Game-theoretical Analysis for Resource Allocating

Bo-Yao Wang and Yu-Hsien Liao

Abstract—**Resource allocation within management systems is the strategic distribution of resources to enhance both efficiency and productivity. This process must remain adaptable to fluctuating demands, which helps reduce waste and idle time, ultimately contributing to cost-effectiveness and profitability. Given its complexity, effective allocation requires a nuanced understanding of factors such as participant roles, behavioral patterns, and strategic interactions. This study introduces a novel allocation scheme that emphasizes both participants and their activity behaviours. Grounded in the principle of symmetric treatment, we propose several axiomatic and dynamic analyzes to examine the logical foundation of this scheme, utilizing reduction and variation functions. The game-theoretical perspective presented here offers deeper insights into participant strategy, supporting more effective and equitable resource allocation.**

Index Terms—**Resource allocating, scheme, symmetry for treatment, axiomatic and dynamic analysis.**

I. Introduction

Game-theoretical approaches have become crucial for analyzing complex interactions within real-world management systems, particularly among participants and coalitions, through rigorous mathematical formulations. Resource management increasingly leverages techniques that merge diverse theoretical frameworks to boost system-wide effectiveness. By integrating insights from game theory, both resource allocation schemes and strategic interaction models can achieve greater precision and impact. Within this context, axiomatic analysis defines an allocation scheme using foundational logical principles, often capturing ideas of rationality, fairness, or stability. This analysis is considered complete when these axioms uniquely determine a scheme. Conversely, dynamic analysis examines how a scheme emerges and stabilizes through behavioral learning, strategic adjustments, or convergence toward equilibrium. A scheme that consistently appears as the stable outcome of a reasonable dynamic process gains practical credibility. In fact, some allocation rules can even be interpreted as long-term negotiation equilibria. By combining axiomatic and dynamic approaches, this study offers a dual validation framework: one that clarifies logical soundness and another that demonstrates adaptive feasibility under evolving conditions.

The concept of symmetry is central to promoting both fairness and operational efficiency in resource allocation processes. Here, symmetry doesn't just mean identical treatment; it refers to an equitable approach that reflects each participant's functional role and contribution. This is particularly important in multi-choice environments where individuals may participate at varying intensities or engagement levels. In such contexts, dynamic allocation mechanisms must preserve this symmetry, ensuring adjustments are proportional to participants' comparable roles or contributions. When allocation systems are designed with these principles in mind, they're more likely to foster cooperation, reduce disputes, and channel resources toward outcomes that maximize collective value.

Under traditional game-theoretical studies of interaction situations, a characteristic function is typically defined over all sub-collections of the set of participators. This framework implies that each participator is restricted to a binary choice: either fully joining a coalition or not participating at all. However, in practical resource allocating systems, participators often demonstrate varying degrees of activity, each of which can significantly influence the allocation outcome. The concept of multi-choice interaction situations therefore serves as a natural extension of traditional coalition structures, accommodating diverse levels of engagement among participators. Several symmetric schemes have been developed within such multi-choice contexts. For instance, Cheng et al. [4], Hwang and Liao [11], Liao [20], [21], Nouweland et al. [30], and Wei et al. [36] have proposed different schemes by applying symmetric concepts such as the core, the equal allocation of non-separable costs (EANSC), and the Shapley value.

This research centers on the pseudo equal allocation of non-separable costs (PEANSC) defined by Hsieh and Liao [9]. Within traditional coalition situations, Hsieh and Liao [9] provided axiomatic and dynamic results demonstrating that the PEANSC constitutes a symmetric and stabilizing scheme. These findings naturally give rise to a compelling question within the framework of multi-choice situations:

- Can existing axiomatic and dynamic results related to the PEANSC be extended and enhanced under multi-choice situations and within resource allocating management contexts?

This study aims to address this question. The main contributions are outlined as follows.

1) Inspired by the work of Hwang and Liao [13], we propose a multi-choice generalization of the PEANSC, termed the efficient individual achieved-efficacy (EIAE), which simultaneously accounts for participators and their activity behaviours, as introduced in Section 2.
2) In Section 3, we introduce alternative properties of symmetry and conformance, extending the framework developed by Hsieh and Liao [9], to characterize the EIAE under multi-choice conditions. The symmetry property for treatment, as formulated in this study,

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captures the principle that participators with identical marginal contributions should receive identical allocations.

3) Symmetry, however, permeates more deeply into the allocation process. In Section 4, we employ the symmetric concept embedded in the variation function to derive a dynamic result for the EIAE within multi-choice contexts. The variation function, which evaluates allocation stability, also reflects symmetry: when participators' contributions are structurally mirrored, their 'dissatisfaction' should correspondingly align.
4) In Section 5, the game-theoretical results are applied to examine a symmetric allocating scheme in the setting of resource allocating management systems. Additional insights and comparisons are presented in Section 6.

By embedding symmetry into the core of the proposed model, this study not only advances fairness in resource allocating but also improves overall efficiency. The sections that follow explore how the EIAE scheme operationalizes these symmetry principles, thereby contributing to both theoretical understanding and practical strategies for resource allocating.

II. The efficient individual achieved-efficacy

Let $\overline{UZ}$ be the universe of participators. For $m \in \overline{UZ}$ and $\vec{b}_m \in \mathbb{N}$, $\overline{B}_m = \{0, 1, \cdots, \vec{b}_m\}$ could be treated as the activity behaviour collection of participator i and $\overline{B}_m^+ = \overline{B}_m \setminus \{0\}$, where 0 represents no participation. Let $\overline{Z} \subseteq \overline{UZ}$ and $\overline{B}^{\overline{Z}} = \prod_{m \in \overline{Z}} \overline{B}_m$ be the product collection of the activity behaviour collections of all participators of $\overline{Z}$. for each $\overline{T} \subseteq \overline{Z}$, we define $\vec{\varsigma}^{\overline{T}} \in \overline{B}^{\overline{Z}}$ is the vector with $\vec{\varsigma}^{\overline{T}}_m = 1$ if $m \in \overline{T}$, and $\vec{\varsigma}^{\overline{T}}_m = 0$ if $m \in \overline{Z} \setminus \overline{T}$. Denote $0_{\overline{Z}}$ the zero vector in $\mathbb{R}^{\overline{Z}}$.

A **multi-choice situation** is a triple $(\overline{Z}, \vec{b}, \hat{\mathbf{A}})$, where $\overline{Z}$ is a non-empty and finite collection of participators, $\vec{b} = (\vec{b}_m)_{m \in \overline{Z}}$ is the vector that presents the highest activity behaviour for each participator, and $\hat{\mathbf{A}} : \overline{B}^{\overline{Z}} \to \mathbb{R}$ is a characteristic mapping with $\hat{\mathbf{A}}(0_{\overline{Z}}) = 0$ which assigns to every $\vec{\lambda} = (\vec{\lambda}_m)_{m \in \overline{Z}} \in \overline{B}^{\overline{Z}}$ the worth that the participators can gain when every participator m participates at behaviour $\vec{\lambda}_m$. As $d \in \mathbb{R}$ is fixed throughout this research, we write $(\overline{Z}, \hat{\mathbf{A}})$ rather than $(\overline{Z}, \vec{b}, \hat{\mathbf{A}})$.

Given a multi-choice situation $(\overline{Z}, \hat{\mathbf{A}})$ and $\vec{\mu} \in \overline{B}^{\overline{Z}}$, we write $\hat{\mathbf{J}}(\vec{\mu}) = \{m \in \overline{Z} | \vec{\mu}_m \neq 0\}$ and $\vec{\mu}_{\overline{T}}$ to be the restriction of $\vec{\mu}$ at $\overline{T}$ for each $\overline{T} \subseteq \overline{Z}$. Denote the family of total multi-choice situations by $\overline{\mathbf{MCS}}$.

Given $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$, let $P^{\overline{Z}} = \{(m, k_m) \mid m \in \overline{Z}, k_m \in \overline{B}_m^+\}$. A **scheme** on $\overline{\mathbf{MCS}}$ is a map $\hat{\eta}$ assigning to every $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$ an element

$$\hat{\eta}(\overline{Z}, \hat{\mathbf{A}}) = \Big(\hat{\eta}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}})\Big)_{(m,k_m) \in P^{\overline{Z}}} \in \mathbb{R}^{P^{\overline{Z}}}.$$

Here $\hat{\eta}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}})$ is the power index or the value of the participator m if it participates with activity behaviour k_m under situation $\hat{\mathbf{A}}$.

Subsequently, we provide a generalized analogue of the pseudo equal allocation of non-separable costs under multi-choice situations as follows.

Definition 1: The **efficient individual achieved-efficacy (EIAE)** of multi-choice situations, $\hat{\mathbf{\Psi}}$, is the function on $\overline{\mathbf{MCS}}$ which associates to every $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$, every participator $m \in \overline{Z}$ and every $k_m \in \overline{B}_m$ the value

$$\begin{aligned} & \hat{\mathbf{\Psi}}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) \\ = \ & \hat{\psi}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) + \tfrac{1}{|\overline{Z}|} \cdot \Big[\hat{\mathbf{A}}(\vec{b}) - \sum_{n \in \overline{Z}} \hat{\psi}_{n,\vec{b}_n}(\overline{Z}, \hat{\mathbf{A}})\Big], \end{aligned}$$

where $\hat{\psi}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) = \hat{\mathbf{A}}\big(\vec{b}_m, 0_{\overline{Z} \setminus \{m\}}\big) - \hat{\mathbf{A}}\big(k_m - 1, 0_{\overline{Z} \setminus \{m\}}\big)$ is the **individual achieved-efficacy** of the participator m and its behaviour k_m.

III. Axiomatic results

A. Conformance property and related axiomatic analysis

In this section, we establish that a specific form of reduction and an associated conformance property can be used to characterize the EIAE.

Let $\hat{\eta}$ be a scheme on $\overline{\mathbf{MCS}}$.

- $\hat{\eta}$ fits **scheme completeness (SCOM)** if for each $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$, $\sum_{m \in \overline{Z}} \hat{\eta}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) = \hat{\mathbf{A}}(\vec{b})$.
- $\hat{\eta}$ fits **principle for two-person situations (PFTS)** if for each $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$ with $|\overline{Z}| \leq 2$, it holds that $\hat{\eta}(\overline{Z}, \hat{\mathbf{A}}) = \hat{\mathbf{\Psi}}(\overline{Z}, \hat{\mathbf{A}})$.

The SCOM property is a widely accepted criterion in resource allocation frameworks, ensuring that the total value is fully allocated when all participators are fully engaged. The PFTS property generalizes the two-person axiomatic foundation originally proposed by Hart and Mas-Colell [8], requiring that the scheme align with $\hat{\mathbf{\Psi}}$ in all cases involving one or two agents. In the remainder of this section, we will prove that EIAE satisfies both SCOM and PFTS.

Lemma 1: The EIAE fits SCOM.

