# Two-Grid Domain Decomposition Methods for the Steady-State Navier-Stokes-Darcy Model

Hao Zheng, Liyun Zuo

*Abstract*—**We develop a family of robin-type domain decomposition techniques, employing a two-grid methodology, for the efficient solution of the Navier-Stokes-Darcy system. Our methodology commences by utilizing the well-established robin-type domain decomposition technique to generate initial coarse grid approximations. Following this, we enhance the fine grid problem formulation within the two grid methodologys framework, by substituting certain interface conditions with data derived from the coarse grid, thereby refining the overall solution process.**

*Index Terms*—**Navier-Stokes equations, Domain decomposition method, Two-grid method, Darcy's law.**

## I. Introduction

THE coupled problems are increasingly prevalent in a wide range of natural and industrial applications. Notable examples include the analysis of groundwater fluid flow, the transport of materials in industrial filtration systems, and the study of blood flow dynamics within arteries. This study investigates the Navier-Stokes-Darcy system, integrating free fluid flow principles from the Navier-Stokes equations with porous media flow behavior as described by Darcy's law. This integration is achieved through an interface that connects two distinct subdomains.

Extensive research has focused on the creation and evaluation of numerical methods for addressing the coupled problem. The methods encompass a broad spectrum of approaches, including: two-grid or multi-grid methods [1], [2], [3], [4], [5], local and parallel finite element methods [6], [7], [8], domain decomposition method [9], [10], [11], [12], [13], [14], [23], lagrange multiplier method [15], [16], discontinuous Galerkin method [17], [18], [19], [20], and several other ways [21], [22], [24], [25], [26], [27], [28]. These methodologies represent the forefront of computational techniques aimed at addressing the complex interactions between fluid flows and porous media, thereby enhancing our ability to model and understand phenomena across a wide range of engineering and scientific fields.

Among the various methods considered, the two-grid approach is particularly noteworthy for its effectiveness in addressing the inherent difficulties associated with the strong interdependencies between different models across separate domains. Similarly, the domain decomposition technique, augmented by robin-type interface boundary conditions, has proven highly effective in tackling the complexities of multi-domain problems and integrating diverse physical processes. The efficacy of this method is underpinned by the availability of robust solvers for the individual, disentangled subproblems.

Sun recently proposed the two-grid domain decomposition method [23] for solving coupled problems, enhancing solution accuracy and efficiency via robin-type domain decomposition within a two-grid framework.

We have performed an in-depth analysis of the interaction between the two equations within their coupled system. Leveraging the domain decomposition method [9] and the two-grid domain decomposition method [23], we introduce a novel two-grid domain decomposition technique. We incorporate the BJ interface condition for the model. The proposed method begins by employing a robin-type domain decomposition technique to get the solution on the coarse grid. Afterward, the interface terms are replaced by the solution from the coarse grid. Thereby obtaining a corrected fine grid problem. The method combines the advantages of both domain decomposition and two-grid techniques. An error analysis has been conducted to demonstrate the convergence of the method.

The subsequent sections of this paper are outlined below. Section II gives the Coupled Navier-Stokes-Darcy problem. In section III, the essential method is outlined. Section IV introduces the newly proposed method. Section V provides a clear error analysis of the newly method, demonstrating its superiorities compared to other algorithms. In section VI, we discuss the prospective applications and future advancements of the two-grid domain decomposition method.

## II. Coupled Navier-Stokes-Darcy Problem

We introduce a coupled Navier-Stokes-Darcy system within a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, consisting of a fluid region $\Omega_f$ and a porous medium region $\Omega_p$, separated by the interface $\Gamma = \partial\Omega_f \cap \partial\Omega_p$. Here, $\Omega_f \cap \Omega_p = \emptyset$, $\overline{\Omega}_f \cup \overline{\Omega}_p = \overline{\Omega}$. Let $\Gamma_f = \Omega_f \setminus \Gamma$, $\Gamma_p = \Omega_p \setminus \Gamma$.

Within the $\Omega_f$, the Navier-Stokes equations govern the fluid flow:

$$\begin{cases} \overrightarrow{u}_f \cdot \nabla \overrightarrow{u}_f - \nabla \cdot (\boldsymbol{T}(\overrightarrow{u}_f, p_f)) = \overrightarrow{f}_1, \\ \nabla \cdot \overrightarrow{u}_f = 0, \end{cases} \tag{1}$$

among them

$$\boldsymbol{T}(\overrightarrow{u}_f, p_f) = -p_f \boldsymbol{I} + 2\nu \boldsymbol{D}(\overrightarrow{u}_f),$$
$$\boldsymbol{D}(\overrightarrow{u}_f) = \frac{1}{2}(\nabla^T \overrightarrow{u}_f + \nabla \overrightarrow{u}_f),$$

$\overrightarrow{u}_f$ and $p_f$ denote the fluid velocity and kinematic pressure within $\Omega_f$. Additionally, $\overrightarrow{f}_1$ represents the external body force, and $\boldsymbol{T}(\overrightarrow{u}_f, p_f)$ is the stress tensor, which involves the identity matrix $\boldsymbol{I}$ and the fluid's kinematic viscosity $\nu > 0$.

