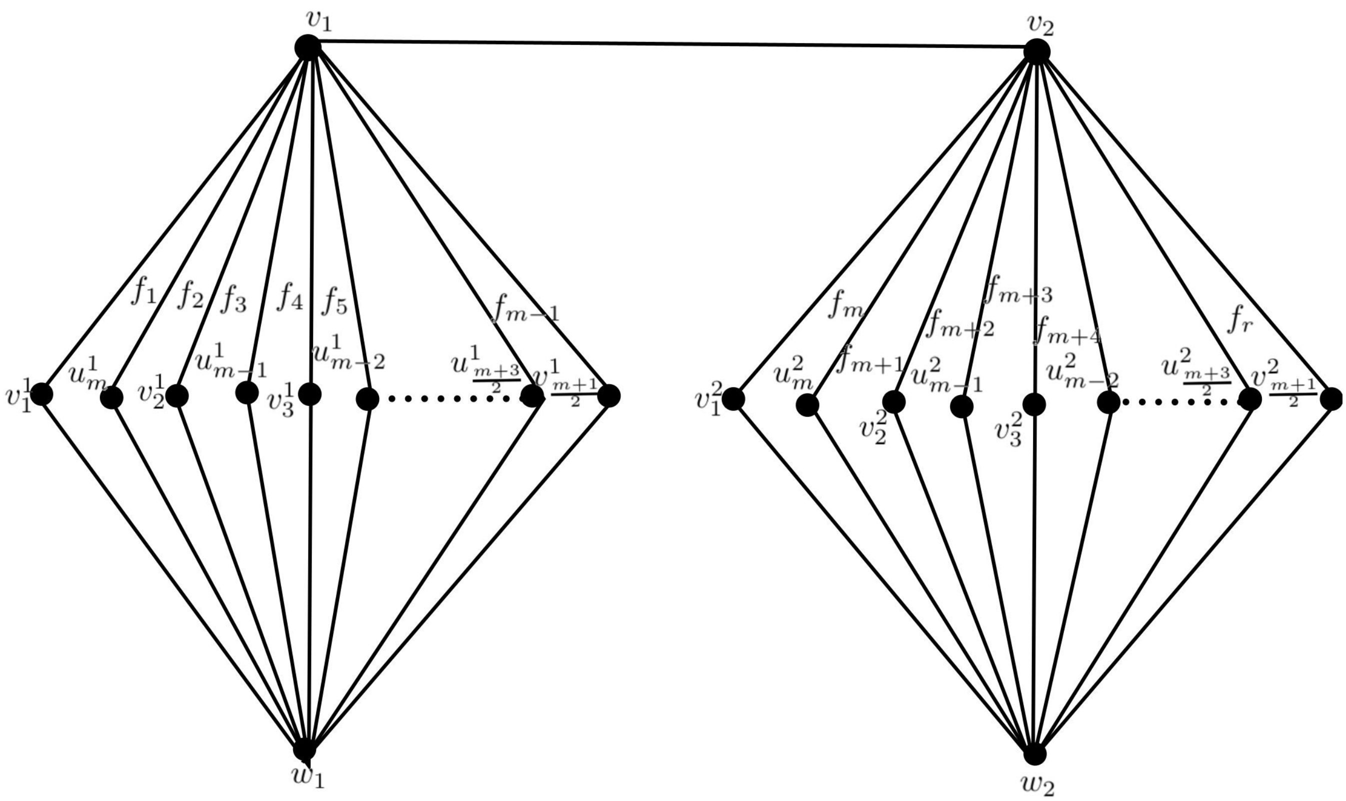

Fig 5. $P_2 \circ K_{2,m}$, where m is odd

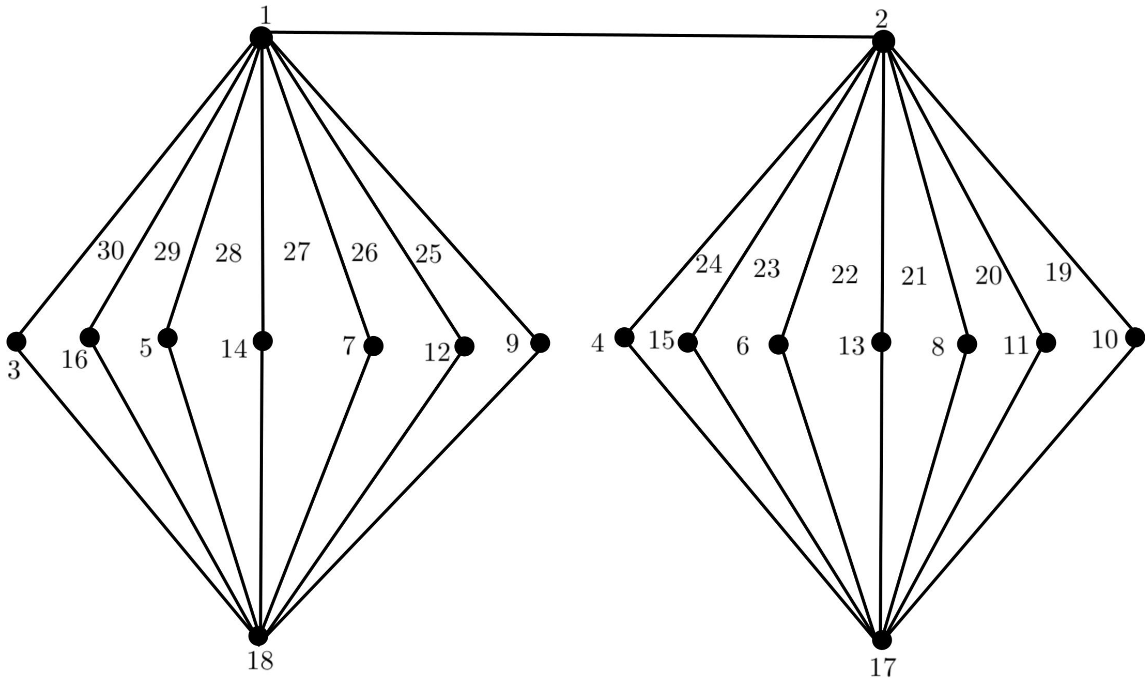

Fig 6. A super 1-antimagic labeling of type (1,0,1) of $P_2 \circ K_{2,7}$

Example: The graph $P_2 \circ K_{2,7}$ in Fig. 6 satisfies super 1-antimagic labeling of type $(1, 0, 1)$. The weight of all the faces is computed using the expression given in Theorem 2. The face weights obtained using the expression range from 58 to 69, which are the same as the face weights obtained by summing the vertex labels and face labels around each face of the graph shown in Fig. 6.

IV. Conclusion

This study has investigated super bimagic labeling and super 1-antimagic labeling in $P_n \circ K_{2,m}$, revealing the interplay between vertices, edges, and faces of a planar graph. By analyzing these labeling techniques, this research highlights their significance in characterizing the structural properties of graphs. This research contributes to the growing body of knowledge in graph theory, enhancing the understanding of labeling methodologies and providing a foundation for future studies in this field. The ongoing evolution of graph labeling theories continues to strengthen theoretical foundations while fostering advancements in real-world applications. Face labeling can serve as a valuable technique in cryptographic applications, offering a structured approach to secure data representation and encryption methods.