Proof: Let $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$. To verify that the scheme $\hat{\mathbf{\Psi}}$ satisfies the Strong Collective Output Matching (SCOM) condition, we compute:

$$\begin{aligned} & \sum_{m \in \overline{Z}} \hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) \\ = \ & \sum_{m \in \overline{Z}} \hat{\psi}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) + \sum_{m \in \overline{Z}} \tfrac{1}{|\overline{Z}|} \cdot \Big[\hat{\mathbf{A}}(\vec{b}) - \sum_{n \in \overline{Z}} \hat{\psi}_{n,\vec{b}_n}(\overline{Z}, \hat{\mathbf{A}})\Big] \\ = \ & \sum_{m \in \overline{Z}} \hat{\psi}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) + \tfrac{|\overline{Z}|}{|\overline{Z}|} \cdot \Big[\hat{\mathbf{A}}(\vec{b}) - \sum_{n \in \overline{Z}} \hat{\psi}_{n,\vec{b}_n}(\overline{Z}, \hat{\mathbf{A}})\Big] \\ = \ & \sum_{m \in \overline{Z}} \hat{\psi}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) + \hat{\mathbf{A}}(\vec{b}) - \sum_{n \in \overline{Z}} \hat{\psi}_{n,\vec{b}_n}(\overline{Z}, \hat{\mathbf{A}}) \\ = \ & \hat{\mathbf{A}}(\vec{b}). \end{aligned}$$

Thus, the total assigned values under $\hat{\mathbf{\Psi}}$ exactly recover the total available output $\hat{\mathbf{A}}(\vec{b})$, completing the proof. ■

Lemma 2: The EIAE fits PFTS.

Proof: This follows directly from the definitions of the EIAE and PFTS. The scheme $\hat{\mathbf{\Psi}}$ is constructed in a manner consistent with the functional form prescribed by PFTS. Hence, the result holds immediately. ■

A natural analogue of the reduction due to Hsieh and Liao [9] on multi-choice situations is as follows. Given $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$, $\overline{S} \subseteq \overline{Z}$ and a scheme $\hat{\eta}$, the **reduced situation** $(\overline{S}, \hat{\mathbf{A}}^{\hat{\eta}}_{\overline{S}})$ **with respect to** $\overline{S}$ **and** $\hat{\eta}$ is defined by for each $\vec{\mu} \in \overline{B}^{\overline{S}}$,

$$\begin{aligned} & \hat{\mathbf{A}}^{\hat{\eta}}_{\overline{S}}(\vec{\mu}) \\ = \ & \begin{cases} 0 & \vec{\mu} = 0_{\overline{S}}, \\ \hat{\mathbf{A}}(\vec{\mu}_m, 0_{\overline{Z} \setminus \{m\}}) & |\overline{S}| \geq 2,\ |\hat{\mathbf{J}}(\vec{\mu})| = 1, \\ \hat{\mathbf{A}}(\vec{\mu}, \vec{b}_{\overline{Z} \setminus \overline{S}}) - \sum_{m \in \overline{Z} \setminus \overline{S}} \hat{\eta}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) & \text{otherwise.} \end{cases} \end{aligned}$$

The bilateral conformance property in resource allocation can be described as follows: Consider a scheme $\hat{\eta}$ operating within a situation $\overline{\mathbf{MCS}}$. For any pair of participators, a "reduced situation" is defined by allocating the payoffs prescribed by $\hat{\eta}$ to all other participators and considering the remaining resources available to the pair. The scheme $\hat{\eta}$ is deemed bilaterally consistent if, when applied to any such reduced situation, it consistently yields the same payoffs for the pair as in the original situation. Formally, a scheme $\hat{\eta}$ fits **bilateral conformance (BCFE)** if for each $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$ with $|\overline{Z}| \geq 3$, for each $\overline{S} \subseteq \overline{Z}$ with $|\overline{S}| = 2$ and for each $(m, k_m) \in A^{\overline{S}}$, $\hat{\eta}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) = \hat{\eta}_{m,k_m}(\overline{S}, \hat{\mathbf{A}}^{\hat{\eta}}_{\overline{S}})$.

Lemma 3: The EIAE $\hat{\mathbf{\Psi}}$ fits BCFE.

Proof: Let $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$ with $|\overline{Z}| \geq 3$ and $\overline{S} = \{m, n\} \subseteq \overline{Z}$. By the definition of $\hat{\mathbf{\Psi}}$, for each $(p, k_p) \in A^{\overline{S}}$,

$$\begin{aligned} & \hat{\mathbf{\Psi}}_{p,k_p}(\overline{S}, \hat{\mathbf{A}}^{\hat{\mathbf{\Psi}}}_{\overline{S}}) \\ = \; & \hat{\psi}_{p,k_p}(\overline{S}, \hat{\mathbf{A}}^{\hat{\mathbf{\Psi}}}_{\overline{S}}) + \tfrac{1}{|\overline{S}|} \cdot \Big[\hat{\mathbf{A}}^{\hat{\mathbf{\Psi}}}_{\overline{S}}(\vec{b}_{\overline{S}}) - \sum_{t \in \overline{S}} \hat{\psi}_{t,\vec{b}_t}(\overline{S}, \hat{\mathbf{A}}^{\hat{\mathbf{\Psi}}}_{\overline{S}}) \Big]. \end{aligned} \tag{1}$$

By definitions of $\hat{\psi}$ and $\hat{\mathbf{A}}^{\hat{\mathbf{\Psi}}}_{\overline{S}}$, for each $k_m \in \overline{B}_m$,

$$\begin{aligned} & \hat{\psi}_{m,k_m}(\overline{S}, \hat{\mathbf{A}}^{\hat{\mathbf{\Psi}}}_{\overline{S}}) \\ = \; & \hat{\mathbf{A}}^{\hat{\mathbf{\Psi}}}_{\overline{S}}(\vec{b}_m, 0) - \hat{\mathbf{A}}^{\hat{\mathbf{\Psi}}}_{\overline{S}}(k_m - 1, 0) \\ = \; & \hat{\mathbf{A}}\big(\vec{b}_m, 0_{\overline{Z} \setminus \{m\}}\big) - \hat{\mathbf{A}}\big(k_m - 1, 0_{\overline{Z} \setminus \{m\}}\big) \\ = \; & \hat{\psi}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}). \end{aligned} \tag{2}$$

Hence, by equations (1), (2) and definitions of $\hat{\mathbf{A}}^{\hat{\mathbf{\Psi}}}_{\overline{S}}$ and $\hat{\mathbf{\Psi}}$,

$$\begin{aligned} & \hat{\mathbf{\Psi}}_{m,k_m}(\overline{S}, \hat{\mathbf{A}}^{\hat{\mathbf{\Psi}}}_{\overline{S}}) \\ = \; & \hat{\psi}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) + \tfrac{1}{|\overline{S}|} \cdot \Big[\hat{\mathbf{A}}^{\hat{\mathbf{\Psi}}}_{\overline{S}}(\vec{b}_{\overline{S}}) - \sum_{t \in \overline{S}} \hat{\psi}_{t,\vec{b}_t}(\overline{Z}, \hat{\mathbf{A}}) \Big] \\ = \; & \hat{\psi}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) \\ & + \tfrac{1}{|\overline{S}|} \cdot \Big[\hat{\mathbf{A}}(\vec{b}) - \sum_{t \in \overline{Z} \setminus \overline{S}} \hat{\mathbf{\Psi}}_{t,\vec{b}_t}(\overline{Z}, \hat{\mathbf{A}}) - \sum_{t \in \overline{S}} \hat{\psi}_{t,\vec{b}_t}(\overline{Z}, \hat{\mathbf{A}}) \Big] \\ = \; & \hat{\psi}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) + \tfrac{1}{|\overline{S}|} \cdot \Big[\sum_{t \in \overline{S}} \hat{\mathbf{\Psi}}_{t,\vec{b}_t}(\overline{Z}, \hat{\mathbf{A}}) - \sum_{t \in \overline{S}} \hat{\psi}_{t,\vec{b}_t}(\overline{Z}, \hat{\mathbf{A}}) \Big] \\ & \Big(\textbf{SCOM of } \hat{\mathbf{\Psi}}\Big) \\ = \; & \hat{\psi}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) + \tfrac{1}{|\overline{S}|} \cdot \Big[\tfrac{|\overline{S}|}{|\overline{Z}|} \cdot [\hat{\mathbf{A}}(\vec{b}) - \sum_{t \in N} \hat{\psi}_{t,\vec{b}_t}(\overline{Z}, \hat{\mathbf{A}})] \Big] \\ & \Big(\textbf{Definition 1}\Big) \\ = \; & \hat{\psi}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) + \tfrac{1}{|\overline{Z}|} \cdot \Big[\hat{\mathbf{A}}(\vec{b}) - \sum_{t \in N} \hat{\psi}_{t,\vec{b}_t}(\overline{Z}, \hat{\mathbf{A}}) \Big] \\ = \; & \hat{\mathbf{\Psi}}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}). \end{aligned}$$

Similarly, $\hat{\mathbf{\Psi}}_{n,k_n}(\overline{S}, \hat{\mathbf{A}}^{\hat{\mathbf{\Psi}}}_{\overline{S}}) = \hat{\mathbf{\Psi}}_{n,k_n}(\overline{Z}, \hat{\mathbf{A}})$ for each $k_n \in \overline{B}_n$. So, the EIAE fits BCFE. ■

Lemma 4: If a scheme $\hat{\eta}$ fits PFTS and BCFE then it also fits SCOM.

Proof: Let $\hat{\eta}$ be a scheme on $\overline{\mathbf{MCS}}$ satisfy PFTS and BCFE, and $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$. It is trivial for $|\overline{Z}| \leq 2$ by PFTS. Assume that $|\overline{Z}| \geq 3$. Let $n \in \overline{Z}$, consider the reduced situation $\big(\{n\}, \hat{\mathbf{A}}^{\hat{\eta}}_{\{n\}}\big)$. By definition of $\hat{\mathbf{A}}^{\hat{\eta}}_{\{n\}}$, $\hat{\mathbf{A}}^{\hat{\eta}}_{\{n\}}(\vec{b}_n) = \hat{\mathbf{A}}(\vec{b}) - \sum_{m \in \overline{Z} \setminus \{n\}} \hat{\eta}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}})$. Since $\hat{\eta}$ fits BCFE, $\hat{\eta}_{n,k_n}(\overline{Z}, \hat{\mathbf{A}}) = \hat{\eta}_{n,k_n}\big(\{n\}, \hat{\mathbf{A}}^{\hat{\eta}}_{\{n\}}\big)$ for each $k_n \in \overline{B}_n$. In particular, $\hat{\eta}_{n,\vec{b}_n}(\overline{Z}, \hat{\mathbf{A}}) = \hat{\eta}_{n,\vec{b}_n}\big(\{n\}, \hat{\mathbf{A}}^{\hat{\eta}}_{\{n\}}\big)$. On the other hand, by PFTS of $\hat{\eta}$, $\hat{\eta}_{n,\vec{b}_n}(\overline{Z}, \hat{\mathbf{A}}) = \hat{\mathbf{A}}^{\hat{\eta}}_{\{n\}}(\vec{b}_n)$. Hence, $\sum_{m \in \overline{Z}} \hat{\eta}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) = \hat{\mathbf{A}}(\vec{b})$, i.e., $\hat{\eta}$ fits SCOM. ■

Theorem 1: A scheme $\hat{\eta}$ on $\overline{\mathbf{MCS}}$ fits PFTS and BCFE if and only if $\hat{\eta} = \hat{\mathbf{\Psi}}$.