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Within the $\Omega_p$, the Darcy equation govern the porous medium flow:

$$\begin{cases} \nabla \cdot \vec{u}_p = f_2, \\ \vec{u}_p = -\boldsymbol{K}\nabla\phi_p, \end{cases} \quad (2)$$

in the porous medium domain $\Omega_p$, $\vec{u}_p$ denotes the fluid discharge rate. Without loss of generality, the hydraulic conductivity tensor is considered isotropic which is represented by $\boldsymbol{K}$. And $\phi_p = z + \frac{p_p}{\rho g}$ means the piezometric head. Involving the dynamic pressure $p_p$, height $z$, fluid density $\rho$, and gravitational acceleration $g$. Additionally, the source term $f_2$ satisfies the solvability condition.

$$\int_{\Omega_p} f_2 = 0.$$

By integrating Darcy's law with (2), we obtain the elliptic partial differential equation:

$$-\nabla \cdot (\boldsymbol{K}\nabla\phi_p) = f_2. \quad (3)$$

Assume the $\vec{u}_f$ and $\phi_p$ satisfying homogeneous Dirichlet boundary conditions:

$$\vec{u}_f = 0 \text{ on } \Gamma_f, \qquad \phi_p = 0 \text{ on } \Gamma_p.$$

The interface $\Gamma$ is subject to three interface conditions, stated as:

$$\begin{cases} \vec{u}_f \cdot \vec{n}_f + \vec{u}_p \cdot \vec{n}_p = 0, \\ \frac{1}{2}\vec{u}_f \cdot \vec{u}_f - (\boldsymbol{T}(\vec{u}_f, p_f) \cdot \vec{n}_f) \cdot \vec{n}_f \\ \qquad = g\phi_p - gz, \\ -(\boldsymbol{T}(\vec{u}_f, p_f) \cdot \vec{n}_f) \cdot \vec{\tau}_i \\ \qquad = \frac{\alpha\nu\sqrt{d}}{\sqrt{trace(\prod)}} \cdot \vec{\tau}_i \cdot (\vec{u}_f - \vec{u}_p), \end{cases} \quad (4)$$

where $\vec{n}_f$ and $\vec{n}_p$ are the unit outer normals at the interface $\Gamma$, respectively. The vectors $\vec{\tau}_i$ $(i = 1, \cdots, d-1)$ are mutually orthogonal unit tangential vectors to $\Gamma$, $\alpha$ is a constant parameter, and $\prod = \frac{\boldsymbol{K}\nu}{g}$.

To introduce the weak formulation of the mixed model, we set

$$H_f = \{\vec{v}_f \in (H^1(\Omega_f))^d : \vec{v}_f = 0 \text{ on } \Gamma_f\},$$
$$H_p = \{\psi_p \in H^1(\Omega_p) : \psi_p = 0 \text{ on } \Gamma_p\},$$
$$Q_f = L^2(\Omega_f).$$

We use $(\cdot,\cdot)_{\Omega_X}$ and $\|\cdot\|_{L^2(\Omega_X)}$ to denote the standard $L^2$-scalar product of the spaces $L^2(\Omega_X)(X = f, p)$ and the associated $L^2$-norms of the space $L^2(\Omega_X)$, respectively.

## III. Domain Decomposition Method

We provide an overview of the domain decomposition method as described in [9]. This approach breaks down the Navier-Stokes-Darcy into two distinct subproblems, which are solved in parallel within the $\Omega_f$, and $\Omega_p$. Through the use of domain decomposition, the computational problem is effectively downsized, allowing for the leveraging of established software packages to solve each subproblem independently.

We will present the key robin-type interface conditions. For two predetermined positive constants $\xi_f$ and $\xi_p$, there exist corresponding functions $g_f$, $g_p$, and $g_{f\tau}$ on the interface $\Gamma$ which satisfy the following relationship:

$$(\boldsymbol{T}(\vec{u}_f, p_f) \cdot \vec{n}_f) \cdot \vec{n}_f - \frac{1}{2}\vec{u}_f \cdot \vec{u}_f + \xi_f \vec{u}_f \cdot \vec{n}_f = g_f, \quad (5)$$

$$\xi_p \boldsymbol{K}\nabla\phi_p \cdot \vec{n}_p + g\phi_p = g_p, \quad (6)$$

$$-\frac{\alpha\nu\sqrt{d}}{\sqrt{trace(\prod)}} P_\tau \vec{u}_f - P_\tau(\boldsymbol{T}(\vec{u}_f, p_f) \cdot \vec{n}_f) = g_{f\tau}. \quad (7)$$

By (4), we can get

$$g_f = \xi_f \vec{u}_f \cdot \vec{n}_f - g\phi_p + gz \quad \text{on } \Gamma, \quad (8)$$
$$g_p = \xi_p \vec{u}_f \cdot \vec{n}_f + g\phi_p \quad \text{on } \Gamma, \quad (9)$$
$$g_{f\tau} = \frac{\alpha\nu\sqrt{d}}{\sqrt{trace(\prod)}} P_\tau(\boldsymbol{K}\nabla\phi_p) \quad \text{on } \Gamma. \quad (10)$$

It is easy to verify that the interface conditions (4) are equivalent to the aforementioned robin-type conditions (5)-(7) if and only if the functions $g_f$, $g_p$, and $g_{f\tau}$ fulfill the compatibility requirements on the interface $\Gamma$.

Then the weak formulation as follows: for three given functions $g_f$, $g_p$, and $g_{f\tau}$ and two normal numbers $\xi_f, \xi_p$, find $(\vec{u}_f, p_f) \in H_f \times Q_f$, $\phi_p \in H_p$ such that

$$a_p(\phi_p, \psi_p) + \langle \frac{g\phi_p}{\xi_p}, \psi_p \rangle = \langle \frac{g_p}{\xi_p}, \psi_p \rangle + (f_2, \psi_p)_{\Omega_p}, \quad \forall\, \psi_p \in H_p, \quad (11)$$

$$\begin{aligned} &\delta b_f(\vec{u}_f, \vec{u}_f, \vec{v}_f) + a_f(\vec{u}_f, \vec{v}_f) - \delta d_f(\vec{v}_f, p_f) \\ &+ \delta d_f(\vec{u}_f, q_f) + \delta\xi_f \langle \vec{u}_f \cdot \vec{n}_f, \vec{v}_f \cdot \vec{n}_f \rangle \\ &+ \frac{\alpha\sqrt{d}}{\sqrt{trace(\prod)}} \langle P_\tau \vec{u}_f, P_\tau \vec{v}_f \rangle \\ &= \delta(\vec{f}_1, \vec{v}_f)_{\Omega_f} + \delta\langle g_f, \vec{v}_f \cdot \vec{n}_f \rangle - \delta\langle g_{f\tau}, P_\tau \vec{v}_f \rangle, \\ &\qquad \forall\, (\vec{v}_f, q_f) \in H_f \times Q_f, \end{aligned} \quad (12)$$

where

$$\delta = \frac{1}{\nu},$$
$$P_\tau \vec{u}_f = \sum_{j=1}^{d-1} (\vec{u}_f \cdot \vec{\tau}_j) \vec{\tau}_j$$

$P_\tau \vec{u}_f$ is the projection onto the tangent space on $\Gamma$.

