Lump-Type Wave Solutions to the (4+1)-Dimensional Fokas Equation

Chun-Xue Zhao

Abstract— **Studying exact solutions to the high-dimensional nonlinear evolution equations provides valuable insights into the wave behaviors and interactions. This work employs the bilinear method to explore exact solutions of the (4+1)-dimensional Fokas equation. Specifically, three distinct types of lump solutions are constructed through the application of fourth-order symmetric matrices, exponential functions and trigonometric functions. Additionally, the characteristics of these lump solutions are analyzed and graphically represented to facilitate a deeper understanding of their properties.**

Keywords: Fokas equation, Hirota bilinear method, lump-type solutions, symmetric matrices

1 Introduction

Investigating integrable properties and the construction of exact solutions for nonlinear evolution equations (NLEEs) are of paramount importance in understanding nonlinear phenomena in physics and mathematics [1–4]. Various methods have been developed to obtain special solutions of NLEEs, including the inverse scattering transformation (IST) [5], Lie group method [6], Darboux transformation (DT) [7], and Hirota bilinear method [8], among others, in which, Hirota bilinear method, in particular, stands out as a canonical approach for investigating exact solutions of NLEEs.

Among the diverse types of exact solutions [9, 10], lump solutions are particularly noteworthy. Lump solutions, which are localized in all spatial directions, represent a class of rational function solutions. It has been demonstrated that many soliton equations [11], including the B-type Kadomtsev-Petviashvili (BKP) equation [12], the Davey-Stewartson (DS) equation [13], and so on.

Higher-dimensional integrable models are increasingly recognized for their significance in mathematical physics. This study concentrates on a (4+1)-dimensional nonlinear Fokas equation [14], which is expressed as:

$$4u_{x_1t} - u_{x_1x_1x_1x_2} + u_{x_1x_2x_2x_2} + 6(u^2)_{x_1x_2} - 6u_{y_1y_2} = 0, \quad (1.1)$$

where $u = u(x_1, x_2, y_1, y_2, t)$.

Due to the significant applications of higher-dimensional equations in practical scenarios, diverse solutions to Eq.(1.1) have been thoroughly investigated. Yang et al. [15] discussed the Lie point symmetries and extracted doubly periodic wave solutions. Lee et al. [16] employed three exact methods to derive some exact solutions of the proposed equation. Wang et al. [17] obtained rogue wave solutions of the equation. Cao et al. [18] investigated various wave solutions for Eq.(1.1), including localized solitary waves, breather-type waves, multi-solitons and rogue waves. Despite the extensive studies conducted on this equation, our findings in this work represent previously unreported results in this field. The primary objective of this study is to seek three different kinds of lump solutions using an appropriate linear transformation and analyze the characteristics of the solutions.

This paper is organized as follows. We first present the Hirota bilinear forms of the (4+1)-dimensional Fokas equation using a suitable linear transformation. Subsequently, we derive three different kinds of lump-type solutions for the dimensionally reduced equation and depict their characteristics through graphical representations in sections 2- 4. Section 5 provides a concluding summary of our key results.

2 Type I lump solutions

This study utilizes the following transformation

$$\begin{aligned} x &= k_1x_1 + k_2x_2, \\ y &= k_3y_1 + k_4y_2, \\ u &= (k_2^2 - k_1^2)(\ln f)_{xx}, \end{aligned} \quad (2.1)$$

the bilinear form of Eq.(1.1) is

$$(4D_xD_t + k_2(k_2^2 - k_1^2)D_x^4 - \frac{6k_3k_4}{k_1}D_y^2)f \cdot f = 0, \quad (2.2)$$

where D represents Hirota bilinear operator defined as

$$\begin{aligned} D_\delta^m D_\sigma^n \alpha \cdot \beta = (\frac{\partial}{\partial \delta} - \frac{\partial}{\partial \delta'})^m (\frac{\partial}{\partial \sigma} - \frac{\partial}{\partial \sigma'})^n \alpha(\delta, \sigma) \\ \cdot \beta(\delta', \sigma')|_{\delta'=\delta, \sigma'=\sigma}, \end{aligned} \quad (2.3)$$

where $\alpha, \beta \in C^\infty(R^2)$. At this point, Eq.(1.1) transforms into the KP-type equation.