Proof: By Lemma 2, $\hat{\mathbf{\Psi}}$ fits PFTS. By Lemma 3, $\hat{\mathbf{\Psi}}$ fits BCFE.

To prove uniqueness, suppose $\hat{\eta}$ fits PFTS and BCFE on $\overline{\mathbf{MCS}}$. By Lemma 4, $\hat{\eta}$ also fits SCOM. Let $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$. If $|\overline{Z}| \leq 2$, then by PFTS of $\hat{\eta}$, $\hat{\eta}(\overline{Z}, \hat{\mathbf{A}}) = \hat{\mathbf{\Psi}}(\overline{Z}, \hat{\mathbf{A}})$. The case $|\overline{Z}| > 2$: Let $m \in \overline{Z}$ and $\overline{S} = \{m, n\}$ for some $n \in \overline{Z} \setminus \{m\}$, then for each $k_m \in \overline{B}_m, \; k_n \in \overline{B}_n$,

$$\begin{aligned} & \hat{\eta}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) - \hat{\eta}_{n,k_n}(\overline{Z}, \hat{\mathbf{A}}) \\ = \; & \hat{\eta}_{m,k_m}(\overline{S}, \hat{\mathbf{A}}^{\hat{\eta}}_{\overline{S}}) - \hat{\eta}_{n,k_n}(\overline{S}, \hat{\mathbf{A}}^{\hat{\eta}}_{\overline{S}}) \\ & \Big(\textbf{BCFE of } \hat{\eta}\Big) \\ = \; & \hat{\mathbf{\Psi}}_{m,k_m}(\overline{S}, \hat{\mathbf{A}}^{\hat{\eta}}_{\overline{S}}) - \hat{\mathbf{\Psi}}_{n,k_n}(\overline{S}, \hat{\mathbf{A}}^{\hat{\eta}}_{\overline{S}}) \\ & \Big(\textbf{PFTS of } \hat{\eta}\Big) \\ = \; & \hat{\psi}_{m,k_m}(\overline{S}, \hat{\mathbf{A}}^{\hat{\eta}}_{\overline{S}}) - \hat{\psi}_{n,k_n}(\overline{S}, \hat{\mathbf{A}}^{\hat{\eta}}_{\overline{S}}) \\ & \Big(\textbf{Definition 1}\Big) \\ = \; & \Big[\hat{\mathbf{A}}^{\hat{\eta}}_{\overline{S}}(\vec{b}_m, 0) - \hat{\mathbf{A}}^{\hat{\eta}}_{\overline{S}}(k_m - 1, 0) \\ & \quad - \hat{\mathbf{A}}^{\hat{\eta}}_{\overline{S}}(0, \vec{b}_n) + \hat{\mathbf{A}}^{\hat{\eta}}_{\overline{S}}(0, k_n - 1) \Big] \\ & \Big(\textbf{Definition 1}\Big) \\ = \; & \Big[\hat{\mathbf{A}}(\vec{b}_m, 0_{\overline{Z} \setminus \{m\}}) - \hat{\mathbf{A}}(k_m - 1, 0_{\overline{Z} \setminus \{m\}}) \\ & \quad - \hat{\mathbf{A}}(\vec{b}_n, 0_{\overline{Z} \setminus \{n\}}) + \hat{\mathbf{A}}(k_n - 1, 0_{\overline{Z} \setminus \{n\}}) \Big] \\ & \Big(\textbf{Definition of } \hat{\mathbf{A}}^{\hat{\eta}}_{\overline{S}}\Big) \end{aligned} \tag{3}$$

Similarly taking, $\hat{\mathbf{\Psi}}$ instead of $\hat{\eta}$ in equation (3), we can derive that

$$\begin{aligned} & \hat{\mathbf{\Psi}}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) - \hat{\mathbf{\Psi}}_{n,k_n}(\overline{Z}, \hat{\mathbf{A}}) \\ = \; & \Big[\hat{\mathbf{A}}(\vec{b}_m, 0_{\overline{Z} \setminus \{m\}}) - \hat{\mathbf{A}}(k_m - 1, 0_{\overline{Z} \setminus \{m\}}) \\ & \quad - \hat{\mathbf{A}}(\vec{b}_n, 0_{\overline{Z} \setminus \{n\}}) + \hat{\mathbf{A}}(k_n - 1, 0_{\overline{Z} \setminus \{n\}}) \Big]. \end{aligned} \tag{4}$$

Hence, by equations (3) and (4),

$$\begin{aligned} & \hat{\eta}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) - \hat{\eta}_{n,k_n}(\overline{Z}, \hat{\mathbf{A}}) \\ = \; & \hat{\mathbf{\Psi}}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) - \hat{\mathbf{\Psi}}_{n,k_n}(\overline{Z}, \hat{\mathbf{A}}). \end{aligned}$$

This implies that $\hat{\eta}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) - \hat{\mathbf{\Psi}}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) = t$ for each (m, k_m). It remains to show that $t = 0$. By SCOM of $\hat{\eta}$ and $\hat{\mathbf{\Psi}}$,

$$\begin{aligned} 0 & = \Big[\hat{\mathbf{A}}(\vec{b}) - \hat{\mathbf{A}}(\vec{b}) \Big] \\ & = \sum_{m \in \overline{Z}} \Big[\hat{\eta}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) \Big] \\ & = |\overline{Z}| \cdot t. \end{aligned}$$

Hence, $t = 0$. ■

Example 1: Apply a scheme $\hat{\eta}$ on $\overline{\mathbf{MCS}}$ by for each $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$ and for each $(m, k_m) \in P^{\overline{Z}}$,

$$\hat{\eta}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) = \begin{cases} \hat{\mathbf{\Psi}}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) & \text{if } |\overline{Z}| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\hat{\eta}$ fits PFTS, but it does not fit BCFE.

Example 2: Apply a scheme $\hat{\eta}$ on $\overline{\mathbf{MCS}}$ by for each $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$ and for each $(m, k_m) \in P^{\overline{Z}}$, $\hat{\eta}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) = \frac{\hat{\mathbf{A}}(\vec{b})}{|\overline{Z}|}$. It is clear that $\hat{\eta}$ fits BCFE, but it does not fit PFTS.

B. Symmetry for treatment and related axiomatic analysis

In this section, we characterize the EIAE by introducing and analyzing a specific symmetry-based property.

Let $\hat{\eta}$ be a scheme on $\overline{\mathbf{MCS}}$.

- $\hat{\eta}$ fits **symmetry for treatment (SYMT)** if, for each $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$, and for some $(m, k_m), (n, k_n) \in P^{\overline{Z}}$ and each $\vec{\mu} \in \overline{B}^{\overline{Z}\setminus\{m,n\}}$, if $\hat{\mathbf{A}}(\vec{\mu}, \vec{b}_m, 0) - \hat{\mathbf{A}}(\vec{\mu}, k_m - 1, 0) = \hat{\mathbf{A}}(\vec{\mu}, 0, \vec{b}_n) - \hat{\mathbf{A}}(\vec{\mu}, 0, k_n - 1)$, then it must be that $\hat{\eta}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) = \hat{\eta}_{n,k_n}(\overline{Z}, \hat{\mathbf{A}})$.
- $\hat{\eta}$ fits **synchronized regulation (SYRE)** if, for each pair $(\overline{Z}, \hat{\mathbf{A}}), (\overline{Z}, \hat{\mathbf{U}}) \in \overline{\mathbf{MCS}}$ with $\hat{\mathbf{A}}(\vec{\mu}) = \hat{\mathbf{U}}(\vec{\mu}) + \sum_{m\in\hat{\mathbf{J}}(\vec{\mu})} \sum_{q=1}^{\vec{\mu}_m} \zeta_{m,q}$, for some $\zeta \in \mathbb{R}^{P^{\overline{Z}}}$ and each $\vec{\mu} \in \overline{B}^{\overline{Z}}$, then $\hat{\eta}(\overline{Z}, \hat{\mathbf{A}}) = \hat{\eta}(\overline{Z}, \hat{\mathbf{U}}) + (\sum_{q=k_m}^{\vec{b}_m} \zeta_{m,q})_{(m,k_m)\in P^{\overline{Z}}}$.

The SYMT property formalizes the idea of functional symmetry: if two participators exhibit identical marginal contributions under specified behaviours, their allocated outcomes should be equal. The SYRE axiom captures a form of responsiveness to global resource variation, requiring the allocation scheme to adjust proportionally when aggregate value changes. In the remainder of this subsection, we prove that the EIAE satisfies both SYMT and SYRE, and examine how these properties, together with SCOM and BCFE, logically imply PFTS and jointly characterize the EIAE scheme uniquely.

Lemma 5: The EIAE fits SYMT.

Proof: Let $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$. Assume that $\hat{\mathbf{A}}(\vec{\mu}, \vec{b}_m, 0) - \hat{\mathbf{A}}(\vec{\mu}, k_m - 1, 0) = \hat{\mathbf{A}}(\vec{\mu}, 0, \vec{b}_n) - \hat{\mathbf{A}}(\vec{\mu}, 0, k_n - 1)$ holds for some $(m, k_m), (n, k_n) \in P^{\overline{Z}}$ and each $\vec{\mu} \in \overline{B}^{\overline{Z}\setminus\{m,n\}}$. Take $\vec{\mu} = 0_{\overline{Z}\setminus\{m,n\}}$. Then,

$$\begin{aligned} & \hat{\mathbf{A}}(\vec{b}_m, 0_{\overline{Z}\setminus\{m\}}) - \hat{\mathbf{A}}(k_m - 1, 0_{\overline{Z}\setminus\{m\}}) \\ = \; & \hat{\mathbf{A}}(\vec{b}_n, 0_{\overline{Z}\setminus\{n\}}) - \hat{\mathbf{A}}(k_n - 1, 0_{\overline{Z}\setminus\{n\}}), \end{aligned}$$

which implies $\hat{\psi}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) = \hat{\psi}_{n,k_n}(\overline{Z}, \hat{\mathbf{A}})$. Therefore,

$$\begin{aligned} & \hat{\mathbf{\Psi}}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) \\ = \; & \hat{\psi}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) + \tfrac{1}{|\overline{Z}|} \cdot \Big[\hat{\mathbf{A}}(\vec{b}) - \sum_{t\in\overline{Z}} \hat{\psi}_{t,\vec{b}_t}(\overline{Z}, \hat{\mathbf{A}})\Big] \\ = \; & \hat{\psi}_{n,k_n}(\overline{Z}, \hat{\mathbf{A}}) + \tfrac{1}{|\overline{Z}|} \cdot \Big[\hat{\mathbf{A}}(\vec{b}) - \sum_{t\in\overline{Z}} \hat{\psi}_{t,\vec{b}_t}(\overline{Z}, \hat{\mathbf{A}})\Big] \\ = \; & \hat{\mathbf{\Psi}}_{n,k_n}(\overline{Z}, \hat{\mathbf{A}}). \end{aligned}$$

Hence, the scheme $\hat{\mathbf{\Psi}}$ fits SYMT. ■

Lemma 6: The EIAE fits SYRE.