The bilinear forms are

$$a_f(\vec{u}_f, \vec{v}_f) = (\nabla \vec{u}_f, \nabla \vec{v}_f)_{\Omega_f},$$
$$a_p(\phi_D, \psi_D) = (\boldsymbol{K}\nabla\phi_p, \nabla\psi_p)_{\Omega_p},$$
$$d_f(\vec{v}_f, q_f) = (\nabla \cdot \vec{v}_f, q_f)_{\Omega_f},$$

and the trilinear form is

$$\begin{aligned} b_f(\vec{u}_f, \vec{u}_f, \vec{v}_f) = &(\vec{u}_f \cdot \nabla \vec{u}_f, \vec{v}_f)_{\Omega_f} \\ &- \frac{1}{2}\langle \vec{u}_f \cdot \vec{u}_f, \vec{v}_f \cdot \vec{n}_f \rangle \\ &+ \frac{1}{2}((\nabla \cdot \vec{u}_f)\vec{u}_f, \vec{v}_f)_{\Omega_f}. \end{aligned}$$

Because $b_f(\vec{u}_f, \vec{u}_f, \vec{v}_f)$ is continuous on the space triplet

$H_f \times H_f \times H_f$, we have

$$\begin{aligned} & b_f(\vec{u}_f, \vec{u}_f, \vec{v}_f) \\ & = (\vec{u}_f \cdot \nabla \vec{u}_f, \vec{v}_f)_{\Omega_f} + \frac{1}{2}((\nabla \cdot \vec{u}_f)\vec{u}_f, \vec{v}_f)_{\Omega_f} \\ & - \frac{1}{2}\langle \vec{u}_f \cdot \vec{u}_f, \vec{v}_f \cdot \vec{n}_f \rangle \\ & = \frac{1}{2}(\vec{u}_f \cdot \nabla \vec{u}_f, \vec{v}_f)_{\Omega_f} - \frac{1}{2}(\vec{u}_f \cdot \nabla \vec{v}_f, \vec{u}_f)_{\Omega_f} \\ & + \frac{1}{2}\langle \vec{u}_f \cdot \vec{v}_f, \vec{u}_f \cdot \vec{n}_f \rangle - \frac{1}{2}\langle \vec{u}_f \cdot \vec{u}_f, \vec{v}_f \cdot \vec{n}_f \rangle \\ & \qquad \forall \vec{u}_f, \vec{v}_f \in H_f. \end{aligned}$$

According to the value of $\frac{1}{2}((\nabla\cdot\vec{u}_f)\vec{u}_f, \vec{v}_f)_{\Omega_f}$ being equal to 0, the trilinear form is suitable for the current problem (1)-(4). Also, it satisfy the following identity:

$$b_f(\vec{u}_f, \vec{u}_f, \vec{u}_f) = 0 \qquad \forall \vec{u}_f \in H_f.$$

He and others have proved the well-posedness of above weak formulation (11)-(12) in [9].

The following finite element discretization methodology is crucial for the domain decomposition approach. Let $\mathcal{T}_h$ be a regular quasi-uniform triangulation of $\overline{\Omega}$ with mesh size $h > 0$. In addition, $\mathcal{B}_h$ represents the segmentation of $\Gamma$ derived by $\mathcal{T}_h$.

Let $H_{f,h} \subset H_f$, $Q_{f,h} \subset Q_f$ and $H_{p,h} \subset H_p$ be the finite element subspaces defined on the partition $\mathcal{T}_h$. The P2-P1 to Navier-Stokes problem finite element, Darcy by matching the P2 finite element problem.

$$\begin{aligned} H_{f,h} &= \{\overrightarrow{v_{f,h}} \in (H^1(\Omega_f))^d : \overrightarrow{v_{f,h}} \mid_T \in (\mathbb{P}_2(T))^d \\ & \qquad \forall T \in \mathcal{T}_{f,h}, \overrightarrow{v_{f,h}} \mid_{\Gamma_f} = 0\}, \\ Q_{f,h} &= \{q_{f,h} \in L^2(\Omega_f) : q_{f,h} \mid_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_{f,h}\}, \\ H_{p,h} &= \{\psi_{p,h} \in H^1(\Omega_p) : \psi_{p,h} \mid_T \in \mathbb{P}_2(T) \\ & \qquad \forall T \in \mathcal{T}_{p,h}, \psi_{p,h} \mid_{\Gamma_p} = 0\}, \end{aligned}$$

the Spaces $H_{f,h}$ and $Q_{f,h}$ satisfy the inf-sup condition.

The discrete trace space on the interface is defined below.:

$$\begin{aligned} X_h &= \{g_h \in L^2(\Gamma) : \ g_h \mid_\tau \in \mathbb{P}_2(\tau) \ \forall \tau \in \mathcal{B}_h, \ g_h \mid_{\partial\Gamma} = 0\} \\ &= H_{f,h} \mid_\Gamma \cdot \vec{n}_f = H_{p,h} \mid_\Gamma . \end{aligned}$$

Drawing upon the robin conditions for the Navier-Stokes-Darcy equation and compatibility conditions (8)-(10), we can outline the method as described in [9]:

1) Initial values of $g_f^0$, $g_p^0$,and $g_{f\tau}^0$ are guessed.