To obtain new lump solutions, we take the following assumption

$$f = X^TAX + \mu_0, \quad (2.4)$$

where $X = (1, x, y, t)^T, A = (a_{ij})$ is a fourth-order symmetric matrix, and $a_{ij}(i, j = 1, 2, 3, 4), \mu_0$ are real constants to be determined. Then, we insert Eq.(2.4) into

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Eq.(2.2), let all the coefficients of different polynomial of x, y, t be zero. We obtain two cases of a_{ij}, μ_0 as follows

Case I:

$$\begin{aligned}
a_{11} &= \frac{B_1}{32 a_{33} a_{44}^2 k_1^3}, \\
a_{12} &= -\frac{3 a_{14} a_{33} k_3 k_4}{2 a_{44} k_1}, \\
a_{13} &= a_{13}, a_{14} = a_{14}, \\
a_{22} &= \frac{9 a_{33}^2 k_3^2 k_4^2}{4 a_{44} k_1^2}, a_{23} = 0, \\
a_{24} &= -\frac{3 a_{33} k_3 k_4}{2 k_1} \\
a_{33} &= a_{33}, a_{34} = 0, \\
a_{44} &= a_{44}, \mu_0 = \mu_0, \qquad (2.5)
\end{aligned}$$

where $a_{13}, a_{14}, a_{33}, a_{44}, \mu_0$ are free parameters,

$$\begin{aligned}
B_1 = &-81 a_{33}^4 k_3^3 k_4^3 k_2 k_1^2 + 81 a_{33}^4 k_3^3 k_4^3 k_2^3 \\
&+ 32 a_{13}^2 a_{44}^2 k_1^3 + 32 a_{14}^2 a_{33} a_{44} k_1^3 \\
&- 32 a_{33} \mu_0 a_{44}^2 k_1^3.
\end{aligned}$$

Case II:

$$\begin{aligned}
a_{11} &= \frac{B_2}{2(a_{22} a_{33} - a_{23}^2) k_3 k_4}, \\
a_{12} &= a_{12}, a_{13} = a_{13}, \\
a_{14} &= -\frac{3 k_3 k_4 (a_{12} a_{33} - 2 a_{13} a_{23})}{2 a_{22} k_1}, \\
a_{22} &= a_{22}, a_{23} = a_{23}, \\
a_{24} &= -\frac{3 k_3 k_4 (a_{22} a_{33} - 2 a_{23}^2)}{2 a_{22} k_1}, \\
a_{33} &= a_{33}, a_{34} = \frac{3 a_{23} a_{33} k_3 k_4}{2 a_{22} k_1}, \\
a_{44} &= \frac{9 a_{33}^2 k_3^2 k_4^2}{4 a_{22} k_1^2}, \mu_0 = \mu_0, \qquad (2.6)
\end{aligned}$$

where $a_{12}, a_{13}, a_{22}, a_{23}, a_{33}, \mu_0$ are free parameters,

$$\begin{aligned}
B_2 = &-a_{22}^3 k_1^3 k_2 + a_{22}^3 k_1 k_2^3 + 2 a_{12}^2 a_{33} k_3 k_4 \\
&- 4 a_{12} a_{13} a_{23} k_3 k_4 + 2 a_{13}^2 a_{22} k_3 k_4 \\
&- 2 a_{22} a_{33} k_3 k_4 \mu_0 + 2 a_{23}^2 k_3 k_4 \mu_0.
\end{aligned}$$

For case I, the solutions in (2.5) are analytic if the parameters satisfy that

$$a_{33} \neq 0, a_{44} \neq 0, k_1 \neq 0.$$

Let

$$\begin{aligned}
&a_{13} = 1, a_{14} = 2, a_{33} = 1, a_{44} = 2, \mu_0 = 1, \\
&k_1 = 1, k_2 = 2, k_3 = 1, k_4 = 1,
\end{aligned}$$

we get the solution of Eq.(1.1) as follows

$$u = \frac{27}{4 A_1} - \frac{3(-3 + \frac{9x}{4} - 3t)^2}{A_1^2}, \qquad (2.7)$$