Proof: Let $(\overline{Z}, \hat{\mathbf{A}}), (\overline{Z}, \hat{\mathbf{U}}) \in \overline{\mathbf{MCS}}$ be such that $\hat{\mathbf{A}}(\vec{\mu}) = \hat{\mathbf{U}}(\vec{\mu}) + \sum_{m\in\hat{\mathbf{J}}(\vec{\mu})} \sum_{q=1}^{\vec{\mu}_m} \zeta_{m,q}$ for each $\vec{\mu} \in \overline{B}^{\overline{Z}}$ and for some $\zeta \in \mathbb{R}^{P^{\overline{Z}}}$. Then, for each $(m, k_m) \in P^{\overline{Z}}$

$$\begin{aligned} & \hat{\psi}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) \\ = \; & \hat{\mathbf{A}}(\vec{b}_m, 0_{\overline{Z}\setminus\{m\}}) - \hat{\mathbf{A}}(k_m - 1, 0_{\overline{Z}\setminus\{m\}}) \\ = \; & \hat{\mathbf{U}}(\vec{b}_m, 0_{\overline{Z}\setminus\{m\}}) + \sum_{q=1}^{\vec{b}_m} \zeta_{m,q} - \hat{\mathbf{U}}(k_m - 1, 0_{\overline{Z}\setminus\{m\}}) \\ & - \sum_{q=1}^{k_m-1} \zeta_{m,q} \\ = \; & \hat{\psi}_{m,k_m}(\overline{Z}, \hat{\mathbf{U}}) + \sum_{q=k_m}^{\vec{b}_m} \zeta_{m,q}. \end{aligned}$$

Thus,

$$\begin{aligned} & \hat{\mathbf{\Psi}}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) \\ = \; & \hat{\psi}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) + \tfrac{1}{|\overline{Z}|} \cdot \Big[\hat{\mathbf{A}}(\vec{b}) - \sum_{n\in\overline{Z}} \hat{\psi}_{n,\vec{b}_n}(\overline{Z}, \hat{\mathbf{A}})\Big] \\ = \; & \hat{\psi}_{m,k_m}(\overline{Z}, \hat{\mathbf{U}}) + \sum_{q=k_m}^{\vec{b}_m} \zeta_{m,q} \\ & + \tfrac{1}{|\overline{Z}|} \cdot \Big[\hat{\mathbf{U}}(\vec{b}) - \sum_{n\in\overline{Z}} \hat{\psi}_{n,\vec{b}_n}(\overline{Z}, \hat{\mathbf{U}})\Big] \\ = \; & \hat{\mathbf{\Psi}}_{m,k_m}(\overline{Z}, \hat{\mathbf{U}}) + \sum_{q=k_m}^{\vec{b}_m} \zeta_{m,q}. \end{aligned}$$

Therefore, the EIAE satisfies SYRE. ■

Lemma 7: If a scheme $\hat{\eta}$ on $\overline{\mathbf{MCS}}$ fits SCOM, SYMT and SYRE, then $\hat{\eta}$ fits PFTS.

Proof: Assume that a scheme $\hat{\eta}$ satisfies the axioms SCOM, SYMT, and SYRE. Consider a two-player setting $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$ with $\overline{Z} = \{m, n\}$ and $m \neq n$. We construct a new situation $(\overline{Z}, \hat{\mathbf{U}})$ where, for each $\vec{\lambda} \in \overline{B}^{\overline{Z}}$, the function $\hat{\mathbf{U}}$ is defined as $\hat{\mathbf{U}}(\vec{\lambda}) = \hat{\mathbf{A}}(\vec{\lambda}) - \sum_{i\in\hat{\mathbf{J}}(\vec{\lambda})} \sum_{q=1}^{\vec{\lambda}_i} [\hat{\mathbf{A}}(q, 0_{\overline{Z}\setminus\{i\}}) - \hat{\mathbf{A}}(q-1, 0_{\overline{Z}\setminus\{i\}})]$. By construction, $\hat{\mathbf{U}}$ removes each player's marginal contributions from the original $\hat{\mathbf{A}}$ allocation. For instance, for each $k_m \in \overline{B}_m$, we have

$$\begin{aligned} & \hat{\mathbf{U}}(k_m, 0) \\ = \; & \hat{\mathbf{A}}(k_m, 0) - \sum_{q=1}^{k_m} [\hat{\mathbf{A}}(q, 0_{\overline{Z}\setminus\{i\}}) - \hat{\mathbf{A}}(q-1, 0_{\overline{Z}\setminus\{i\}})] \\ = \; & \hat{\mathbf{A}}(k_m, 0) - \hat{\mathbf{A}}(k_m, 0) \\ = \; & 0. \end{aligned}$$

Similarly, $\hat{\mathbf{U}}(0, k_n) = 0$ for all $k_n \in \overline{B}_n$. Furthermore, $\hat{\mathbf{U}}(\vec{b}_m, 0) - \hat{\mathbf{U}}(k_m - 1, 0) = 0 = \hat{\mathbf{U}}(0, \vec{b}_n) - \hat{\mathbf{U}}(0, k_n - 1)$, implying that all marginal differences in $\hat{\mathbf{U}}$ vanish. By the SYMT property of $\hat{\eta}$, we therefore have $\hat{\eta}_{m,k_m}(\overline{Z}, \hat{\mathbf{U}}) = \hat{\eta}_{n,k_n}(\overline{Z}, \hat{\mathbf{U}})$. Invoking SCOM for $\hat{\eta}$ in the modified situation, the total allocation satisfies

$$\hat{\mathbf{U}}(\vec{b}) = \hat{\eta}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{U}}) + \hat{\eta}_{n,\vec{b}_n}(\overline{Z}, \hat{\mathbf{U}}) = 2 \cdot \hat{\eta}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{U}}). \quad (5)$$

Therefore, by Equation (5) and the definition of $\hat{\mathbf{U}}$, we obtain

$$\begin{aligned} & \hat{\eta}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{U}}) \\ = \; & \tfrac{\hat{\mathbf{U}}(\vec{b})}{2} \\ = \; & \tfrac{1}{2} \cdot \big[\hat{\mathbf{A}}(\vec{b}) - \hat{\psi}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \hat{\psi}_{n,\vec{b}_n}(\overline{Z}, \hat{\mathbf{A}})\big]. \end{aligned}$$

Now, by applying the SYRE property of $\hat{\eta}$, we recover the original allocation as

$$\begin{aligned} & \hat{\eta}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) \\ = \; & \hat{\eta}_{m,k_m}(\overline{Z}, \hat{\mathbf{U}}) + \hat{\psi}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) \\ = \; & \tfrac{1}{2} \cdot [\hat{\mathbf{A}}(\vec{b}) - \hat{\psi}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \hat{\psi}_{n,\vec{b}_n}(\overline{Z}, \hat{\mathbf{A}})] \\ & + \hat{\psi}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}) \\ = \; & \hat{\mathbf{\Psi}}_{m,k_m}(\overline{Z}, \hat{\mathbf{A}}). \end{aligned}$$

By symmetric reasoning, we similarly obtain $\hat{\eta}_{n,k_n}(\overline{Z}, \hat{\mathbf{A}}) = \hat{\mathbf{\Psi}}_{n,k_n}(\overline{Z}, \hat{\mathbf{A}})$ for each $k_n \in \overline{B}_n$. This confirms that $\hat{\eta}$ fits PFTS. ■

Theorem 2: A scheme $\hat{\eta}$ on $\overline{\mathbf{MCS}}$ fits SCOM, SYMT, SYRE and BCFE if and only if $\hat{\eta} = \hat{\mathbf{\Psi}}$.

Proof: By Lemmas 1, 3, 5, and 6, the scheme $\hat{\mathbf{\Psi}}$ satisfies all four axioms: SCOM, SYMT, SYRE, and BCFE. The uniqueness follows directly from Theorem 1 and Lemma 7, completing the characterization. ■

The following four schemes are constructed to demonstrate the logical independence of each axiom appearing in Theorem 2.

Example 3: Apply a scheme $\hat{\eta}$ on $\overline{\mathbf{MCS}}$ defined as $\hat{\eta}_{m,k_m}(\overline{Z},\hat{\mathbf{A}}) = \hat{\psi}_{m,k_m}(\overline{Z},\hat{\mathbf{A}})$. Then $\hat{\eta}$ clearly satisfies SYMT, SYRE, and BCFE, but it fails to satisfy SCOM and consequently PFTS.

Example 4: Define $\hat{\eta}$ as the uniform distribution scheme $\hat{\eta}_{m,k_m}(\overline{Z},\hat{\mathbf{A}}) = \frac{\hat{\mathbf{A}}(\vec{b})}{|\overline{Z}|}$. This scheme satisfies SCOM, SYMT, and BCFE, but does not respect the SYRE property, as it neglects marginal payoff structures.

Example 5: Define $\hat{\eta}$ as

$$\begin{aligned} & \hat{\eta}_{m,k_m}(\overline{Z},\hat{\mathbf{A}}) \\ = \; & \big[\hat{\mathbf{A}}(\vec{b}) - \hat{\mathbf{A}}(\vec{b}_{\overline{Z}\setminus\{m\}},0)\big] \\ & + \tfrac{1}{|\overline{Z}|}\cdot\Big[\hat{\mathbf{A}}(\vec{b}) - \sum_{k\in\overline{Z}}\big[\hat{\mathbf{A}}(\vec{b}) - \hat{\mathbf{A}}(\vec{b}_{\overline{Z}\setminus\{k\}},0)\big]\Big]. \end{aligned}$$

This $\hat{\eta}$ satisfies SCOM, SYRE, and BCFE, but fails to satisfy SYMT due to asymmetric treatment of player indices.