2) For n=0,1,2, find $\phi_{p,h}^n \in H_{p,h}$ satisfy

$$a_p(\phi_{p,h}^n, \psi_p) + \langle \frac{g\phi_{p,h}^n}{\xi_p}, \psi_p \rangle = \langle \frac{g_{p,h}^n}{\xi_p}, \psi_p \rangle + (f_2, \psi_p)_{\Omega_p}, \quad \forall \ \psi_p \in H_{p,h}, \quad (13)$$

and $(\vec{u}_{f,h}^n, p_{f,h}^n) \in H_{f,h} \times Q_{f,h}$ satisfy

$$\begin{aligned} & \delta b_f(\vec{u}_{f,h}^n, \vec{u}_{f,h}^n, \vec{v}_f) + a_f(\vec{u}_{f,h}^n, \vec{v}_f) - \delta d_f(\vec{v}_f, p_{f,h}^n) \\ & + \delta d_f(\vec{u}_{f,h}^n, q_f) + \delta\xi_f\langle \vec{u}_{f,h}^n \cdot \vec{n}_f, \vec{v}_f \cdot \vec{n}_f \rangle \\ & + \frac{\alpha\sqrt{d}}{\sqrt{trace(\prod)}}\langle P_\tau \vec{u}_{f,h}^n, P_\tau \vec{v}_f \rangle \\ & = \delta(\vec{f}_1, \vec{v}_f)_{\Omega_f} + \delta\langle g_{f,h}^n, \vec{v}_f \cdot \vec{n}_f \rangle - \delta\langle g_{f\tau,h}^n, P_\tau \vec{v}_f \rangle, \\ & \qquad \forall \ (\vec{v}_f, q_f) \in H_{f,h} \times Q_{f,h}, \quad (14) \end{aligned}$$

respectively.

3) Update $g_{f,h}^{n+1}, g_{p,h}^{n+1}, g_{f\tau,h}^{n+1}$ by the following way:

$$\begin{aligned} g_{f,h}^{n+1} &= \frac{\xi_f}{\xi_p} g_{p,h}^n - (1 + \frac{\xi_f}{\xi_p}) g\phi_{p,h}^n + gz, \\ g_{p,h}^{n+1} &= -g_{f,h}^n + (\xi_f + \xi_p)\vec{u}_{f,h}^n \cdot \vec{n}_f + gz, \\ g_{f\tau,h}^{n+1} &= \frac{\alpha\nu\sqrt{d}}{\sqrt{trace(\prod)}} P_\tau(\boldsymbol{K}\nabla\phi_{p,h}^n) \end{aligned}$$

The convergence analysis of domain decomposition method has been obtained in [9]. Additionally, the method attains an error estimate that is independent of the mesh size h when $\xi_f < \xi_p$, provided that the parameters $\xi_f$ and $\xi_p$ are selected judiciously, in accordance with specific control criteria. The robin-robin method also approximates the decoupled Navier-Stokes-Darcy problem with FEM and boundary conditions (5)-(7). Specifically, for given functions $g_{f,h}$, $g_{p,h}$, $g_{f\tau,h}$ and two normal numbers $\xi_f$, $\xi_p$, the method seeks to find $(\vec{u}_{f,h}, p_{f,h}) \in H_{f,h} \times Q_{f,h}$ and $\phi_{p,h} \in H_{p,h}$ such that

$$a_p(\phi_{p,h}, \psi_p) + \langle \frac{g\phi_{p,h}}{\xi_p}, \psi_p \rangle = \langle \frac{g_{p,h}}{\xi_p}, \psi_p \rangle + (f_2, \psi_p)_{\Omega_p}, \quad \forall \ \psi_p \in H_{p,h}, \quad (15)$$

$$\begin{aligned} & \delta b_f(\vec{u}_{f,h}, \vec{u}_{f,h}, \vec{v}_f) + a_f(\vec{u}_{f,h}, \vec{v}_f) - \delta d_f(\vec{v}_f, p_{f,h}) \\ & + \delta d_f(\vec{u}_{f,h}, q_f) + \frac{\alpha\sqrt{d}}{\sqrt{trace(\prod)}}\langle P_\tau \vec{u}_{f,h}, P_\tau \vec{v}_f \rangle \\ & + \delta\xi_f\langle \vec{u}_{f,h} \cdot \vec{n}_f, \vec{v}_f \cdot \vec{n}_f \rangle \\ & = \delta(\vec{f}_1, \vec{v}_f)_{\Omega_f} + \delta\langle g_{f,h}, \vec{v}_f \cdot \vec{n}_f \rangle - \delta\langle g_{f\tau,h}, P_\tau \vec{v}_f \rangle, \\ & \qquad \forall \ (\vec{v}_f, q_f) \in H_{f,h} \times Q_{f,h}, \quad (16) \end{aligned}$$

with the compatibility conditions:

$$g_{f,h} = \xi_f \vec{u}_{f,h} \cdot \vec{n}_f + gz - g\phi_{p,h} \quad \text{on} \ \Gamma, \quad (17)$$

$$g_{p,h} = \xi_p \vec{u}_{f,h} \cdot \vec{n}_f + g\phi_{p,h} \quad \text{on} \ \Gamma, \quad (18)$$

$$g_{f\tau,h} = \frac{\alpha\nu\sqrt{d}}{\sqrt{trace(\prod)}} P_\tau(\boldsymbol{K}\nabla\phi_{p,h}) \quad \text{on} \ \Gamma. \quad (19)$$

## IV. The two-grid domain decomposition method

This section focuses on a decoupling strategy for solving the coupled equations. Inspired by the method from the previous section III, we propose a tailored two-grid domain decomposition method specifically designed for the Navier-Stokes-Darcy model.

The two-grid domain decomposition method for solving the coupled problem proceeds in two successive steps, as outlined below.

1) On a coarse grid with mesh size H, we recall domain decomposition method to solve problems (15)-(16). Then we obtain the coarse grid result $g_{f,H}$, $g_{p,H}$, $g_{f\tau,H}$.