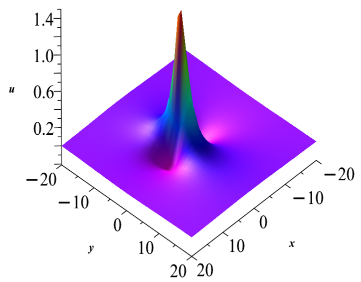

Figure 1: Lump solution.

where

$$\begin{aligned}
A_1 = &\frac{435}{64} - \frac{3x}{2} + y + 2t + (-\frac{3}{2} + \frac{9x}{8} - \frac{3t}{2})x \\
&+ (y+1)y + (2 - \frac{3x}{2} + 2t)t.
\end{aligned}$$

The localized, wave-like structure of the lump solutions is depicted in Fig.1. The density plot(Fig.2) provides a visual representation of the solution's intensity distribution, with warmer colors indicating higher amplitudes. The contour plot(Fig.3) offers a clear view of the solution's level curves, emphasizing its spatial symmetry and peak locations. Fig.4 shows the lump solution maintains its shape over time, highlighting its soliton nature, which reveals the temporal evolution of the lump solution, demonstrating its stability and propagation characteristics over time.

For case II, the solutions in (2.6) are analytic if the parameters satisfy that

$$a_{22} \neq 0, k_1 \neq 0, k_3 \neq 0, k_4 \neq 0, a_{22} a_{33} - a_{23}^2 \neq 0.$$

Let

$$\begin{aligned}
&a_{12} = 1, a_{13} = 1, a_{22} = 2, a_{23} = 1, a_{33} = 1, \\
&\mu_0 = 1, k_1 = 1, k_2 = 2, k_3 = 1, k_4 = 1,
\end{aligned}$$

we obtain the solution of Eq.(1.1) as follows

$$u = \frac{12}{A_2} - \frac{3(2 + 4x + 2y)^2}{A_2^2}, \qquad (2.8)$$

where

$$\begin{aligned}
A_2 = &x + y + \frac{3t}{4} + 25 + (2x + y + 1)x \\
&+ (x + y + \frac{3t}{4} + 1)y + (\frac{3}{4} + \frac{3y}{4} + \frac{9t}{8})t.
\end{aligned}$$

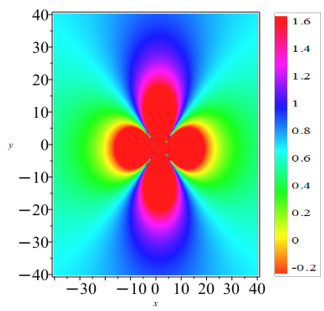

Figure 2: Density plot.

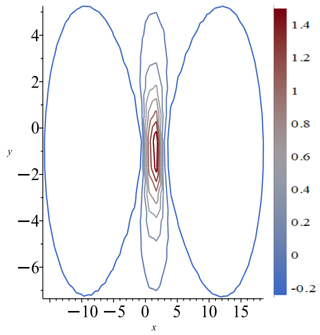

Figure 3: Contour plot.

As $a_{22} = 2$, Figs.5-6 depict the lump solution and a graphical illustration of the solution's intensity distribution, respectively. Subsequently, we modify the parameter $a_{22} = 200$, and the outcomes are displayed in Figs.7-8. These results reveal that the symmetry of the solution has changed.

3 Type II lump solutions

To obtain the second type of lump solution, we take the following assumption

$$f = X^T AX + b_1 \exp(b_2 x + b_3 y + b4t + b5) + \mu_0, \quad (3.1)$$

where $X = (1, x, y, t)^T, A = (a_{ij})$ is a fourth-order symmetric matrix, and $a_{ij}(i, j = 1, 2, \ldots, 4), b_i(i = 1, 2, \ldots, 5), \mu_0$ are real constants to be determined.

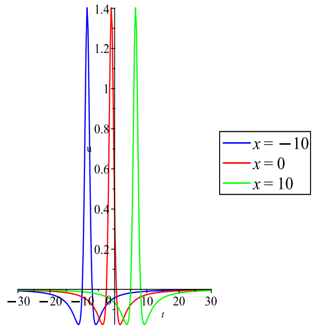

Figure 4: Dynamic analysis.