Example 6: Let $\hat{\eta}$ be defined as

$$\begin{aligned} & \hat{\eta}_{m,k_m}(\overline{Z},\hat{\mathbf{A}}) \\ = \; & \sum_{\substack{\overline{S}\subseteq\overline{Z} \\ m\in\overline{S}}} \tfrac{(|\overline{S}|-1)!(|\overline{Z}|-|\overline{S}|)!}{|\overline{Z}|!} \\ & \cdot\big[\hat{\mathbf{A}}\big(\vec{b}_{\overline{S}\setminus\{m\}},k_m,0_{\overline{Z}\setminus\overline{S}}\big) - \hat{\mathbf{A}}\big(\vec{b}_{\overline{S}\setminus\{m\}},0,0_{\overline{Z}\setminus\overline{S}}\big)\big]. \end{aligned}$$

This scheme satisfies SCOM, SYMT, and SYRE, but fails to satisfy BCFE due to its lack of bilateral consistency under reduction.

In the following, more definitions, related axioms and alternative characterizations would be also provided. A multi-choice situation $(\overline{Z},\vec{b},\hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$ is said to be *$\vec{\alpha}$-trivial* if $\sum_{m\in\hat{\mathbf{J}}(\vec{\mu})} \vec{\alpha}_{m,\vec{\mu}_m} = \hat{\mathbf{A}}(\vec{\mu})$ for every $\vec{\mu}\in\overline{B}^{\overline{Z}}$, where $\vec{\alpha}\in\mathbb{R}^{P^{\overline{Z}}}$. Let $\hat{\eta}$ be a scheme.

- $\hat{\eta}$ satisfies weak scheme completeness (WSCOM) if $\hat{\eta}$ satisfies scheme completeness under all one-participator multi-choice situations.
- $\hat{\eta}$ satisfies trivial-situations condition (TSCN) if $\hat{\eta}(\overline{Z},\vec{b},\hat{\mathbf{A}}) = \vec{\alpha}$ under every $\vec{\alpha}$-trivial situation $(\overline{Z},\vec{b},\hat{\mathbf{A}})$, where $\vec{\alpha}\in\mathbb{R}^{P^{\overline{Z}}}$.
- $\hat{\eta}$ satisfies weak scheme completeness (WTSCN) if $\hat{\eta}$ satisfies trivial-situations condition under all one-participator multi-choice situations.

Lemma 8: The EIAE fits WSCOM.

Proof: It is shown that the EIAE fits SCOM. Based on definitions of SCOM and WSCOM, the EIAE fits WSCOM absolutely. ■

Lemma 9: A scheme $\hat{\eta}$ satisfies WSCOM if and only if $\hat{\eta}$ satisfies WTSCN.

Proof: Let $\hat{\eta}$ be a scheme.

$$\begin{aligned} & \hat{\eta} \text{ satisfies WSCOM} \\ \Leftrightarrow\; & \hat{\eta}(\overline{Z},\vec{b},\hat{\mathbf{A}}) = \hat{\mathbf{A}}(\overline{Z}) \text{ for all one-participator} \\ & \text{multi-choice situation } (\overline{Z},\vec{b},\hat{\mathbf{A}}) \\ \Leftrightarrow\; & \hat{\eta}(\overline{Z},\vec{b},\hat{\mathbf{A}}) = \hat{\mathbf{A}}(\overline{Z}) \text{ and } \hat{\mathbf{A}}(\overline{Z}) = \hat{\mathbf{A}}(\overline{Z}) \text{ for all} \\ & \text{one-participator multi-choice situation } (\overline{Z},\vec{b},\hat{\mathbf{A}}) \\ \Leftrightarrow\; & \hat{\eta} \text{ satisfies WTSCN.} \end{aligned}$$

■

Lemma 10: The EIAE fits WTSCN.

Proof: The proofs of this lemma can be finished via above two lemmas. ■

Lemma 11: If a scheme $\hat{\eta}$ fits WSCOM and BCFE then it also fits SCOM.

Proof: Let $\hat{\eta}$ be a scheme on $\overline{\mathbf{MCS}}$ satisfy WSCOM and BCFE, and $(\overline{Z},\hat{\mathbf{A}})\in\overline{\mathbf{MCS}}$. It is trivial for $|\overline{Z}|=1$ by WSCOM. Assume that $|\overline{Z}|\geq 2$. Let $n\in\overline{Z}$, consider the reduced situation $\big(\{n\},\hat{\mathbf{A}}^{\hat{\eta}}_{\{n\}}\big)$. By definition of $\hat{\mathbf{A}}^{\hat{\eta}}_{\{n\}}$, $\hat{\mathbf{A}}^{\hat{\eta}}_{\{n\}}(\vec{b}_n) = \hat{\mathbf{A}}(\vec{b}) - \sum_{m\in\overline{Z}\setminus\{n\}} \hat{\eta}_{m,\vec{b}_m}(\overline{Z},\hat{\mathbf{A}})$. Since $\hat{\eta}$ fits BCFE, $\hat{\eta}_{n,k_n}(\overline{Z},\hat{\mathbf{A}}) = \hat{\eta}_{n,k_n}\big(\{n\},\hat{\mathbf{A}}^{\hat{\eta}}_{\{n\}}\big)$ for each $k_n\in\overline{B}_n$. In particular, $\hat{\eta}_{n,\vec{b}_n}(\overline{Z},\hat{\mathbf{A}}) = \hat{\eta}_{n,\vec{b}_n}\big(\{n\},\hat{\mathbf{A}}^{\hat{\eta}}_{\{n\}}\big)$. On the other hand, by WSCOM of $\hat{\eta}$, $\hat{\eta}_{n,\vec{b}_n}(\overline{Z},\hat{\mathbf{A}}) = \hat{\mathbf{A}}^{\hat{\eta}}_{\{n\}}(\vec{b}_n)$. Hence, $\sum_{m\in\overline{Z}} \hat{\eta}_{m,\vec{b}_m}(\overline{Z},\hat{\mathbf{A}}) = \hat{\mathbf{A}}(\vec{b})$, i.e., $\hat{\eta}$ fits SCOM. ■

Lemma 12: If a scheme $\hat{\eta}$ fits WTSCN and BCFE then it also fits SCOM.

Proof: The proofs of this lemma can be finished via above four lemmas. ■

Theorem 3:

- A scheme $\hat{\eta}$ on $\overline{\mathbf{MCS}}$ fits WSCOM, SYMT, SYRE and BCFE if and only if $\hat{\eta} = \hat{\mathbf{\Psi}}$.
- A scheme $\hat{\eta}$ on $\overline{\mathbf{MCS}}$ fits WTSCN, SYMT, SYRE and BCFE if and only if $\hat{\eta} = \hat{\mathbf{\Psi}}$.

Proof: The proofs of this lemma can be finished via above five lemmas and Theorem 2. ■

IV. Dynamic Result

In order to provide a dynamic formulation of the EIAE, we begin by establishing a representation for the EIAE using the *variation* function. Let $(\overline{Z},\hat{\mathbf{A}})\in\overline{\mathbf{MCS}}$ and $\vec{x}\in\mathbb{R}^{P^{\overline{Z}}}$. The **variation** of $\vec{\mu}\in\overline{B}^{\overline{Z}}$ at point $\vec{x}$ is defined as the real-valued expression $\hat{\mathbf{v}}(\vec{\mu},\hat{\mathbf{A}},\vec{x}) = \big[\hat{\mathbf{A}}(\vec{\mu}) - \sum_{m\in\hat{\mathbf{J}}(\vec{\mu})} \hat{\mathbf{A}}(\vec{\mu}_m - 1, 0_{\overline{Z}\setminus\{m\}})\big] - \vec{x}(\vec{\mu})$, where $\vec{x}(\vec{\mu}) = \sum_{m\in\hat{\mathbf{J}}(\vec{\mu})} \vec{x}_{m,\vec{\mu}_m}$. Further, we define the constraint set $\overline{X}(\overline{Z},\hat{\mathbf{A}}) = \{\vec{x}\in\mathbb{R}^{P^{\overline{Z}}} \mid \sum_{m\in\overline{Z}} \vec{x}_{m,\vec{b}_m} = \hat{\mathbf{A}}(\vec{b})\}$. This variation function quantifies the discrepancy between the realized payoff from a specific collective action and the payoff allocation prescribed by $\vec{x}$. In this sense, it captures the local imbalance from a dynamic or adjustment-based viewpoint in allocation processes.

In what follows, we investigate the connection between the variation function and the axiom SYMT. Intuitively, if a resource allocation scheme treats any two participants symmetrically, then their induced variations, measured under the same decision profile, should also reflect this symmetry.

Lemma 13: Let $(\overline{Z},\hat{\mathbf{A}})\in\overline{\mathbf{MCS}}$ and $\vec{x}\in\overline{X}(\overline{Z},\hat{\mathbf{A}})$. Then

$$\begin{aligned} & \hat{\mathbf{v}}((\vec{b}_m,0_{\overline{Z}\setminus\{m\}}),\hat{\mathbf{A}},\vec{x}) = \hat{\mathbf{v}}((\vec{b}_n,0_{\overline{Z}\setminus\{n\}}),\hat{\mathbf{A}},\vec{x}) \\ \Longleftrightarrow\; & \vec{x}_{m,\vec{b}_m} = \hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z},\hat{\mathbf{A}}) \;\forall\, m,n\in\overline{Z}. \end{aligned}$$

Proof: Let $(\overline{Z},\hat{\mathbf{A}})\in\overline{\mathbf{MCS}}$ and $\vec{x}\in\overline{X}(\overline{Z},\hat{\mathbf{A}})$. For each pair $\{m,n\} \subseteq \overline{Z}$, we compute

$$\begin{array}{ll} & \hat{\mathbf{v}}((\vec{b}_m, 0_{\overline{Z}\setminus\{m\}}), \hat{\mathbf{A}}, \vec{x}) = \hat{\mathbf{v}}((\vec{b}_n, 0_{\overline{Z}\setminus\{n\}}), \hat{\mathbf{A}}, \vec{x}) \\ \iff & \hat{\mathbf{A}}(\vec{b}_m, 0_{\overline{Z}\setminus\{m\}}) - \hat{\mathbf{A}}(\vec{b}_m - 1, 0_{\overline{Z}\setminus\{m\}}) - \vec{x}_{m,\vec{b}_m} \\ & = \hat{\mathbf{A}}(\vec{b}_n, 0_{\overline{Z}\setminus\{n\}}) - \hat{\mathbf{A}}(\vec{b}_n - 1, 0_{\overline{Z}\setminus\{n\}}) - \vec{x}_{n,\vec{b}_n} \\ \iff & \vec{x}_{m,\vec{b}_m} - \vec{x}_{n,\vec{b}_n} \\ & = \big[\hat{\mathbf{A}}(\vec{b}_m, 0_{\overline{Z}\setminus\{m\}}) - \hat{\mathbf{A}}(\vec{b}_m - 1, 0_{\overline{Z}\setminus\{m\}})\big] \\ & -\big[\hat{\mathbf{A}}(\vec{b}_n, 0_{\overline{Z}\setminus\{n\}}) - \hat{\mathbf{A}}(\vec{b}_n - 1, 0_{\overline{Z}\setminus\{n\}})\big]. \end{array} \quad (6)$$