2) An modified fine grid problem is constructed and solved

by finding $(\overrightarrow{u}_f^h, p_f^h) \in H_{f,h} \times Q_{f,h}$, $\phi_p^h \in H_{p,h}$, such that

$$a_p(\phi_p^h, \psi_p) + \langle \frac{g\phi_p^h}{\xi_p}, \psi_p\rangle = \langle \frac{g_{p,H}}{\xi_p}, \psi_p\rangle + (f_2, \psi_p)_{\Omega_p}, \quad \forall\ \psi_p \in H_{p,h}, \quad (20)$$

$$\begin{aligned} &\delta b_f(\overrightarrow{u}_f^h, \overrightarrow{u}_f^h, \overrightarrow{v}_f) + a_f(\overrightarrow{u}_f^h, \overrightarrow{v}_f) - \delta d_f(\overrightarrow{v}_f, p_f^h) \\ &+ \delta d_f(\overrightarrow{u}_f^h, q_f) + \delta\xi_f\langle \overrightarrow{u}_f^h \cdot \overrightarrow{n}_f, \overrightarrow{v}_f \cdot \overrightarrow{n}_f\rangle \\ &+ \frac{\alpha\sqrt{d}}{\sqrt{trace(\prod)}}\langle P_\tau \overrightarrow{u}_f^h, P_\tau \overrightarrow{v}_f\rangle \\ &= \delta(\overrightarrow{f}_1, \overrightarrow{v}_f)_{\Omega_f} + \delta\langle g_{f,H}, \overrightarrow{v}_f \cdot \overrightarrow{n}_f\rangle - \delta\langle g_{f\tau,H}, P_\tau \overrightarrow{v}_f\rangle, \\ &\qquad \forall\ (\overrightarrow{v}_f, q_f) \in H_{f,h} \times Q_{f,h}. \quad (21)\end{aligned}$$

The two-grid domain decomposition method utilizes the advantages of both the two-grid strategy and the domain decomposition framework to enhance its performance. It seamlessly navigates the complex coupling between disparate models across separate domains. Additionally, this method is particularly well-suited for addressing multi-domain and multi-physics coupling challenges. For decoupled solutions, it significantly amplifies the efficiency of computations.

## V. Error analysis

We will echoes the rationale presented in [9] to demonstrate the convergence of the proposed method. For the sake of brevity, we introduce the notation $x \lesssim y$ to indicate that $x$ is less than or comparable to $Cy$. $C$ is a generic constant that may take on various values depending on the context. We now revisit the error estimates for the decoupled scheme as discussed in [9]:

$$\begin{aligned} &\|\overrightarrow{u}_f - \overrightarrow{u}_{f,h}\|_1 \lesssim h^2, && \|\overrightarrow{u}_f - \overrightarrow{u}_{f,h}\| \lesssim h^3, \\ &\|\phi_p - \phi_{p,h}\|_1 \lesssim h^2, && \|\phi_p - \phi_{p,h}\| \lesssim h^3, \\ &\|p_f - p_{f,h}\| \lesssim h^2. \end{aligned}$$

For the finite element approximation given by equations (15)-(16), we express the error functions, which are related to the discrepancies between the solution components on coarse and fine meshes, as follows:

$$\begin{aligned} &\sigma_{f,H} = g_{f,h} - g_{f,H}, && \sigma_{p,H} = g_{p,h} - g_{p,H}, \\ &\sigma_{f\tau,H} = g_{f\tau,h} - g_{f\tau,H}, && \zeta_{f,H} = p_{f,h} - p_{f,H}, \\ &\theta_{p,H} = \phi_{p,h} - \phi_{p,H}, && \overrightarrow{\theta}_{f,H} = \overrightarrow{u}_{f,h} - \overrightarrow{u}_{f,H}. \end{aligned}$$

Then, by means of the triangle inequality, we can easily get several basic error estimates about the numerical solution of (15)-(16) on coarse and fine meshes

$$\begin{aligned} &\|\overrightarrow{\theta}_{f,H}\|_1 \lesssim H^2, && \|\overrightarrow{\theta}_{f,H}\| \lesssim H^3, \\ &\|\theta_{p,H}\|_1 \lesssim H^2, && \|\theta_{p,H}\| \lesssim H^3, \\ &\|\zeta_{f,H}\| \lesssim H^2. && \quad (22) \end{aligned}$$

To implement the error estimation, the following lemma is essential:

*Lemma 1:* Along the interface $\Gamma$, error estimates for $\sigma_{f,H}$ and $\sigma_{p,H}$ associated with the interface conditions are given by

$$\|\sigma_{f,H}\|_\Gamma \lesssim (\xi_f + g)H^{\frac{5}{2}}, \quad (23)$$

$$\|\sigma_{p,H}\|_\Gamma \lesssim (\xi_p + g)H^{\frac{5}{2}}, \quad (24)$$

$$\|\sigma_{f\tau,H}\|_\Gamma \lesssim \frac{\alpha\nu\sqrt{d}\boldsymbol{K}}{\sqrt{trace(\prod)}}H^{\frac{5}{2}}. \quad (25)$$

*Proof:* According to the definition of $\sigma_{f,H}$ , $\sigma_{p,H}$, $\sigma_{f\tau,H}$ and (17)-(19), we can derive the following formula

$$\begin{aligned} &\sigma_{f,H} = \xi_f \overrightarrow{\theta}_{f,H} \cdot \overrightarrow{n}_f - g\theta_{p,H}, \\ &\sigma_{p,H} = \xi_p \overrightarrow{\theta}_{f,H} \cdot \overrightarrow{n}_f + g\theta_{p,H}, \\ &\sigma_{f\tau,H} = \frac{\alpha\nu\sqrt{d}}{\sqrt{trace(\prod)}}P_\tau(\boldsymbol{K}\nabla\phi_{p,h} - \boldsymbol{K}\nabla\phi_{p,H}), \end{aligned}$$

Using the Young inequality we can launch

$$\begin{aligned} \|\sigma_{f,H}\|_\Gamma &= \|\xi_f \overrightarrow{\theta}_{f,H} \cdot \overrightarrow{n}_f - g\theta_{p,H}\|_\Gamma \\ &\le \xi_f\|\overrightarrow{\theta}_{f,H} \cdot \overrightarrow{n}_f\|_\Gamma + g\|\theta_{p,H}\|_\Gamma. \end{aligned}$$