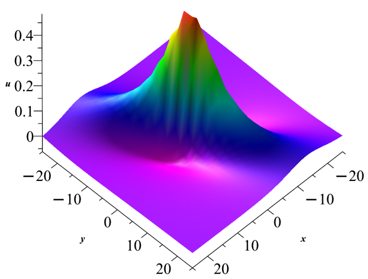

Figure 5: Lump solution.

Then, we insert Eq.(3.1) into Eq.(2.2), let all the coefficients of different polynomial of x, y, t be zero. We obtain three cases of a_{ij}, b_i, μ_0 as follows

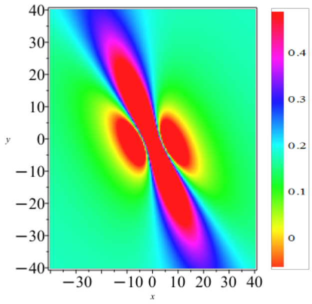

Figure 6: Density plot.

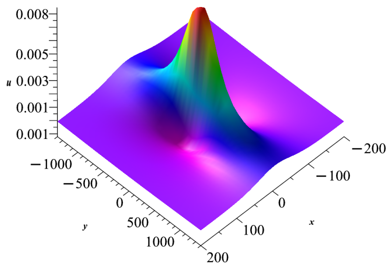

Figure 7: Lump solution.

Case I:

$$a_{11} = \frac{B_3}{a_{22}b_2^2k_1k_2(k_1^2 - k_2^2)},$$
$$a_{12} = a_{12}, a_{13} = a_{13}, a_{14} = \frac{3a_{12}(k_1^2 - k_2^2)k_2b_2^2}{4},$$
$$a_{22} = a_{22}, a_{23} = 0, a_{24} = \frac{3(k_1^2 - k_2^2)k_2a_{22}b_2^2}{4},$$
$$a_{33} = -\frac{1}{2}a_{22}b_2^2k_1^3k_2 + \frac{1}{2}a_{22}b_2^2k_1k_2^3, a_{34} = 0,$$
$$a_{44} = \frac{9a_{22}b_2^4k_2^2(k_1^2 - k_2^2)^2}{16}, b_1 = b_1, b_2 = b_2, b_3 = 0,$$
$$b_4 = \frac{b_2^3k_2(k_1^2 - k_2^2)}{4}, b_5 = b_5, \mu_0 = \mu_0, \tag{3.2}$$

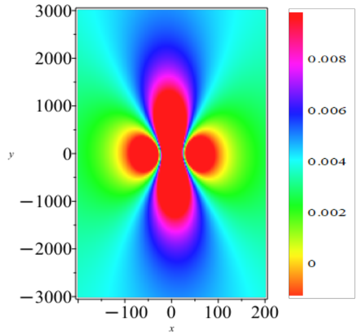

Figure 8: Density plot.

where $a_{12}, a_{13}, a_{22}, b_1, b_2, b_5, \mu_0$ are free parameters,

$$B_3 = b_2^2a_{12}^2k_1^3k_2 - b_2^2a_{12}^2k_1k_2^3 - b_2^2a_{22}k_1^3k_2\mu_0 + b_2^2a_{22}k_1k_2^3\mu_0 + a_{22}^2k_1^3k_2 - a_{22}^2k_1k_2^3 - 2a_{13}^2. \tag{3.3}$$

Case II:

$$a_{11} = \frac{B_4}{a_{22}^3b_2^2k_1k_2(k_1^2 - k_2^2)}, a_{12} = a_{12}, a_{13} = a_{13},$$
$$a_{14} = \frac{3B_5}{4a_{22}^2k_1}, a_{22} = a_{22}, a_{23} = a_{23},$$
$$a_{24} = \frac{3(a_{22}^2b_2^2k_1^3k_2 - a_{22}^2b_2^2k_1k_2^3 + 2a_{23}^2)}{4a_{22}k_1},$$
$$a_{33} = -\frac{a_{22}^2b_2^2k_1^3k_2 - a_{22}^2b_2^2k_1k_2^3 - 2a_{23}^2}{2a_{22}},$$
$$a_{34} = -\frac{3a_{23}(a_{22}^2b_2^2k_1^3k_2 - a_{22}^2b_2^2k_1k_2^3 - 2a_{23}^2)}{4a_{22}^2k_1},$$
$$a_{44} = \frac{9(a_{22}^2b_2^2k_1^3k_2 - a_{22}^2b_2^2k_1k_2^3 - 2a_{23}^2)^2}{16a_{22}^3k_1^2},$$
$$b_1 = b_1, b_2 = b_2, b_3 = \frac{a_{23}b_2}{a_{22}},$$
$$b_4 = \frac{b_2(a_{22}^2b_2^2k_1^3k_2 - a_{22}^2b_2^2k_1k_2^3 + 6a_{23}^2)}{4a_{22}^2k_1},$$
$$b_5 = b_5, \mu_0 = \mu_0, \tag{3.4}$$