By the definition of the EIAE scheme $\hat{\mathbf{\Psi}}$, we also have

$$\begin{array}{ll} & \hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \hat{\mathbf{\Psi}}_{n,\vec{b}_n}(\overline{Z}, \hat{\mathbf{A}}) \\ = & \big[\hat{\mathbf{A}}(\vec{b}_m, 0_{\overline{Z}\setminus\{m\}}) - \hat{\mathbf{A}}(\vec{b}_m - 1, 0_{\overline{Z}\setminus\{m\}})\big] \\ & -\big[\hat{\mathbf{A}}(\vec{b}_n, 0_{\overline{Z}\setminus\{n\}}) - \hat{\mathbf{A}}(\vec{b}_n - 1, 0_{\overline{Z}\setminus\{n\}})\big]. \end{array} \quad (7)$$

From equations (6) and (7), we conclude

$$\vec{x}_{m,\vec{b}_m} - \vec{x}_{n,\vec{b}_n} = \hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \hat{\mathbf{\Psi}}_{n,\vec{b}_n}(\overline{Z}, \hat{\mathbf{A}}).$$

Summing both sides over all $n \neq m$

$$\begin{array}{ll} & \sum_{n\neq m} (\vec{x}_{m,\vec{b}_m} - \vec{x}_{n,\vec{b}_n}) \\ = & \sum_{n\neq m} (\hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \hat{\mathbf{\Psi}}_{n,\vec{b}_n}(\overline{Z}, \hat{\mathbf{A}})). \end{array}$$

This gives

$$\begin{array}{ll} & (|\overline{Z}| - 1) \cdot \vec{x}_{m,\vec{b}_m} - \sum_{n\neq m} \vec{x}_{n,\vec{b}_n} \\ = & (|\overline{Z}| - 1) \cdot \hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \sum_{n\neq m} \hat{\mathbf{\Psi}}_{n,\vec{b}_n}(\overline{Z}, \hat{\mathbf{A}}). \end{array}$$

Because $\vec{x} \in \overline{X}(\overline{Z}, \hat{\mathbf{A}})$ and $\hat{\mathbf{\Psi}}$ satisfies SCOM, we finally obtain

$$|\overline{Z}| \cdot \vec{x}_{m,\vec{b}_m} - \hat{\mathbf{A}}(\vec{b}) = |\overline{Z}| \cdot \hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \hat{\mathbf{A}}(\vec{b}).$$

Thus, it follows that $\vec{x}_{m,\vec{b}_m} = \hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}})$ for each $m \in \overline{Z}$. ■

Based on the notion of Lemma 13, we define a calibration mechanism that serves as the foundation for a dynamic formulation of the EIAE. This calibration operates via an iterative procedure that progressively adjusts an initial allocation $\vec{x}$ to better satisfy fairness conditions derived from the variation function.

Definition 2: Let $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$ with $|\overline{Z}| \geq 2$ and $\vec{x} \in \overline{X}(\overline{Z}, \hat{\mathbf{A}})$. We define the *calibrations* $\hat{\mathbf{g}}_{m,k_m} : \overline{X}(\overline{Z}, \hat{\mathbf{A}}) \to \mathbb{R}$ by for each $(m, k_m) \in P^{\overline{Z}}$,

$$\begin{array}{ll} & \hat{\mathbf{g}}_{m,k_m}(\vec{x}) \\ = & \vec{x}_{m,k_m} + \\ & \alpha \sum_{n\in\overline{Z}\setminus\{m\}} \Big(\hat{\mathbf{v}}((\vec{b}_m, 0_{\overline{Z}\setminus\{m\}}), \hat{\mathbf{A}}, \vec{x}) - \hat{\mathbf{v}}((\vec{b}_n, 0_{\overline{Z}\setminus\{n\}}), \hat{\mathbf{A}}, \vec{x})\Big). \end{array}$$

We denote $\hat{\mathbf{g}} = (\hat{\mathbf{g}}_{m,k_m})_{(m,k_m)\in P^{\overline{Z}}}$ and define an iterative sequence by setting $\vec{x}^0 = \vec{x}$, $\vec{x}^1 = \hat{\mathbf{g}}(\vec{x}^0), \cdots, \vec{x}^q = \hat{\mathbf{g}}(\vec{x}^{q-1})$ for each $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$, for each $\vec{x} \in \overline{X}(\overline{Z}, \hat{\mathbf{A}})$, and for each $q \in \mathbb{N}$.

The purpose of the calibration operator $\hat{\mathbf{g}}$ is to iteratively reduce inter-agent variation in excesses by adjusting allocations. This process tends to steer the allocation toward symmetry-compliant solutions—particularly those satisfying the EIAE conditions.

Lemma 14: Let $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$. If $\vec{x} \in \overline{X}(\overline{Z}, \hat{\mathbf{A}})$, then $\hat{\mathbf{g}}(\vec{x}) \in \overline{X}(\overline{Z}, \hat{\mathbf{A}})$.

Proof: Let $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$, $m, n \in \overline{Z}$ and $\vec{x} \in \overline{X}(\overline{Z}, \hat{\mathbf{A}})$.

$$\begin{array}{ll} & \sum_{n\in\overline{Z}\setminus\{m\}} \Big(\hat{\mathbf{v}}((\vec{b}_m, 0_{\overline{Z}\setminus\{m\}}), \hat{\mathbf{A}}, \vec{x}) - \hat{\mathbf{v}}((\vec{b}_n, 0_{\overline{Z}\setminus\{n\}}), \hat{\mathbf{A}}, \vec{x})\Big) \\ = & \sum_{n\in\overline{Z}\setminus\{m\}} \Big(\hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \hat{\mathbf{\Psi}}_{n,\vec{b}_n}(\overline{Z}, \hat{\mathbf{A}}) - \vec{x}_{m,\vec{b}_m} + \vec{x}_{n,\vec{b}_n}\Big) \\ & \Big(\text{equations (6) and (7)}\Big) \\ = & \Big((|\overline{Z}| - 1) \cdot (\hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \vec{x}_{m,\vec{b}_m}) \\ & - \sum_{n\in\overline{Z}\setminus\{m\}} \hat{\mathbf{\Psi}}_{n,\vec{b}_n}(\overline{Z}, \hat{\mathbf{A}}) + \sum_{n\in\overline{Z}\setminus\{i\}} \vec{x}_{n,\vec{b}_n}\Big) \\ = & \Big(|\overline{Z}| \cdot (\hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \vec{x}_{m,\vec{b}_m}) - \hat{\mathbf{A}}(\vec{b}) + \hat{\mathbf{A}}(\vec{b})\Big) \\ & \Big(\textbf{SCOM of } \hat{\mathbf{\Psi}}\textbf{,}\ \vec{x} \in \overline{X}(\overline{Z}, \hat{\mathbf{A}})\Big) \\ = & |\overline{Z}| \cdot \Big(\hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \vec{x}_{m,\vec{b}_m}\Big). \end{array} \quad (8)$$

Moreover,

$$\begin{array}{ll} & \sum_{m\in\overline{Z}} \sum_{n\in\overline{Z}\setminus\{m\}} \Big(\hat{\mathbf{v}}((\vec{b}_m, 0_{\overline{Z}\setminus\{m\}}), \hat{\mathbf{A}}, \vec{x}) \\ & \qquad -\hat{\mathbf{v}}((\vec{b}_n, 0_{\overline{Z}\setminus\{n\}}), \hat{\mathbf{A}}, \vec{x})\Big) \\ = & \sum_{m\in\overline{Z}} |\overline{Z}| \cdot \Big(\hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \vec{x}_{m,\vec{b}_m}\Big) \\ = & |\overline{Z}| \cdot \Big(\sum_{m\in\overline{Z}} \hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \sum_{m\in\overline{Z}} \vec{x}_{m,\vec{b}_m}\Big) \\ = & |\overline{Z}| \cdot \big(\hat{\mathbf{A}}(\vec{b}) - \hat{\mathbf{A}}(\vec{b})\big) \\ & \Big(\textbf{SCOM of } \hat{\mathbf{\Psi}}\textbf{,}\ \vec{x} \in \overline{X}(\overline{Z}, \hat{\mathbf{A}})\Big) \\ = & 0. \end{array} \quad (9)$$

So we have that

$$\begin{array}{ll} & \sum_{m\in\overline{Z}} \hat{\mathbf{g}}_{m,\vec{b}_m}(\vec{x}) \\ = & \sum_{m\in\overline{Z}} \Big[\vec{x}_{m,\vec{b}_m} + \\ & \alpha \sum_{n\in\overline{Z}\setminus\{m\}} \Big(\hat{\mathbf{v}}((\vec{b}_m, 0_{\overline{Z}\setminus\{i\}}), \hat{\mathbf{A}}, \vec{x}) - \hat{\mathbf{v}}((\vec{b}_n, 0_{\overline{Z}\setminus\{n\}}), \hat{\mathbf{A}}, \vec{x})\Big)\Big] \\ = & \hat{\mathbf{A}}(\vec{b}). \\ & \Big(\textbf{equation (9) and } \vec{x} \in \overline{X}(\overline{Z}, \hat{\mathbf{A}})\Big) \end{array}$$

Hence, $\hat{\mathbf{g}}(\vec{x}) \in \overline{X}(\overline{Z}, \hat{\mathbf{A}})$ if $\vec{x} \in \overline{X}(\overline{Z}, \hat{\mathbf{A}})$. ■

The next result establishes the convergence of the calibration process. Under a suitable condition on the step-size parameter α, repeated application of $\hat{\mathbf{g}}$ yields a sequence of allocations that converges componentwise to the EIAE.

Theorem 4: Let $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$ with $|\overline{Z}| \geq 2$. If $0 < \alpha < \frac{2}{|\overline{Z}|}$, then $\{\vec{x}^q_{m,\vec{b}_m}\}_{q=1}^{\infty}$ converges to $\hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}})$ for each $\vec{x} \in \overline{X}(\overline{Z}, \hat{\mathbf{A}})$ and for each $m \in \overline{Z}$.