Based on the trace inequality, we are aware that there exists $C$ such that

$$\begin{aligned} &\|\overrightarrow{\theta}_{f,H} \cdot \overrightarrow{n}_f\|_\Gamma \le C\|\theta_{f,H}\|^{\frac{1}{2}}\|\theta_{f,H}\|_1^{\frac{1}{2}}, \\ &\|\theta_{p,H}\|_\Gamma \le C\|\theta_{p,H}\|^{\frac{1}{2}}\|\theta_{p,H}\|_1^{\frac{1}{2}}, \end{aligned}$$

then we can conclude that

$$\begin{aligned} \xi_f\|\overrightarrow{\theta}_{f,H} \cdot \overrightarrow{n}_f\|_\Gamma + g\|\theta_{p,H}\|_\Gamma &\le \xi_f C\|\theta_{f,H}\|^{\frac{1}{2}}\|\theta_{f,H}\|_1^{\frac{1}{2}} \\ &\quad + gC\|\theta_{p,H}\|^{\frac{1}{2}}\|\theta_{p,H}\|_1^{\frac{1}{2}} \\ &\le \xi_f CH^{\frac{3}{2}}H + gCH^{\frac{3}{2}}H \\ &\lesssim (\xi_f + g)H^{\frac{5}{2}}. \end{aligned}$$

The error estimate of $\|\sigma_{p,H}\|_\Gamma$, $\|\sigma_{f\tau,H}\|_\Gamma$ can be obtained in the same way. ■

Furthermore, building on the groundwork described above, we can derive the error estimate for the new domain decomposition method, as follows.

**Theorem 1:** Let $(\overrightarrow{u}_{f,h}, p_{f,h}, \phi_{p,h})$ be the solution comes from domain decomposition method, and assume that $(\overrightarrow{u}_f^h, p_f^h, \phi_p^h)$ is the solution derived from two-grid domain decomposition method, the following error estimates hold:

$$\|\phi_{p,h} - \phi_p^h\|_1 \lesssim \frac{\xi_p + g}{\boldsymbol{K}\xi_p}H^{\frac{5}{2}}, \quad (26)$$

$$\|\overrightarrow{u}_{f,h} - \overrightarrow{u}_f^h\|_1 \lesssim R_1 H^{\frac{5}{2}}, \quad (27)$$

$$\|p_{f,h} - p_f^h\| \lesssim R_2 H^{\frac{5}{2}}, \quad (28)$$

where

$$R_1 = \frac{C_0^2\delta\sqrt{2\nu}\big((\xi_f + g)\sqrt{trace(\prod)} + \alpha\nu\sqrt{d}\boldsymbol{K}\big)}{(2\sqrt{2\nu} - C_0^2\delta)\sqrt{trace(\prod)}},$$

$$\begin{aligned} R_2 = &\Big((\delta\sqrt{trace(\prod)} + \sqrt{trace(\prod)}\sqrt{2\nu}C_1\xi_f \\ &+ \sqrt{2\nu}C_2^2\alpha\sqrt{d})\frac{C_0^2\big((\xi_f + g)\sqrt{trace(\prod)} + \alpha\nu\sqrt{d}\boldsymbol{K}\big)}{(2\sqrt{2\nu} - C_0^2\delta)(\sqrt{trace(\prod)})^2} \\ &+ \xi_f + g + \frac{\alpha\nu\sqrt{d}\boldsymbol{K}}{\sqrt{trace(\prod)}}\Big). \end{aligned}$$

*Proof:* On the fine grid, subtracting (20)-(21) from (13)-(14) yields

$$\begin{aligned} &a_p(\phi_{p,h} - \phi_p^h, \psi_p) + \langle \frac{g(\phi_{p,h} - \phi_p^h)}{\xi_p}, \psi_p\rangle \\ &\qquad = \langle \frac{g_{p,h} - g_{p,H}}{\xi_p}, \psi_p\rangle, \quad \forall\ \psi_p \in H_{p,h}, \quad (29) \end{aligned}$$

$$
\begin{aligned}
&\delta b_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,\overrightarrow{u}_{f,h},\overrightarrow{v}_f)\\
&+\delta b_f(\overrightarrow{u}_{f,h},\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,\overrightarrow{v}_f)\\
&-\delta b_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,\overrightarrow{v}_f)\\
&+a_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,\overrightarrow{v}_f)-\delta d_f(\overrightarrow{v}_f,p_{f,h}-p^h_f)\\
&+\delta d_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,q_f)\\
&+\delta\xi_f\langle(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f)\cdot\overrightarrow{n}_f,\overrightarrow{v}_f\cdot\overrightarrow{n}_f\rangle\\
&+\frac{\alpha\sqrt{d}}{\sqrt{trace(\prod)}}\langle P_\tau(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f),P_\tau\overrightarrow{v}_f\rangle\\
&=\delta\langle(g_{f,h}-g_{f,H}),\overrightarrow{v}_f\cdot\overrightarrow{n}_f\rangle\\
&-\delta\langle(g_{f\tau,h}-g_{f\tau,H}),P_\tau\overrightarrow{v}_f\rangle.\\
&\qquad\qquad\forall\ (\overrightarrow{v}_f,q_f)\in H_{f,h}\times Q_{f,h}. \quad (30)
\end{aligned}
$$

Let $\psi_p=\phi_{p,h}-\phi^h_p\in H_{p,h}$ in (29), get

$$
\begin{aligned}
&a_p(\phi_{p,h}-\phi^h_p,\phi_{p,h}-\phi^h_p)+\langle\frac{g(\phi_{p,h}-\phi^h_p)}{\xi_p},(\phi_{p,h}-\phi^h_p)\rangle\\
&\qquad=\langle\frac{g_{p,h}-g_{p,H}}{\xi_p},(\phi_{p,h}-\phi^h_p)\rangle. \quad (31)
\end{aligned}
$$