where $a_{12}, a_{13}, a_{22}, a_{23}, b_1, b_2, b_5, \mu_0$ are free parameters,

$$B_4 = b_2^2a_{12}^2a_{22}^2k_1^3k_2 - b_2^2a_{12}^2a_{22}^2k_1k_2^3 - b_2^2a_{22}^3k_1^3k_2\mu_0 + b_2^2a_{22}^3k_1k_2^3\mu_0 + a_{22}^4k_1^3k_2 - a_{22}^4k_1k_2^3 - 2a_{12}^2a_{23}^2 + 4a_{12}a_{13}a_{23}a_{22} - 2a_{13}^2a_{22}^2,$$
$$B_5 = a_{12}a_{22}^2b_2^2k_1^3k_2 - a_{12}a_{22}^2b_2^2k_1k_2^3 - 2a_{12}a_{23}^2 + 4a_{13}a_{22}a_{23}.$$

Case III:

$$\begin{aligned}
a_{11} &= \frac{a_{12}^2 b_2^2 - a_{22} b_2^2 \mu_0 + a_{22}^2}{a_{22} b_2^2},\\
a_{12} &= a_{12}, a_{13} = \frac{a_{12} b_3}{b_2},\\
a_{14} &= \frac{3a_{12}(b_2^4 k_1^3 k_2 - b_2^4 k_1 k_2^3 + 2b_3^2)}{4k_1 b_2^2},\\
a_{22} &= a_{22}, a_{23} = \frac{a_{22} b_3}{b_2},\\
a_{24} &= \frac{3a_{22}(b_2^4 k_1^3 k_2 - b_2^4 k_1 k_2^3 + 2b_3^2)}{4k_1 b_2^2},\\
a_{33} &= -\frac{a_{22}(b_2^4 k_1^3 k_2 - b_2^4 k_1 k_2^3 - 2b_3^2)}{2b_2^2},\\
a_{34} &= -\frac{3a_{22} b_3 (b_2^4 k_1^3 k_2 - b_2^4 k_1 k_2^3 - 2b_3^2)}{4b_2^3 k_1},\\
a_{44} &= \frac{9a_{22}(b_2^4 k_1^3 k_2 - b_2^4 k_1 k_2^3 - 2b_3^2)^2}{16 b_2^4 k_1^2},\\
b_1 &= b_1, b_2 = b_2, b_3 = b_3,\\
b_4 &= \frac{b_2^4 k_1^3 k_2 - b_2^4 k_1 k_2^3 + 6b_3^2}{4b_2 k_1},\\
b_5 &= b_5, \mu_0 = \mu_0,
\end{aligned}$$

where $a_{12}, a_{22}, b_1, b_2, b_3, b_5, \mu_0$ are free parameters.

We take case I as an example, the solutions in (3.2) are analytic if

$$a_{22} b_2^2 k_1 k_2 (k_1^2 - k_2^2) \neq 0.$$

Let

$$a_{12} = 1, a_{13} = 1, a_{22} = 1, b_1 = 1, b_2 = 1,$$

$$b_5 = 1, \mu_0 = 1, k_1 = 1, k_2 = 2, k_3 = 1, k_4 = 1,$$

we get the solution of Eq.(1.1) as follows

$$u = \frac{3(2 + \exp(x - \frac{3t}{2} + 1)}{A_3} - \frac{3(2 + 2x - 9t + \exp(x - \frac{3t}{2} + 1))^2}{A_3^2}, \tag{3.5}$$

where

$$A_3 = \frac{7}{3} + x + y - \frac{9t}{2} + (1 + x - \frac{9t}{2})x + (1 + 3y)y + (-\frac{9}{2} - \frac{9x}{2} + \frac{81t}{4})t + \exp(x - \frac{3t}{2} + 1).$$

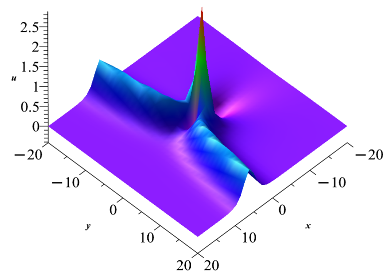

Figure 9: Lump solution.