Proof: Let $(\overline{Z}, \hat{\mathbf{A}}) \in \overline{\mathbf{MCS}}$ with $|\overline{Z}| \geq 2$, $m \in \overline{Z}$ and $\vec{x} \in \overline{X}(\overline{Z}, \hat{\mathbf{A}})$. By equation (8) and definition of $\hat{\mathbf{g}}$,

$$\begin{array}{ll} & \hat{\mathbf{g}}_{m,\vec{b}_m}(\vec{x}) - \vec{x}_{m,\vec{b}_m} \\ = & \alpha \sum_{n\in\overline{Z}\setminus\{m\}} \Big(\hat{\mathbf{v}}((\vec{b}_m, 0_{\overline{Z}\setminus\{i\}}), \hat{\mathbf{A}}, \vec{x}) - \hat{\mathbf{v}}((\vec{b}_n, 0_{\overline{Z}\setminus\{n\}}), \hat{\mathbf{A}}, \vec{x})\Big) \\ = & \alpha \cdot |\overline{Z}| \cdot \Big(\hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \vec{x}_{m,\vec{b}_m}\Big). \end{array}$$

Hence,

$$\begin{array}{ll} & \hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \hat{\mathbf{g}}_{m,\vec{b}_m}(\vec{x}) \\ = & \hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \vec{x}_{m,\vec{b}_m} + \vec{x}_{m,\vec{b}_m} - \hat{\mathbf{g}}_{m,\vec{b}_m}(\vec{x}) \\ = & \hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \vec{x}_{m,\vec{b}_m} - \alpha \cdot |\overline{Z}| \cdot \Big[\hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \vec{x}_{m,\vec{b}_m}\Big] \\ = & \Big(1 - \alpha \cdot |\overline{Z}|\Big)\Big[\hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \vec{x}_{m,\vec{b}_m}\Big]. \end{array}$$

So, for each $q \in \mathbb{N}$,

$$\begin{array}{ll} & \hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \vec{x}^q_{m,\vec{b}_m} \\ = & \Big(1 - \alpha \cdot |\overline{Z}|\Big)^q \Big[\hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z}, \hat{\mathbf{A}}) - \vec{x}_{m,\vec{b}_m}\Big]. \end{array}$$

If $0<\alpha<\frac{2}{|\overline{Z}|}$, then $-1<\left(1-\alpha\cdot|\overline{Z}|\right)<1$ and $\{\vec{x}^q_{m,\vec{b}_m}\}_{q=1}^{\infty}$ converges to $\hat{\mathbf{\Psi}}_{m,\vec{b}_m}(\overline{Z},\hat{\mathbf{A}})$. ■

Inspired by Liao [21], we also define a notion of completeness that connects allocation outcomes with marginal coalition behaviour. A payoff vector $\vec{x}$ fits **plurality-scheme completeness (PSCOM)** under $(\overline{Z},\hat{\mathbf{A}})\in\overline{\mathbf{MCS}}$ if

$$\vec{x}_{m,k_m}+\sum_{n\in\overline{Z}\setminus\{m\}}\vec{x}_{n,\vec{b}_n}=\hat{\mathbf{A}}\big(\vec{b}_{\overline{Z}\setminus\{m\}},k_m\big)$$

for each $(m,k_m)\in P^{\overline{Z}}$. Note that every payoff vector satisfying PSCOM automatically satisfies the constraint $\vec{x}\in\overline{X}(\overline{Z},\hat{\mathbf{A}})$.

Theorem 5: Let $(\overline{Z},\hat{\mathbf{A}})\in\overline{\mathbf{MCS}}$ with $|\overline{Z}|\geq 2$. If $0<\alpha<\frac{2}{|\overline{Z}|}$, then $\{\vec{x}^q\}_{q=1}^{\infty}$ converges to $\hat{\mathbf{\Psi}}(\overline{Z},\hat{\mathbf{A}})$ for each payoff vector $\vec{x}$ which fits PSCOM in $(\overline{Z},\hat{\mathbf{A}})$.

Proof: This result follows directly from Theorem 4, together with the observation that PSCOM implies $\vec{x}\in\overline{X}(\overline{Z},\hat{\mathbf{A}})$. ■

V. Application on Resource Allocating Management Systems

The increasing complexity of resource allocation in practical management systems highlights the critical need for integrating robust game-theoretical principles. In this section, we apply several previously developed results, particularly the EIAE scheme, to industrial and organizational resource allocation. Our goal is to enhance the effectiveness and equity of managerial decisions in operational environments.

Consider an organization comprising multiple departments, each responsible for a set of operational behaviours. Beyond their primary functional tasks, departments often engage in auxiliary activities, such as developing derivative services or collaborating on projects that contribute to the organization's overall utility. Let $\overline{Z}$ denote the set of all departments. When these departments engage in collective actions represented by activity behaviour vectors $\vec{\mu}=(\vec{\mu}_m)_{m\in\overline{Z}}$, the total generated utility is captured by the function $\hat{\mathbf{A}}(\vec{\mu})$.

Each $\vec{\mu}_m$ represents the activity behaviour chosen by department m, and $\hat{\mathbf{A}}$ defines the achievable resource output under any such behavior profile. Thus, this organizational structure naturally fits the framework of a multi-choice transferable-utility (TU) situation, with $\hat{\mathbf{A}}$ acting as its characteristic function. To concretely illustrate this modeling, consider the following numerical application.

Example 7: Consider an organization $\overline{Z}$ consisting of three departments

- Manufacturing (M)
- Marketing (K)
- Research and Development (R)

Each department can participate at different levels of activity. Let $\vec{b}_M=2$, $\vec{b}_K=2$ and $\vec{b}_R=1$. The characteristic function $\hat{\mathbf{A}}$ specifies the organization's utility under various behaviour profiles

$$\begin{array}{rclrcl}
\hat{\mathbf{A}}(2,2,1)&=&108, & \hat{\mathbf{A}}(2,2,0)&=&90,\\
\hat{\mathbf{A}}(2,1,1)&=&130, & \hat{\mathbf{A}}(2,1,0)&=&65,\\
\hat{\mathbf{A}}(2,0,1)&=&80, & \hat{\mathbf{A}}(2,0,0)&=&70,\\
\hat{\mathbf{A}}(1,2,1)&=&80, & \hat{\mathbf{A}}(1,2,0)&=&45,\\
\hat{\mathbf{A}}(1,1,1)&=&65, & \hat{\mathbf{A}}(1,1,0)&=&40,\\
\hat{\mathbf{A}}(1,0,1)&=&95, & \hat{\mathbf{A}}(1,0,0)&=&50,\\
\hat{\mathbf{A}}(0,2,1)&=&95, & \hat{\mathbf{A}}(0,2,0)&=&50,\\
\hat{\mathbf{A}}(0,1,1)&=&30, & \hat{\mathbf{A}}(0,1,0)&=&90,\\
\hat{\mathbf{A}}(0,0,1)&=&80, & \hat{\mathbf{A}}(0,0,0)&=&0.
\end{array}$$

The EIAE scheme allocates payoffs using the formula:

$$\begin{aligned}&\hat{\mathbf{\Psi}}_{m,k_m}(\overline{Z},\hat{\mathbf{A}})\\ =\;&\hat{\psi}_{m,k_m}(\overline{Z},\hat{\mathbf{A}})+\tfrac{1}{|\overline{Z}|}\left[\hat{\mathbf{A}}(\vec{b})-\textstyle\sum_{n\in\overline{Z}}\hat{\psi}_{n,\vec{b}_n}(\overline{Z},\hat{\mathbf{A}})\right],\end{aligned}\tag{10}$$

where

$$\hat{\psi}_{m,k_m}(\overline{Z},\hat{\mathbf{A}})=\hat{\mathbf{A}}(\vec{b}_m,0_{\overline{Z}\setminus\{m\}})-\hat{\mathbf{A}}(k_m-1,0_{\overline{Z}\setminus\{m\}}).\tag{11}$$

First, compute the marginal efficacy components $\hat{\psi}_{m,k_m}$:

$$\begin{array}{rclcr}
\hat{\psi}_{M,2}(\overline{Z},\hat{\mathbf{A}})&=&\hat{\mathbf{A}}(2,0,0)-\hat{\mathbf{A}}(1,0,0)&=&20,\\
\hat{\psi}_{M,1}(\overline{Z},\hat{\mathbf{A}})&=&\hat{\mathbf{A}}(2,0,0)-\hat{\mathbf{A}}(0,0,0)&=&70,\\
\hat{\psi}_{K,2}(\overline{Z},\hat{\mathbf{A}})&=&\hat{\mathbf{A}}(0,2,0)-\hat{\mathbf{A}}(0,1,0)&=&-40,\\
\hat{\psi}_{K,1}(\overline{Z},\hat{\mathbf{A}})&=&\hat{\mathbf{A}}(0,2,0)-\hat{\mathbf{A}}(0,0,0)&=&50,\\
\hat{\psi}_{R,1}(\overline{Z},\hat{\mathbf{A}})&=&\hat{\mathbf{A}}(0,0,1)-\hat{\mathbf{A}}(0,0,0)&=&80.
\end{array}$$

Then, compute the EIAE payoffs as follows.

$$\hat{\mathbf{\Psi}}_{M,2}(\overline{Z},\hat{\mathbf{A}})=20+\frac{1}{3}\left(108-(20-40+80)\right)=36,$$
$$\hat{\mathbf{\Psi}}_{M,1}(\overline{Z},\hat{\mathbf{A}})=70+\frac{1}{3}\left(108-(20-40+80)\right)=86,$$
$$\hat{\mathbf{\Psi}}_{K,2}(\overline{Z},\hat{\mathbf{A}})=-40+\frac{1}{3}\left(108-(20-40+80)\right)=-24,$$
$$\hat{\mathbf{\Psi}}_{K,1}(\overline{Z},\hat{\mathbf{A}})=50+\frac{1}{3}\left(108-(20-40+80)\right)=66,$$
$$\hat{\mathbf{\Psi}}_{R,1}(\overline{Z},\hat{\mathbf{A}})=80+\frac{1}{3}\left(108-(20-40+80)\right)=96.$$

This computation illustrates that the EIAE provides a consistent, marginally grounded, and globally balanced allocation of resources based on departments' activity behaviours.

The efficient individual achieved-efficacy (EIAE) is tailored to support both equitable and efficient outcomes across a range of participatory profiles in resource allocating systems. Beyond establishing its theoretical existence, the EIAE enables precise valuation of each participant's role. To clarify its institutional significance, we highlight the game-theoretical principles that connect directly to managerial applications

1) **Scheme completeness** ensures that the full value of resources is allocated among participants, eliminating waste or inefficiency.
2) **Principle for two-person situations** acknowledges that pairwise collaboration often drives strategic shifts in organizations; the scheme preserves fairness even in these foundational units.
3) **Bilateral conformance** provides robustness: if a subset of agents re-evaluates their allocation, the scheme

guarantees their original position remains stable, contributing to systemic resilience.

4) **Symmetry for treatment** enforces equal reward for equal contribution. Operational equivalence implies allocative parity—an essential tenet in fair management.
5) **Synchronized regulation** asserts that shifts in overall output should be reflected proportionally across departments, ensuring coherence in adaptation and strategy.

Taken together, these principles confirm that resource allocating management systems are aptly modeled as multi-choice TU settings. As demonstrated in Section 3, the EIAE scheme is the unique one that satisfies all five aforementioned axioms. Consequently, Theorems 1, 2, and 3 validate the EIAE as an appropriate and rigorous framework for managing dynamic and distributed industrial resource allocation.