Utilizing the Cauchy-Schwarz inequality in conjunction with the trace inequality, we can conclude that

$$
\begin{aligned}
&\|\phi_{p,h}-\phi^h_p\|_1^2\le\frac{1}{\boldsymbol{K}}a_p(\phi_{p,h}-\phi^h_p,\phi_{p,h}-\phi^h_p),\\
&a_p(\phi_{p,h}-\phi^h_p,\phi_{p,h}-\phi^h_p)\\
&\qquad\le a_p(\phi_{p,h}-\phi^h_p,\phi_{p,h}-\phi^h_p)+\frac{g}{\xi_p}\|\phi_{p,h}-\phi^h_p\|^2_\Gamma,
\end{aligned}
$$

then, from Lemma 1 and (31), we get the following inequality,

$$
\begin{aligned}
\|\phi_{p,h}-\phi^h_p\|_1^2&\le\frac{1}{\boldsymbol{K}}[a_p(\phi_{p,h}-\phi^h_p,\phi_{p,h}-\phi^h_p)\\
&+\langle\frac{g(\phi_{p,h}-\phi^h_p)}{\xi_p},(\phi_{p,h}-\phi^h_p)\rangle]\\
&\le\frac{1}{\boldsymbol{K}}\langle\frac{g_{p,h}-g_{p,H}}{\xi_p},(\phi_{p,h}-\phi^h_p)\rangle\\
&\le\frac{1}{\boldsymbol{K}\xi_p}\|\sigma_{p,H}\|_\Gamma\|\phi_{p,h}-\phi^h_p\|_\Gamma\\
&\lesssim\frac{\xi_p+g}{\boldsymbol{K}\xi_p}H^{\frac{5}{2}}\|\phi_{p,h}-\phi^h_p\|_1. \quad (32)
\end{aligned}
$$

We can get (26) by eliminating $\|\phi_{p,h}-\phi^h_p\|_1$ from (32).

Setting $(\overrightarrow{v}_f,q_f)=(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,p_{f,h}-p^h_f)\in H_{f,h}\times Q_{f,h}$ and substituting into (30), we have

$$
\begin{aligned}
&\delta b_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,\overrightarrow{u}_{f,h},\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f)\\
&+\delta b_f(\overrightarrow{u}_{f,h},\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f)\\
&+a_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f)\\
&-\delta d_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,p_{f,h}-p^h_f)\\
&+\delta d_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,p_{f,h}-p^h_f)\\
&+\delta\xi_f\langle(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f)\cdot\overrightarrow{n}_f,(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f)\cdot\overrightarrow{n}_f\rangle\\
&+\frac{\alpha\sqrt{d}}{\sqrt{trace(\prod)}}\langle P_\tau(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f),P_\tau(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f)\rangle\\
&=\delta\langle(g_{f,h}-g_{f,H}),(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f)\cdot\overrightarrow{n}_f\rangle\\
&-\delta\langle(g_{f\tau,h}-g_{f\tau,H}),P_\tau(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f)\rangle, \quad (33)
\end{aligned}
$$

for the trilinear terms in (33), we have

$$
\begin{aligned}
&b_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,\overrightarrow{u}_{f,h},\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f)\\
&+b_f(\overrightarrow{u}_{f,h},\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f)\\
&\lesssim\frac{1}{\sqrt{2\nu}}\|\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f\|_1^2,
\end{aligned}
$$

by the $Korn's$ inequality, there exists $C_0$ that makes

$$
\|\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f\|_1^2\le\frac{C_0^2}{2}a_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f),
$$

then we can get

$$
\begin{aligned}
&\|\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f\|_1^2\\
&\le\frac{C_0^2}{2}[a_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f)\\
&+\delta\xi_f\langle(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f)\cdot\overrightarrow{n}_f,(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f)\cdot\overrightarrow{n}_f\rangle\\
&+\frac{\alpha\sqrt{d}}{\sqrt{trace(\prod)}}\langle P_\tau(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f),P_\tau(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f)\rangle]\\
&\le\frac{C_0^2}{2}[\delta\langle(g_{f,h}-g_{f,H}),(u_{f,h}-u^h_f)\cdot n_f\rangle\\
&+\delta|b_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,\overrightarrow{u}_{f,h},\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f)|\\
&+\delta|b_f(\overrightarrow{u}_{f,h},\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f)|\\
&+\delta|\langle(g_{f\tau,h}-g_{f\tau,H}),P_\tau(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f)\rangle|]\\
&\lesssim\frac{C_0^2\delta}{2}[\|\sigma_{f,H}\|_\Gamma\|\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f\|_\Gamma\\
&+\frac{1}{\sqrt{2\nu}}\|\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f\|_1^2+\|\sigma_{f\tau,H}\|_\Gamma\|\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f\|_\Gamma].
\end{aligned}
\quad (34)
$$

Thanks to Lemma 1 we know

$$
\begin{aligned}
&\|\sigma_{f,H}\|_\Gamma\|\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f\|_\Gamma\lesssim\|\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f\|_1(\xi_f+g)H^{\frac{5}{2}},\\
&\|\sigma_{f\tau,H}\|_\Gamma\|\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f\|_\Gamma\lesssim\|\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f\|_1\frac{\alpha\nu\sqrt{d}\boldsymbol{K}}{\sqrt{trace(\prod)}}H^{\frac{5}{2}},
\end{aligned}
$$

by subdividing (34) and simplifying it, we get

$$
\begin{aligned}
&\|\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f\|_1\\
&\qquad\lesssim\frac{C_0^2\delta\sqrt{2\nu}\big((\xi_f+g)\sqrt{trace(\prod)}+\alpha\nu\sqrt{d}\boldsymbol{K}\big)}{(2\sqrt{2\nu}-C_0^2\delta)\sqrt{trace(\prod)}}H^{\frac{5}{2}}.
\end{aligned}
$$

Let $q_f=p_{f,h}-p^h_f\in Q_{f,h}$, there exist $\overrightarrow{v}_f\in H_{f,h}$ such that

$$
\|p_{f,h}-p^h_f\|\le\frac{d_f(\overrightarrow{v}_f,p_{f,h}-p^h_f)}{\|\overrightarrow{v}_f\|_1}.
$$