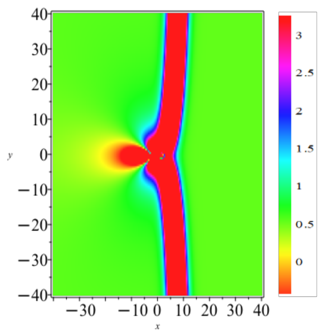

Figure 10: Density plot.

Figs.9-12 depict the solution for Case I. From Fig.9, both solitary waves and lump waves are observed simultaneously. The solitary waves are characterized by their stable, localized wave packets that propagate without changing shape, reflecting their robust nature in nonlinear systems. On the other hand, the lump waves appear as localized, two-dimensional structures with algebraic decay in all spatial directions, showcasing their unique properties in higher-dimensional settings. The interaction between these two types of waves is clearly illustrated, highlighting their coexistence and the distinct features of their waveforms. Figs.10-12 show density plot, contour plot, and dynamic analysis, respectively. Figs.13-15 present the solutions at three distinct moments, from which it is observed that the solitary wave and the lump wave coincide at a certain instant before separating. Subsequently, the distance between them continues to increase, indicating that the two waves propagate at different velocities.

4 Type III lump solutions

To obtain the lump-periodic solution, we take the following assumption

$$f = X^T A X + b_1 \sin(b_2 x + b_3 y + b_4 t + b_5) + \mu_0, \tag{4.1}$$

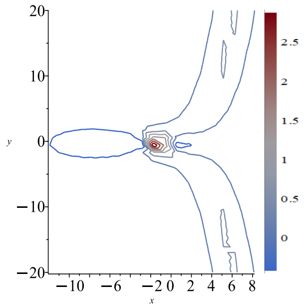

Figure 11: Contour plot.

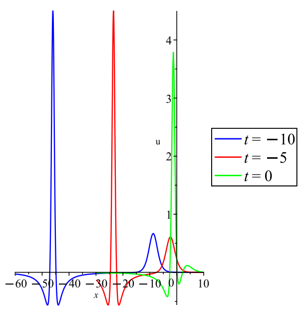

Figure 12: Dynamic analysis.

where $X = (1, x, y, t)^T, A = (a_{ij})$ is a fourth-order symmetric matrix, and $a_{ij}(i, j = 1, 2, \ldots, 4), b_i(i = 1, 2, \ldots, 5), \mu_0$ are real constants to be determined. Then, we substitute Eq.(4.1) in Eq.(2.2) and look up for the coefficients of different polynomials of x, y, t and trigonometric functions. Then, we set each coefficient to zero and get

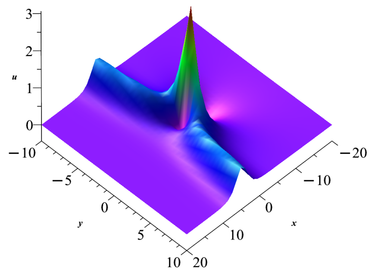

Figure 13: Lump solution as $t = 0$.