References

[1] Joseph A Gallian, "A dynamic survey of graph labeling", *Electronic Journal of Combinatorics, vol. 1 (DynamicSurveys)*, DS6-Dec, 2018.

[2] Ko-Wei Lih, "On magic and consecutive labelings of plane graphs", *Utilitas Mathematica*, vol. 24, pp. 165–197, 1983.

[3] Martin Baca, "On consecutive labeling of plane graphs", *Journal of the Franklin Institute*, vol. 328, no. 2-3, pp. 249–253, 1991.

[4] Martin Baca and Mirka Miller, "On d-antimagic labelings of type (1,1,1) for prisms", *Jounal of Combinatorial Mathematics and Combinatorial Computing*, vol. 44, pp. 199–208, 2003.

[5] Jia-Bao Liu, Hafiz Usman Afzal and Muhammad Javaid, "Computing edge weights of magic labeling on rooted products of graphs", *Mathematical Problems in Engineering*, vol. 2020, pp. 1–16, 2020.

[6] Mohammed Ali Ahmed and J. Baskar Babujee, "On Face Bimagic Labeling of Graphs", *IOSR Journal of Mathematics*, vol. 12, no. 6, pp. 1–7, 2016.

[7] Medini H. R., Sabitha D'Souza, Devadas Nayak C and Pradeep G. Bhat, "Exploring secure communication through artistic graph creation", *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 28, no. 1, 143–159, 2025.

[8] Medini H. R., Sabitha D'Souza, Devadas Nayak C and Pradeep G. Bhat, "Multifaceted coding of messages using the concepts of graph theory", *IAENG International Journal of Computer Science*, vol. 51, no. 2, pp. 143-153, 2024.