VI. Concluding Remarks

This paper introduces a rigorous extension of the pseudo equal allocation of non-separable costs (PEANSC) to multi-choice transferable-utility (TU) situations. By explicitly incorporating both participants and their respective activity behaviours, we proposed the efficient individual achieved-efficacy (EIAE) and established its axiomatic foundation. In addition to the theoretical development, the proposed model has been shown to be applicable in resource allocating management, with a focus on fairness, operational symmetry, and dynamic convergence.

1) **Summary of Key Contributions**
   - We generalized the PEANSC to a multi-choice TU framework, allowing participants to engage at varying degrees of intensity, thereby broadening the class of cooperative environments in which fair allocation can be studied.
   - The EIAE was axiomatized by five key properties: scheme completeness, symmetry for treatment, bilateral conformance, synchronized regulation, and the principle for two-person situations. Together, these properties uniquely determine the EIAE.
   - A dynamic calibration process was developed to compute the EIAE iteratively. The convergence result ensures that the scheme is not only theoretically sound but also practically computable.
   - The theoretical structure was implemented in a resource allocating management context. This application demonstrates the real-world relevance of EIAE, particularly in organizational settings involving decentralized decision-making and cooperative value generation.
2) **Symmetry as a Central Principle**
   - The notion of functional symmetry—expressed formally via the symmetry for treatment axiom—played a central role in ensuring that participants contributing equally (in marginal terms) are treated equally. This reflects a fairness standard often required in organizational or industrial environments.
   - The variation function, defined to capture dissatisfaction under given allocations, further supports this symmetry by aligning deviations with marginal equity. It also provides a mechanism through which the EIAE can be dynamically approached.
   - The iterative adjustment process proposed for the EIAE is structurally symmetric, ensuring that the scheme remains fair not only in static outcomes but also during transitional phases of resource realignment.
3) **Comparisons with Existing Research** A meaningful comparison arises with the work of Hwang and Liao [13], who explored extensions of the Shapley value [32] in fuzzy game settings. The present paper differs in several essential aspects:
   - While Hwang and Liao [13] considered fuzzy environments, this study operates within multi-choice TU situations, capturing a different form of agent heterogeneity.
   - Their extension of the Shapley value was based on the reduction framework of Hart and Mas-Colell [8], whereas this work builds on the reduction principle introduced by Hsieh and Liao [9], targeting the PEANSC rather than the Shapley value.
   - Our results contribute to practical domains, specifically resource allocating management systems, through theoretical application and numerical illustration, a dimension not addressed in Hwang and Liao [13].
4) **Future Research Directions** The contributions presented in this paper point toward several potential extensions and refinements:
   - **Alternative Solution Concepts:** Beyond the PEANSC, it would be valuable to explore how the axioms introduced here can characterize or approximate other well-known solutions, such as the bargaining set, the kernel, or the nucleolus.
   - **Axiomatic Flexibility:** The bilateral conformance axiom was central to our uniqueness results. Investigating allocations under relaxed conformance conditions could reveal new classes of solution schemes or partial consensus principles.
   - **Dynamic and Computational Implementations:** The calibration method underlying the EIAE can be further formalized into real-time algorithms, facilitating applications in decentralized and large-scale computational settings, particularly in logistics and network coordination problems.
   - **Cross-Domain Applications:** The EIAE framework may also be adapted for other environments requiring resource coordination, such as distributed AI systems, collaborative project scheduling, or joint decision-making under uncertainty.
5) **Conclusion** In summary, this paper provides a

mathematically grounded and operationally meaningful allocation scheme tailored for multi-choice cooperative environments. By extending the PEANSC and rigorously characterizing the EIAE, we contribute significantly to both the theory and practice of resource allocation. The integration of fairness, symmetry, and dynamic adjustability strengthens the relevance of our model across a range of organizational and systemic applications. Future work may expand on these insights by developing broader solution families, relaxing existing axioms, and embedding the EIAE into algorithmic or multi-agent infrastructures.

References

[1] J.F. Banzhaf, "Weighted Voting Doesn't Work: A Mathematical Analysis," *Rutgers Law Review*, vol. 19, pp317-343, 1965

[2] R van den. Brink and G van der. Lann, "Axiomatizations of the Normalized Banzhaf Value and the Shapley Value," *Social Choice and Welfare*, vol. 15, pp567-582, 1998

[3] E. Calvo and J.C. Santos, "A Value for Multichoice Games," *Mathematical Social Sciences*, vol. 40, pp341-354, 2000

[4] C.Y. Cheng, E.C. Chi, K. Chen and Y.H. Liao, "A Power Mensuration and its Normalization under Multicriteria Situations," *IAENG International Journal of Applied Mathematics*, vol. 50, no. 2, pp262-267, 2020

[5] P. Dubey and L.S. Shapley, "Mathematical Properties of the Banzhaf Power Index," *Mathematics of Operations Research*, vol. 4, pp99-131, 1979

[6] K.H.C. Chen, J.C Huang and Y.H. Liao, "Sustainable Combination Mechanism for Catalysts: A Game-theoretical Approach," *Catalysts*, vol. 11, pp345, 2022

[7] H. Haller, "Collusion Properties of Values," *International Journal of Game Theory*, vol. 23, pp261-281, 1994

[8] S. Hart and A. Mas-Colell, "Potential, Value and Consistency," *Econometrica*, vol. 57, pp589-614, 1989

[9] Y.L. Hsieh and Y.H. Liao, *The Pseudo EANSC: Axiomatization and Dynamic Process*, National Pingtung University Press, Taiwan, 2016

[10] Y.A. Hwang, "Fuzzy Games: A Characterization of the Core," *Fuzzy Sets and Systems*, vol.158, pp. 2480-2493, 2007.

[11] Y.A. Hwang and Y.H. Liao, "The Unit-level-core for Multi-choice Games: The Replicated Core for Games," *J. Glob. Optim.*, vol. 47, pp161-171, 2010

[12] Y.A. Hwang and Y.H. Liao, "Note: Natural Extensions of the Equal Allocation of Non-Separable Costs," *Engineering Letters*, vol. 28, no. 4, pp1325-1330, 2020

[13] Y.A. Hwang and Y.H. Liao, "The Core Configuration for Fuzzy Games," *Journal of Intelligent and Fuzzy Systems*, vol. 27, pp. 3007-3014, 2014.

[14] J.C Huang, K.H.C. Chen and Y.H. Liao, "Symmetric and Asymmetric Allocating Concepts under Multiple-goals," *Symmetry*, vol. 16, pp1428, 2024

[15] R.R. Huang, H.C. Wei, C.Y. Huang and Y.H. Liao, "Axiomatic Analysis for Scaled Allocating Rule," *IAENG International Journal of Applied Mathematics*, vol. 52, no. 1, pp254-260, 2022

[16] F. Klijn, M. Slikker and J. Zazuelo, "Characterizations of a Multi-choice Value," *International Journal of Game Theory*, vol. 28, pp521-532, 1999

[17] E. Lehrer, "An Axiomatization of the Banzhaf Value," *International Journal of Game Theory*, vol. 17, pp89-99, 1988

[18] S. Li and Q. Zhang, "A Simplified Expression of the Shapley Function for Fuzzy Game," *European Journal of Operational Research*, vol. 196, pp. 234-245, 2009.

[19] C.H Li, J.W. Chiang, E.C. Chi and Y.H. Liao, "Axiomatic and Dynamic Processes for A Symmetric Allocation Rule," *Symmetry*, vol. 13, no. 8, pp1474, 2021

[20] Y.H. Liao, "The Maximal Equal Allocation of Non-separable Costs on Multi-choice Games," *Economics Bulletin*, vol. 3, no. 70, pp1-8, 2008

[21] Y.H. Liao, "The Duplicate Extension for the Equal Allocation of Nonseparable Costs," *Operational Research: An International Journal*, vol. 13, pp385-397, 2012

[22] Y.H. Liao, "A Weighted Solution Concept under Replicated Behavior," *Mathematics*, vol. 11, pp150, 2023

[23] Y.H. Liao, C.H. Li, Y.C. Chen, L.Y. Tsai, Y.C. Hsu and C.K. Chen, "Agents, Activity Levels and Utility Distributing Mechanism: Game-theoretical Viewpoint," *IAENG International Journal of Applied Mathematics*, vol. 51, no. 4, pp867-873, 2021

[24] M. Maschler and G. Owen, "The Consistent Shapley Value for Hyperplane Games," *International Journal of Game Theory*, vol. 18, pp389-407, 1989

[25] S. Masuya and M. Inuiguchi, "A Fundamental Study for Partially Defined Cooperative Games," *Fuzzy Optimization and Decision Making*, vol. 15, pp. 281-306, 2016.

[26] F. Meng and Q. Zhang, "The Shapley Value on a Kind of Cooperative Fuzzy Games," *Journal of Computational Information Systems*, vol. 7, pp. 1846-1854, 2011.

[27] H. Moulin, "The Separability Axiom and Equal-sharing Methods," *Journal of Economic Theory*, vol. 36, pp120-148, 1985

[28] S. Muto, S. Ishihara, S. Fukuda, S. Tijs and R. Branzei, "Generalized Cores and Stable Sets for Fuzzy Games," *International Game Theory Review*, vol. 8, pp. 95-109, 2006.

[29] I. Nishizaki and M. Sakawa, *Fuzzy and Multiobjective Games for Conflict Resolution*, Physica-Verlag, Heidelberg, 2001.

[30] A van den. Nouweland, J. Potters, S. Tijs and J.M. Zarzuelo, "Core and Related Solution Concepts for Multi-choice Games," *ZOR-Mathematical Methods of Operations Research*, vol. 41, pp289-311, 1995

[31] J.S. Ransmeier, *The Tennessee Valley Authority*, Vanderbilt University Press, Nashville, 1942.

[32] L.S Shapley, "A Value for n-person Game," in: Kuhn, H.W., Tucker, A.W.(Eds.), *Contributions to the Theory of Games* **II**, Princeton, 1953, pp307-317

[33] L.S. Shapley, *Discussant's Comment*, In: Moriarity S (ed) Joint cost allocation. University of Oklahoma Press, Tulsa, 1982.

[34] R.E. Stearns, "Convergent Transfer Schemes for n-person Games," *Transactions of the American Mathematical Society*, vol. 134, pp449-459, 1968

[35] B. Uapipatanakul, J.C. Huang, K.H.C. Chen, S. Ngammuangpak and Y.H. Liao, "Modeling Pollen Tube Polar Growth Pattern under Asymmetric Consideration and Creating Game-theoretical Model for Ecotoxicity Assessment," *IAENG International Journal of Applied Mathematics*, vol. 55, no. 1, pp16-25, 2025

[36] H.C. Wei, A.T. Li, W.N. Wang and Y.H. Liao, "Game-theoretical Evaluating Rules for Minimizing Losses under Multi-objective Situations," *Engineering Letters*, vol. 33, no. 2, pp265-274, 2025