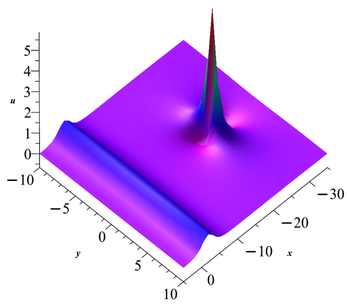

Figure 14: Lump solution as $t = -5$.

$$\begin{aligned}
&a_{11} = \frac{B_6}{2a_{22}a_{33}}, a_{12} = a_{12}, a_{13} = a_{13}, \\
&a_{14} = -\frac{3a_{33}a_{12}}{2a_{22}k_1}, a_{22} = a_{22}, a_{23} = 0, \\
&a_{24} = -\frac{3a_{33}}{2k_1}, a_{33} = a_{33}, a_{34} = 0, \\
&a_{44} = \frac{9a_{33}^2}{4a_{22}k_1^2}, b_1 = 0, b_2 = b_2, \\
&\quad b_3 = b_3, b_4 = b_4, b_5 = b_5, \mu_0 = \mu_0,
\end{aligned} \tag{4.2}$$

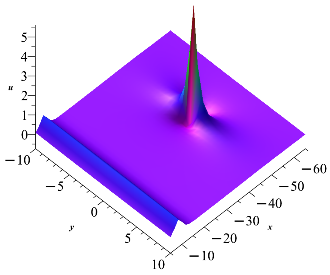

Figure 15: Lump solution as $t = -10$.

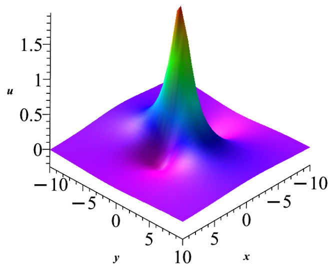

Figure 16: Lump solution.

where $a_{12}, a_{13}, a_{22}, a_{33}, b_2, b_3, b_4, b_5, \mu_0$ are free parameters.

$$B_6 = -a_{22}^3 k_1^3 k_2 + a_{22}^3 k_1 k_2^3 + 2a_{12}^2 a_{33} + 2a_{13}^2 a_{22} - 2a_{22} a_{33} \mu_0. \tag{4.3}$$

The solutions in (4.2) are analytic if

$$a_{22} \neq 0, a_{33} \neq 0, k_1 \neq 0.$$

Let

$$a_{12} = 1, a_{13} = 1, a_{22} = 1, a_{33} = 1, \mu_0 = 1, b_2 = 1,$$

$$b_3 = 1, b_4 = 1, b_5 = 1, k_1 = 1, k_2 = 2, k_3 = 1, k_4 = 1,$$

we the solution of Eq.(1.1) as follows

$$u = \frac{6}{A_4} - \frac{3(2 + 2x - 3t)^2}{A_4^2}, \tag{4.4}$$

where

$$A_4 = 5 + x + y - \frac{3t}{2} + (1 + x - \frac{3t}{2})x + (y + 1)y + (-\frac{3}{2} - \frac{3x}{2} + \frac{9t}{4})t.$$

The structure of the solution (4.3) is visualized in Fig.16. The density and contour plot are depicted in Figs.17-18. From the figures, it can be seen that under the assumption, the localized, wave-like structure of lump solutions and periodic phenomena cannot coexist within the same solution.

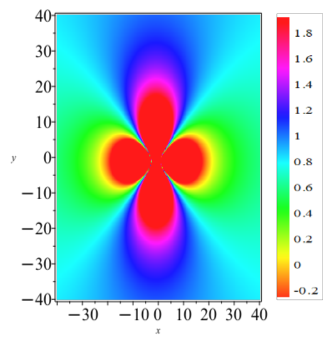

Figure 17: Density plot of solution (4.3).

5 Conclusions

In this paper, we utilize Hirota's bilinear method to investigate three types lump waves of the (4+1)-dimensional Fokas equation. By introducing a novel transformation within the bilinear formalism, we construct unique lump-type solutions and thoroughly analyze their dynamic properties. Through detailed visualizations in both 2D and 3D, we explore the shapes and spatial structures of these solutions, providing a comprehensive analysis of their dynamic evolution. The results not only demonstrate the effectiveness of the proposed method but also provide a foundation for future studies on lump solutions in more complex systems. The methods and insights presented here enrich new avenues for exploring nonlinear wave interactions in multidimensional settings.

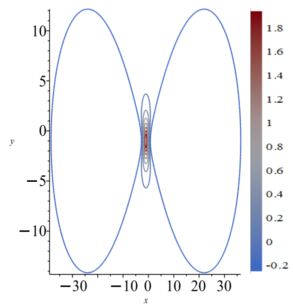

Figure 18: Contour plot of solution (4.3).

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