It can also be inferred from (30) that

$$
\begin{aligned}
\|p_{f,h}-p^h_f\|\le\frac{1}{\|\overrightarrow{v}_f\|_1\delta}\Big[&\delta|\langle(g_{f,h}-g_{f,H}),\overrightarrow{v}_f\cdot\overrightarrow{n}_f\rangle|\\
&+\delta|b_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,\overrightarrow{v}_f)|\\
&+\delta|b_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,\overrightarrow{u}_{f,h},\overrightarrow{v}_f)|\\
&+\delta|b_f(\overrightarrow{u}_{f,h},\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,\overrightarrow{v}_f)|\\
&+|a_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f,\overrightarrow{v}_f)|\\
&+\delta\xi_f|\langle(\overrightarrow{u}_{f,h}-\overrightarrow{u}^h_f)\cdot\overrightarrow{n}_f,\overrightarrow{v}_f\cdot\overrightarrow{n}_f\rangle|
\end{aligned}
$$

$$+\frac{\alpha\sqrt{d}}{\sqrt{trace(\prod)}}|\langle P_\tau(\vec{u}_{f,h}-\vec{u}_f^h),P_\tau\vec{v}_f\rangle|$$
$$+\delta|\langle(g_{f\tau,h}-g_{f\tau,H}),P_\tau\vec{v}_f\rangle|\Big]$$
$$\lesssim\|\sigma_{f,H}\|_\Gamma+\|\sigma_{f\tau,H}\|_\Gamma$$
$$+(\frac{1}{\sqrt{2\nu}}+C_1\xi_f)\|\vec{u}_{f,h}-\vec{u}_f^h\|_1$$
$$+\frac{C_2^2}{\delta}\frac{\alpha\sqrt{d}}{\sqrt{trace(\prod)}}\|\vec{u}_{f,h}-\vec{u}_f^h\|_1.$$

Thus,

$$\|p_{f,h}-p_f^h\|$$
$$\lesssim\frac{C_0^2\delta\big((\xi_f+g)\sqrt{trace(\prod)}+\alpha\nu\sqrt{d}\boldsymbol{K}\big)}{(2\sqrt{2\nu}-C_0^2\delta)\sqrt{trace(\prod)}}H^{\frac{5}{2}}$$
$$+C_1\xi_f\frac{C_0^2\delta\sqrt{2\nu}\big((\xi_f+g)\sqrt{trace(\prod)}+\alpha\nu\sqrt{d}\boldsymbol{K}\big)}{(2\sqrt{2\nu}-C_0^2\delta)\sqrt{trace(\prod)}}H^{\frac{5}{2}}$$
$$+\frac{C_2^2\alpha\sqrt{d}}{\delta\sqrt{trace(\prod)}}\frac{C_0^2\delta\sqrt{2\nu}\big((\xi_f+g)\sqrt{trace(\prod)}+\alpha\nu\sqrt{d}\boldsymbol{K}\big)}{(2\sqrt{2\nu}-C_0^2\delta)\sqrt{trace(\prod)}}H^{\frac{5}{2}}$$
$$+(\xi_f+g)H^{\frac{5}{2}}+\frac{\alpha\nu\sqrt{d}\boldsymbol{K}}{\sqrt{trace(\prod)}}H^{\frac{5}{2}}$$
$$\lesssim\big((\delta\sqrt{trace(\prod)}+\sqrt{trace(\prod)}\sqrt{2\nu}C_1\xi_f$$
$$+\sqrt{2\nu}C_2^2\alpha\sqrt{d})\frac{C_0^2\big((\xi_f+g)\sqrt{trace(\prod)}+\alpha\nu\sqrt{d}\boldsymbol{K}\big)}{(2\sqrt{2\nu}-C_0^2\delta)(\sqrt{trace(\prod)})^2}$$
$$+\xi_f+g+\frac{\alpha\nu\sqrt{d}\boldsymbol{K}}{\sqrt{trace(\prod)}}\big)H^{\frac{5}{2}}$$

which completes the proof of (28). ■

Drawing from Theorem 1, we can establish the error estimate for the solution obtained by the new method in relation to the exact solution as detailed below.

**Corollary 1:** Let $(\vec{u}_f^h,p_f^h)\in H_{f,h}\times Q_{f,h},\phi_p^h\in H_{p,h}$, and $(\vec{u}_f,p_f)\in H_f\times Q_f,\phi_p\in H_p$ be the solution of two-grid domain decomposition method and (9)-(10), respectively. Choosing $H=h^{\frac{4}{5}}$, we have

$$\|\phi_p-\phi_p^h\|_1\lesssim h^2,\qquad\|\vec{u}_f-\vec{u}_f^h\|_1+\|p_f-p_f^h\|\lesssim h^2.$$

The two-grid domain decomposition technique adeptly combines the strengths of the two-grid approach with the domain decomposition paradigm. It masterfully handles the strong interconnections between distinct models spanning multiple domains. Moreover, this innovative method directly addresses the inherent complexities of multi-domain and multi-physics coupling, providing a robust solution to these challenging issues. When it comes to decoupled solutions, it markedly boosts the computational efficiency.

## VI. Conclusion

This paper, targeting the coupled model, improves upon the classical domain decomposition method and proposes a novel two-grid domain decomposition approach. The core idea is to leverage the advantages of domain decomposition methods in conjunction with the two-grid method, thereby enhancing the approach for multi-domain, multi-physics coupling models. Specifically, the domain is divided into two subregions by enforcing the boundary conditions: one representing free flow and the other representing porous medium flow. For each region, an existing algorithm is first employed to get a coarse grid approximate solution. Then, by replacing some interface terms with functions from the coarse grid, an improved fine grid problem is derived. Compared to previous methods, the two-grid domain decomposition method significantly reduces the number of iterative steps and saves computation time. Future developments can extend the two-grid domain decomposition method to more complex multi-physics models; it can also be further researched and improved into a multi-grid domain decomposition method to solve even more complex coupling problems.

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